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A NOTE ON UNIQUENESS OF L-FUNCTIONS WITH REGARD TO MULTIPLICITY

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Abstract. By introducing the concept of multiplicity along with weighted sharing of small functions or fixed points, we investigate the uniqueness problems of meromorphic function and a L-function in the extended Selberg class. The results obtained in this paper improves the results due to Liu-Li-Yi ([14]), Sahoo-Halder ([18]), F. Liu-Li-Yi ([15]) and Wen-Jie Hao and Jun-Fan Chen ([3]).

1. Introduction and main results

Selberg studied that the Riemann hypothesis is also true for L-functions in the Selberg class. An L-function based on Riemann Zeta function as the prototype is defined as Dirichlet series

$$L(s)=\sum_{n=1}^{\infty}a(n)n^{-s}, \tag{1.1}$$

of a complex number $s=\sigma+it$ satisfying the following axioms([19]):

(i) Ramanujan hypothesis: $a(n)\ll n^{\epsilon}$ for every $\epsilon>0$;

(ii) Analytic continuation: There is a non-negative integer m such that $(s-1)^mL(s)$ is an entire function of finite order;

(iii) Functional equation: L satisfies a functional equation of type

$$\Lambda_L(s)=\omega\overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s)=L(s)Q^s\prod_{j=1}^{k}\Gamma(\lambda_js+\nu_j)$$

with positive real numbers Q,λ_j and complex numbers ν_j,ω with $Re\nu_j\geq 0$ and $|\omega|=1$;

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(iv) Euler product hypothesis: $logL(s) = \sum_{n=1}^{\infty}\left(\frac{b(n)}{n^s}\right)$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

The degree d of an L-function L is defined to be

$$d = 2\sum_{j=1}^{k}\lambda_j,$$

where k and λ_j are respectively the positive real numbers defined in axiom (iii) of the definition of L-function.

Also, there are many Dirichlet series only satisfying axioms (i)-(iii)([3]) and are regarded as the extended Selberg class. Here, the study on L-functions is carried out from the extended Selberg class. Thus, the conclusions are also true for L-functions in the Selberg class.

From past few years, there has been an interesting study on value distribution of L-functions related to some meromorphic functions ([23], [6], [11], [12]). The uniqueness result on L-functions was first studied by Steuding ([23]) is given as follows:

Theorem A.([23]) If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c(\neq\infty)$ CM, then $L_1 = L_2$.

Now, since L-functions are analytically continued as meromorphic functions, many researchers are working on an L-function sharing values with an arbitrary meromorphic function.

Recently, the authors Liu-Li-Yi([14], [15]) obtained the following results.

Theorem B.([14]) Let f be a nonconstant meromorphic function, L be an L-function and let n, k be two positive integers such that $n > 3k + 6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.

Theorem C.([14]) Let f be a nonconstant meromorphic function, L be an L-function and let n, k be two positive integers such that $n > 3k + 6$. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share 0 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.

Theorem D.([15]) Let f be a nonconstant meromorphic function, L be an L-function and let n, k be two positive integers. Suppose that $[f^n(f-1)]^{(k)}$ and $[L^n(L-1)]^{(k)}$ share 1 CM. If $n > 3k + 9$ and $k \geq 2$, then $f \equiv L$.

Later, the authors W-J Hao and Jun-Fan Chen ([3]), generalised the above result, by considering the general differential polynomials $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ and obtained the results as follows:

Theorem E.([3]) Let f be a nonconstant meromorphic function, L be an L-function and let n, m and k be three positive integers. Suppose that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share 1 CM. If $n > 3k + m + 6$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.

Theorem F.([3]) Let f be a nonconstant meromorphic function, L be an L-function and let n, m and k be three positive integers. Suppose that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share 1 IM. If $n > 7k + 4m + 11$, $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.

In 2018, the authors Pulak Sahoo and Samer Halder ([18]) considered Theorems B and C, by introducing weighted sharing of values improved the results as follows:

Theorem G.([18]) Let f be a nonconstant meromorphic function, L be an L-function and let n, k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(1, l)$ and one of the following conditions is satisfied:

(i) $l \geq 2$ and $n > 3k + 6$,
(ii) $l = 1$ and $n > \frac{7k}{2} + \frac{13}{2}$,
(iii) $l = 0$ and $n > 7k + 11$,

then $f = tL$ for a constant t satisfying $t^n = 1$.

Theorem H.([18]) Let f be a nonconstant meromorphic function, L be an L-function and let n, k be two positive integers. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share $(0, l)$ and one of the following conditions is satisfied:

(i) $l \geq 2$ and $n > 3k + 6$,
(ii) $l = 1$ and $n > \frac{7k}{2} + \frac{13}{2}$,
(iii) $l = 0$ and $n > 7k + 11$,

then $f = tL$ for a constant t satisfying $t^n = 1$.

The following question is obvious in this direction.

Question 1.1. What can be said about the uniqueness of the functions of the form $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ with regard to multiplicity, when they share a small function z with weight l.

Definition 1.1.([9]) Let l be a non-negative integer or infinity. Let $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_l(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. Also, if $E_l(a; f) = E_l(a; g)$, we say that f, g share the value a with weight l.

We write f, g share (a, l) to mean that f, g share the value a with weight l. Also, if f, g share (a, l) then f, g share (a, p) for any integer p, where $0 \leq p < l$. Also, we note that f, g share the value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

We give a positive answer to the above question, by considering the concept of weighted sharing of values and considering the functions $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ with regard to multiplicity, we obtain results which improve theorems B-H.

The following theorem is our main result.

Theorem 1.1. Let f be a nonconstant meromorphic function, L be an L-function and also let f and L have multiplicities atleast s. Suppose $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share (z, l) and if one of the following conditions is satisfied:

(i) $l \geq 2$ and $s(n + m) > max\{4k + 12, 6k + 12\}$,
(ii) $l = 1$ and $s(n + m) > max\{6k + 14, 7k + 13\}$,
(iii) $l = 0$ and $s(n + m) > max\{14k + 20, 14k + 22\}$,

then either $f \equiv HL$, where H is dth root of unity, where $d = GCD\{n + m, ..., n + m - i, ..., n\}$ or $f^n(f-1)^m \equiv L^n(L-1)^m$.

The following example shows that the conditions obtained in Theorem 1.1. are necessary for the unicity result.

Example 1.1. Let us consider $L = \zeta$ and $f = -\zeta$, where ζ is the Riemann zeta function which has a simple pole. By hypothesis of the theorem $F = [f^n(f-1)^m]^{(k)}$

and $G = [L^n(L-1)^m]^{(k)}$ share (z,l) and the conditions are satisfied for different weights when $s = 1$ and $m = 0$. Then we get $f \equiv HL$, where H is dth root of unity.

Remarks.

(1) Let $s = 2$, $m = 0$ in Theorem 1.1, then for (i) $l \geq 2$, $n > 3k + 6$; (ii) $l = 1$, $n > \frac{7k}{2} + \frac{13}{2}$; (iii) $l = 0$, $n > 7k + 11$, which is the result stated in Theorem H.

(2) Let $s = 2$, $m = 1$ and when $l \geq 2$ i.e., f and L share $(1, \infty)$ in Theorem 1.1, we get $n > 3k + 5$, which improves the result given in ([15]), where $n > 3k + 9$.

(3) Let $s = 2$, $m = 1$ and when $l = 0$ i.e., when f and L share 1 IM in Theorem 1.1, we get $n > 7k + 10$, which improves the result given by (See Corollary 14, [3]), where $n > 7k + 15$.

Let f be a non-constant meromorphic function in the complex plane. The order $\rho(f)$ and the lower order $\mu(f)$ of a meromorphic function f (see [4], [27], [29]) is given by

$$\rho(f) = \lim_{r\to\infty} \sup \frac{logT(r,f)}{logr}, \mu(f) = \lim_{r\to\infty} \inf \frac{logT(r,f)}{logr}.$$

We need the following notations and definitions in the study.

Definition 1.2.([8]) For $a \in \overline{\mathbb{C}}$, let us denote $N(r,a;f| = 1)$ as the counting function of simple a-points of f. For a positive integer p we denote by $N(r,a;f| \leq p)$ the counting function of those a-points of f whose multiplicities are not greater than p. By $\overline{N}(r,a;f| \leq p)$ is the corresponding reduced counting function.

Similarly, the terms $N(r,a;f| \geq p)$ and $\overline{N}(r,a;f| \geq p)$ are defined.

For a positive integer p or infinity. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f| \geq 2) + ... + \overline{N}(r,a;f| \geq p).$$

Clearly, we see that $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.3. Let $a \in \mathbb{C} \cup \{\infty\}$ and let k be an arbitrary non-negative integer. Then, we know that

$$\Theta(a,f) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}(r,a;f)}{T(r,f)}$$

and

$$\delta_k(a,f) = 1 - \lim_{r\to\infty} \sup \frac{N_k(r,a;f)}{T(r,f)}.$$

We also note that,

$$0 \leq \delta_k(a,f) \leq \delta_{k-1}(a,f) \leq ... \leq \delta_1(a,f) \leq \Theta(a,f) \leq 1.$$

2. Lemmas

The following lemmas are used in the sequel.

Lemma 2.1.[25] Suppose that f is a nonconstant meromorphic function and let $a_0, a_1, ..., a_n$ be finite complex numbers such that $a_n \neq 0$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.2.[27] Let f be a nonconstant meromorphic function and k be a positive integer. Then
$$N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,f) + S(r,f),$$
as $r \to \infty$, except possibly outside a set of finite linear measure.

Lemma 2.3.[4] Let f be a nonconstant meromorphic function, k be a positive integer and let c be a nonzero finite complex number. Then
$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f), \end{aligned}$$
where $N_0(r,0;f^{(k+1)})$ is the counting function of those zeros of $f^{(k+1)}$ in $|z|<r$ which are not the zeros of $f(f^{(k)}-c)$ in $|z|<r$.

Lemma 2.4.[5] Let f be a transcendental meromorphic function in the complex plane. Then corresponding to each $\Lambda > 1$, there exists a set $M(\Lambda) \subset (0,+\infty)$, with lower logarithmic density not exceeding the value $d(\Lambda) = 1-(2e^{\Lambda-1}-1)^{-1} > 0$
$$\underline{logdensM(\Lambda)} = \lim_{r\to\infty} \inf \frac{1}{logr}\int_{M(\Lambda)\cap[1,r]} \frac{dt}{t} \leq d(\Lambda),$$
provided that for all $r \notin M(\Lambda)$ and for each positive integer k,
$$\lim_{r\to\infty} \sup \frac{T(r,f)}{T(r,f^{(k)})} \leq 3e\Lambda.$$

Lemma 2.5.[28] Let f be a non-constant meromorphic function, $\alpha \not\equiv 0,\infty$ be a small function of f. Then
$$T(r,f) \leq \overline{N}(r,f) + N(r,0;f) + N(r,0;f^{(k)}-\alpha) - N\left(r,0;\left(\frac{f^{(k)}}{\alpha}\right)'\right) + S(r,f).$$

Lemma 2.6.[30] Let $E \subset (0,+\infty)$ be a set of finite linear measure and let f_1, f_2 be two non-constant meromorphic functions such that $\overline{N}(r,f_j) + \overline{N}(r,0;f_j) = S(r), (j=1,2)$. Then either $\overline{N}_0(r,1;f_1,f_2) = S(r)$ or there exists two integers p and q satisfying $|p|+|q|>0$ such that $f_1^p f_2^q = 1$. Here $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function f the common 1-points of f_1 and f_2 in $|z|<r$, $T(r) = T(r,f_1)+T(r,f_2)$ and $S(r) = o(T(r))$ as $r\to\infty$ and $r \notin E$.

Lemma 2.7.[17] Let F and G be two transcendental meromorphic functions and let $k(\geq 1)$, $l(\geq 0)$ be two integers. Suppose that $F^{(k)}-P$ and $G^{(k)}-P$ share $(0,l)$, where $P \not\equiv 0$ is a polynomial. Then either $F^{(k)}G^{(k)} = P^2$ or $F \equiv G$, whenever F and G satisfies one of the following conditions:
(i) $l \geq 2$ and
$$\Delta_{11} = 2\Theta(\infty,F) + (k+2)\Theta(\infty,G) + \delta_{k+2}(0,F) + \delta_{k+2}(0,G) > k+5$$
and
$$\Delta_{12} = 2\Theta(\infty,G) + (k+2)\Theta(\infty,F) + \delta_{k+2}(0,G) + \delta_{k+2}(0,F) > k+5;$$

(ii) $l=1$ and

$$\begin{aligned}\Delta_{21} &= \left(\frac{k}{2}+\frac{5}{2}\right)\Theta(\infty,F)+(k+2)\Theta(\infty,G)+\frac{1}{2}\delta_{k+1}(0,F)+\delta_{k+2}(0,F)+\delta_{k+2}(0,G)\\ &> \frac{3k}{2}+6\end{aligned}$$

and

$$\begin{aligned}\Delta_{22} &= \left(\frac{k}{2}+\frac{5}{2}\right)\Theta(\infty,G)+(k+2)\Theta(\infty,F)+\frac{1}{2}\delta_{k+1}(0,G)+\delta_{k+2}(0,G)+\delta_{k+2}(0,F)\\ &> \frac{3k}{2}+6;\end{aligned}$$

(iii) $l=0$ and

$$\begin{aligned}\Delta_{31} &= (2k+4)\Theta(\infty,F)+(2k+3)\Theta(\infty,G)+2\delta_{k+1}(0,F)\\ &\qquad +\delta_{k+1}(0,G)+\delta_{k+2}(0,F)+\delta_{k+2}(0,G)\\ &> 4k+11\end{aligned}$$

and

$$\begin{aligned}\Delta_{32} &= (2k+4)\Theta(\infty,G)+(2k+3)\Theta(\infty,F)+2\delta_{k+1}(0,G)+\delta_{k+1}(0,F)\\ &\qquad +\delta_{k+2}(0,G)+\delta_{k+2}(0,F)\\ &> 4k+11.\end{aligned}$$

Lemma 2.8.[16] Let f be a non constant meromorphic function and let

$$F=\frac{\sum_{k=0}^{p}a_kf^k}{\sum_{j=0}^{q}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p\neq 0$ and $b_q\neq 0$. Then $T(r,F)=d_0T(r,f)+O(1)$, where $d_0=max\{p,q\}$.

3. Proof of Theorem 1.1.

Let f and L be meromorphic functions whose zeros and poles are with multiplicities atleast s.

Let $F=[f^n(f-1)^m]^{(k)}$, $G=[L^n(L-1)^m]^{(k)}$.

Also, let $F_1=f^n(f-1)^m$, $G_1=L^n(L-1)^m$. Then $F=(F_1)^{(k)}$, $G=(G_1)^{(k)}$.

By Lemma 2.8, we get

$T(r,F_1)=(n+m)T(r,f)+S(r,f)$, $T(r,G_1)=(n+m)T(r,L)+S(r,L)$.

Let d be the degree of L. Then $d=2\sum_{j=1}^{k}\lambda_j>0$, where k and λ_j are the numbers in the axiom (iii) of the definition of L-function. Thus, by Steuding ([23], p.150), we get

$$T(r,L)=\frac{d}{\pi}rlogr+O(r). \tag{3.1}$$

Let z_0 be any zero of L of multiplicity q_0, then z_0 is also a zero of $\left[\frac{L^n(L-1)^m}{z}\right]^{(k)}$ with multiplicity atleast $(n+m)q_0-k-2$. Also, let z_1 be zero of $\frac{[L^n(L-1)^m]^{(k)}}{z}-1$ of multiplicity q_1, then z_1 is also a zero of $\left(\frac{L^n(L-1)^m}{z}\right)'$ of multiplicity q_1-1.

Since an L-function can have atmost one pole at $z = 1$ in the complex plane, using (3.1), Lemma 2.1 and Lemma 2.5, we get

$$\begin{aligned}
(n+m)T(r,L) &= T(r,G_1) + S(r,f) \\
&= T(r, L^n(L-1)^m) + S(r,f) \\
&\leq N(r,0;L^n(L-1)^m) + N\left(r,0;\frac{(L^n(L-1)^m)^{(k)}}{z} - 1\right) - \\
&\qquad N\left(r,0;\left(\frac{L^n(L-1)^m}{z}\right)'\right) + S(r,f) \\
&\leq N_{k+2}(r,0;L^n(L-1)^m) + \overline{N}\left(r,0;\frac{(L^n(L-1)^m)^{(k)}}{z} - 1\right) - \\
&\qquad N_0\left(r,0;\left(\frac{(L^n(L-1)^m)^{(k)}}{z}\right)'\right) + S(r,f) \\
&\leq (k+2)\overline{N}(r,0;L^n(L-1)^m) + \overline{N}\left(r,0;\frac{(L^n(L-1)^m)^{(k)}}{z} - 1\right) - \\
&\qquad N_0\left(r,0;\left(\frac{(L^n(L-1)^m)^{(k)}}{z}\right)'\right) + S(r,f)
\end{aligned}$$

Since $F_1 = f^n(f-1)^m$, $\frac{(F_1)^{(k)}}{z} - 1$ and $\frac{(G_1)^{(k)}}{z} - 1$ share 0 CM, hence $\frac{(G_1)^{(k)}}{z}$ can be replaced by $\frac{(F_1)^{(k)}}{z}$.

$$\begin{aligned}
(n+m)T(r,L) &\leq (k+2)T(r,L^n(L-1)^m) + \overline{N}\left(r,0;\frac{(L^n(L-1)^m)^{(k)}}{z} - 1\right) + S(r,f) \\
&\leq (k+2)(n+m)T(r,L) + T(r,(F_1)^{(k)}) + S(r,f).
\end{aligned}$$

This gives

$$[(n+m)\{1-(k+2)\}]T(r,L) \leq T(r,(F_1)^{(k)}) + S(r,f). \tag{3.2}$$

By our assumption, zeros and poles of f and L are of multiplicities atleast s, we have

$$\overline{N}(r,f) \leq \frac{1}{s}N(r,f) \leq \frac{1}{s}T(r,f),\ \overline{N}(r,\frac{1}{f}) \leq \frac{1}{s}N(r,\frac{1}{f}) \leq \frac{1}{s}T(r,f).$$

Similarly, we also have

$$\overline{N}(r,L) \leq \frac{1}{s}N(r,L) \leq \frac{1}{s}T(r,L),\ \overline{N}(r,\frac{1}{L}) \leq \frac{1}{s}N(r,\frac{1}{L}) \leq \frac{1}{s}T(r,L).$$

We now find the estimates of the deficiency terms as follows

$$\Theta(\infty, F_1) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}(r, F_1)}{T(r, F_1)}$$

$$\therefore \Theta(\infty, F_1) \geq 1 - \frac{\frac{2}{s}T(r, f)}{(n+m)T(r, f) + O(1)}$$

$$\therefore \Theta(\infty, F_1) \geq 1 - \frac{2}{s(n+m)} \tag{3.3}$$

$$\delta_{k+2}(0, F_1) = 1 - \lim_{r\to\infty} \sup \frac{N_{k+2}(r, 0; F_1)}{T(r, F_1)}$$

$$\therefore \delta_{k+2}(0, F_1) \geq 1 - \frac{2(k+2)}{s(n+m)} \tag{3.4}$$

Also,

$$\delta_{k+1}(0, F_1) \geq 1 - \frac{2(k+1)}{s(n+m)} \tag{3.5}$$

And similarly, we have

$$\delta_{k+2}(0, G_1) = 1 - \lim_{r\to\infty} \sup \frac{N_{k+2}(r, 0; G_1)}{T(r, G_1)}$$

$$\therefore \delta_{k+2}(0, G_1) \geq 1 - \frac{2(k+2)}{s(n+m)} \tag{3.6}$$

Also, since an L-function has atmost one pole $z = 1$ in the complex plane, we have

$$N(r, G_1) \leq logr + O(1).$$

By using (3.1) we deduce that

$$\Theta(\infty, G_1) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}(r, \infty; G_1)}{T(r, G_1)}.$$

Since $\overline{N}(r, \infty; G_1) = S(r, L)$, we have $\Theta(\infty, G_1) = 1$.

Let $F = [f^n(f-1)^m]^{(k)}$, $G = [L^n(L-1)^m]^{(k)}$ in Lemma 2.7, we get the following three cases.

Case 1. Let $l \geq 2$. Then

$$\Delta_{11} = 2\Theta(\infty, F) + (k+2)\Theta(\infty, G) + \delta_{k+2}(0, F) + \delta_{k+2}(0, G) > k+5$$

Now, here

$$\Delta_{11} = 2\Theta(\infty, F_1) + (k+2)\Theta(\infty, G_1) + \delta_{k+2}(0, F_1) + \delta_{k+2}(0, G_1)$$

Substituting (3.3), (3.4), (3.6) and simplifying, we get

$$\Delta_{11} = (k+6) - \frac{4}{s(n+m)} - \frac{4(k+2)}{s(n+m)},$$

since $\Delta_{11} > k+5$, then we get $s(n+m) > 4k+12$.

Also,

$$\Delta_{12} = 2\Theta(\infty, G) + (k+2)\Theta(\infty, F) + \delta_{k+2}(0, G) + \delta_{k+2}(0, F) > k+5;$$

Now, here

$$\Delta_{12} = 2\Theta(\infty, G_1) + (k+2)\Theta(\infty, F_1) + \delta_{k+2}(0, G_1) + \delta_{k+2}(0, F_1)$$

Substituting (3.3), (3.4), (3.6) and simplifying, we get

$$\Delta_{12} = (k+6) - \frac{6(k+2)}{s(n+m)}$$

since $\Delta_{12} > k+5$, then we get $s(n+m) > 6k+12$. We can rewrite this as,

$$s(n+m) > max\{4k+12, 6k+12\}$$

Now, by (i) of Lemma 2.7, we have either $(F_1)^{(k)}(G_1)^{(k)} = z^2$ or $F_1 = G_1$.
Let $F_1 = G_1$. Then

$$f^n(f-1)^m = L^n(L-1)^m \tag{3.7}$$

i.e.,

$$\begin{aligned} f^n(f^m + ... + (-1)^i C_m^{m-i} f^{m-i} + ... + (-1)^m) = \\ L^n(L^m + ... + (-1)^i C_m^{m-i} L^{m-i} + ... + (-1)^m) \end{aligned} \tag{3.8}$$

Let $H = \frac{f}{L}$. If H is a nonconstant meromorphic function, from (3.7), we have

$$f^n(f-1)^m = L^n(L-1)^m$$

. If H is a constant, then from (3.8), we get

$$L^{n+m}(H^{n+m}-1) + ... + (-1)^i C_m^{m-i} L^{n+m-i}(H^{n+m-i}-1) + ... + L^n(H^n - 1) = 0 \tag{3.9}$$

which implies $H^d = 1$, where $d = GCD\{n+m, ..., n+m-i, ..., n\}$. Therefore $H^d = \left(\frac{f}{L}\right)^d$ or $f^d = L^d$ or $f = HL$, where H is the dth root of unity.
Next, we assume that $(F_1)^{(k)}(G_1)^{(k)} = z^2$.

Claim. 0 is a Picard exceptional value of both f and L.
i.e., $f-0$ and $L-0$ has finitely many zeros.

On contrary, let $z_2(\neq 0) \in (C)$ be a zero of f with multiplicity $p_2(\geq 1)$. Then, from our assumption that $(F_1)^{(k)}(G_1)^{(k)} = z^2$, it follows that $z_2 = 1$ is a pole of L with multiplicity $q_2(\geq 1)$ such that $s(n+m)p_2 - k = s(n+m)q_2 + k$ i.e., $s(n+m) \leq 2k$, which is a contradiction to the lower bound of n as given in (i), (ii), (ii) of Theorem 1.1.

Hence, we get the proof for the claim for f. On the similar lines, we can prove the claim for L.

Again, by using (3.1), Lemma 2.1 and Theorem 1.15 ([27]), a result of Whittaker ([24]), the definition of the order of meromorphic function and also by assumption that

$$\frac{(F_1)^{(k)}}{z}\frac{(G_1)^{(k)}}{z} = 1, \text{we get}$$

$$\begin{aligned} \rho(f) = \rho(f^n(f-1)^m) = \rho([f^n(f-1)^m]^{(k)}) \\ = \rho([L^n(L-1)^m]^{(k)}) = \rho(L^n(L-1)^m) = \rho(L) = 1 \end{aligned} \tag{3.10}$$

Now, from (3.10), Lemma 2.2 and

$$\frac{(F_1)^{(k)}}{z}\frac{(G_1)^{(k)}}{z} = 1$$

and the fact that $z = 1$ is the only possible pole of L in $\mathbb{C}$, we get

$$\begin{aligned}(n+m+k)\overline{N}(r,f) &\leq N(r,[f^n(f-1)^m]^{(k)})\\ &\leq N\left(r,\frac{(F_1)^{(k)}}{z}\right)+O(1)\\ &\leq N\left(r,\frac{(G_1)^{(k)}}{z}\right)+O(1)\\ &\leq N\left(r,0;(G_1)^{(k)}\right)+O(1)\\ &\leq N(r,0;G_1)+k\overline{N}(r,G_1)+O(1)\\ &\leq S(r,f) \qquad (3.11)\end{aligned}$$

Also, since $z = 1$ is the only possible pole of L in $\mathbb{C}$, by (3.11) it follows that

$$\overline{N}(r,f)+\overline{N}(r,L)\leq S(r,f) \qquad (3.12)$$

Set

$$\Gamma_1=\frac{F^*}{G^*},\ \Gamma_2=\frac{F^*-1}{G^*-1}, \qquad (3.13)$$

where

$$F^*=\frac{(F_1)^{(k)}}{z},\ G^*=\frac{(G_1)^{(k)}}{z}.$$

By (3.13) and the assumption that f and L are transcendental meromorphic functions, we have

$$\Gamma_1\not\equiv 0,\ \Gamma_2\not\equiv 0.$$

Now, suppose that atleast one of Γ_1 and Γ_2 is a non-zero constant. Then, from (3.13), we observe that F^* and G^* share ∞ CM. Taking this together with the fact that $F^*G^*=1$, we find that ∞ is a Picard exceptional value of both f and L. Next, let us suppose that Γ_1 and Γ_2 are non-constant meromorphic functions.
From (3.13), we deduce that

$$F^*=\frac{\Gamma_1(1-\Gamma_2)}{\Gamma_1-\Gamma_2},\ G^*=\frac{1-\Gamma_2}{\Gamma_1-\Gamma_2} \qquad (3.14)$$

By (3.12), we get that there exists a subset $I\subset(0,+\infty)$ with the infinite linear measure such that $S(r)=o(T(r,f))$ and $T(r,G^*)\leq T(r,F^*)$.
Also,

$$\begin{aligned}T(r,F^*)&\leq 2\{T(r,\Gamma_1)+T(r,\Gamma_2)\}+S(r)\\ &\leq 8T(r,F^*)+S(r), \qquad (3.15)\end{aligned}$$

where $T(r)=T(r,\Gamma_1)+T(r,\Gamma_2)$.
By using (3.10), Lemma 2.2 and 0 is an evP of both f and L, we get

$$\begin{aligned}N(r,0;F^*)&=N\left(r,0;\frac{(F_1)^{(k)}}{z}\right)\\ &\leq k\overline{N}(r,f)+S(r,f). \qquad (3.16)\end{aligned}$$

Now, by (3.11), (3.12) and (3.16), we get

$$N(r,0;F^*)+N(r,0;G^*)\leq S(r,f). \qquad (3.17)$$

Since $F^*G^* = 1$, F^* and G^* share 1 and -1 CM.
Now, when F^* and G^* share 1 CM, by using (3.12), (3.13), (3.17) and since F^* and G^* are transcendental, we have

$$\overline{N}(r, \Gamma_j) + \overline{N}(r, 0; \Gamma_j) = S(r), (j = 1, 2), \tag{3.18}$$

as $r \in E$ and $r \to \infty$.
Now, we prove that $\overline{N_0}(r, 1; \Gamma_1, \Gamma_2) = S(r)$ is not possible.
Since F^* and G^* share -1 CM, from (3.12), (3.13), (3.16) and Nevanlinna's second fundamental theorem

$$\begin{aligned} T(r, F^*) &\leq \overline{N}(r, 0; F^*) + \overline{N}(r, -1; F^*) + \overline{N}(r, F^*) + S(r, F^*) \\ &\leq \overline{N}(r, -1; F^*) + S(r, f) + S(r, F^*) \\ &\leq \overline{N_0}(r, 1; \Gamma_1, \Gamma_2) + S(r, F^*), \end{aligned} \tag{3.19}$$

as $r \in E$ and $r \to \infty$.
If $\overline{N_0}(r, 1; \Gamma_1, \Gamma_2) = S(r)$, we get from (3.15) and (3.19) that $T(r, \Gamma_1) + T(r, \Gamma_2) \leq S(r)$, which is a contradiction. Therefore by Lemma 2.6, (3.12) and (3.18) it follows that there exist two relatively prime integers p and q such that $|p| + |q| > 0$ and $\Gamma_1^p \Gamma_2^q = 1$. Therefore from (3.12) we get that

$$\left(\frac{F^*}{G^*}\right)^p \left(\frac{F^* - 1}{G^* - 1}\right)^q = 1. \tag{3.20}$$

Now we discuss the following two subcases.

Subcase 1.1. Assume that $pq > 0$. From (3.20) we see that F^* and G^* share ∞ CM. Then noting that $F^*G^* = 1$ i.e., $\frac{(F_1)^{(k)}}{z} \frac{(G_1)^{(k)}}{z} = 1$, we obtain that ∞ is a Picard exceptional value of f and L. This together with the fact that 0 is another Picard exceptional value of f and L, and by (3.9), we can write as

$$L(z) = e^{c_1 z + c_2},$$

where $c_1 (\neq 0)$ and c_2 are constants.
Therefore by the result of Hayman [[4], p.7], we get that

$$T(r, L) = T(r, e^{c_1 z + c_2}) = \frac{|c_1| r}{\pi}(1 + o(1)),$$

a contradiction to (3.1).
Subcase 1.2. Assume that $pq < 0$. Without loss of generality let $p > 0$ and $q < 0$ and $q = -q^*$, for some positive integer q^*. Therefore (3.20) reduces to

$$\left(\frac{F^*}{G^*}\right)^p = \left(\frac{F^* - 1}{G^* - 1}\right)^{q^*}. \tag{3.21}$$

From $F^*G^* = 1$ it follows that if z_3 be a pole of F^* of some multiplicity $p_3 (\geq 1)$, then z_3 is also a zero of G^* of multiplicity p_3. Therefore from (3.20) we get $2p = q^* = -q$. This gives, $p = 1$ and $q = -q^* = -2$ as p and q are prime to each other. Hence, we get $F^*(G^* - 1)^2 = G^*(F^* - 1)^2$, which is $F^*G^* = 1$. Now we deduce a contradiction.

Since $z = 1$ is the only possible pole of L and of $(G_1)^{(k)}$, by using (3.17), we get

$$(G_1)^{(k)}(z) = \frac{zP(z)}{(z-1)^m} e^{c_3 z + c_4}, \tag{3.22}$$

where $P(z)$ is a nonzero polynomial, m is a nonnegative integer and $c_3(\neq 0)$, c_4 are constants.

Now using the result of Hayman [[4], p.7], Lemma 2.4 we get from (3.22) that there exists a subset $E \subset (0, +\infty)$ with logarithmic measure $logmeasE = \int_E \frac{dt}{t} = \infty$ such that for any given sufficiently large number $\Lambda > 1$, we have

$$\begin{aligned} T(r, L) &\leq 3e\Lambda T(r, (G_1)^{(k)}) \\ &= \frac{3e\Lambda|c_3|r}{z}(1 + o(1)) + S(r, f), \end{aligned}$$

as $r \in E$ and $r \to \infty$. This is a contradictiion to (3.1).

Case 2. Let $l = 1$. Then by using the deficiencies in (3.3), (3.4), (3.5), (3.6) and (ii) of Lemma 2.7, we get $s(n+m) > max\{6k+14, 7k+13\}$. Then either $(F_1)^{(k)}(G_1)^{(k)} = z^2$ or $F_1 = G_1$. Now proceeding in the same manner as in Case 1, we get the conclusion of the theorem.

Case 3. Let $l = 0$. Then by using the deficiencies in (3.3), (3.4), (3.5), (3.6) and (iii) of Lemma 2.7, we get $s(n+m) > max\{14k+20, 14k+22\}$. Then either $(F_1)^{(k)}(G_1)^{(k)} = z^2$ or $F_1 = G_1$. Now proceeding in the same manner as in Case 1, we get the conclusion of the theorem.

Hence the proof of Theorem 1.1.

4. Conclusion

In this work presented, we can conclude by saying that by introducing the concept of multiplicity to L-functions, our results reduce to previous results given in [3], [18] and also improves the results as the value of s increases irrespective of 'm' value, the value of n decreases.

References

[1] C.-Y. Fang and M.-L. Fang, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl., 44(2002), no. 5-6, 607–617.

[2] M.-L. Fang, Uniqueness and value-sharing of entire functions. Comput. Math. Appl., 44(2002), no. 5-6, 823–831.

[3] W.-J. Hao and J.-F. Chen, Uniqueness of L-functions concerning certain differential polynomials, Discrete Dyn. Nat. Soc., 2018, Art. ID 4673165, 12 pp.

[4] W. K. Hayman, Meromorphic functions, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964 xiv+191 pp.

[5] W. K. Hayman and J. Miles, On the growth of a meromorphic function and its derivatives, Complex Variables Theory Appl., 12(1989), no. 1-4, 245–260.

[6] P. Hu and P. Zhang, A characterization of L-functions in the extended Selberg class, Bull. Korean Math. Soc., 53(2016), no. 6, 1645–1650.

[7] I. Lahiri, Uniqueness of meromorphic functions as governed by their differential polynomials, Yokohama Math. J., 44(1997), no. 2, 147–156.

[8] I. Lahiri, Value distribution of certain differential polynomials, Int. J. Math. Math. Sci., 28(2001), no. 2, 83–91.

[9] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Variables Theory Appl., 46(2001), no. 3, 241–253.

[10] I. Laine, Nevanlinna theory and complex differential equations, De Gruyter Studies in Mathematics, 15, Walter de Gruyter & Co., Berlin, 1993. viii+341 pp. ISBN: 3-11-013422-5

[11] B. Q. Li, A result on value distribution of L-functions, Proc. Amer. Math. Soc., 138(2010), no. 6, 2071–2077.

[12] X.-M. Li and H.-X. Yi, Results on value distribution of L-functions. Math. Nachr., 286(2013), no. 13, 1326–1336.

[13] W.-C. Lin and H.-X. Yi, Uniqueness theorems for meromorphic function, Indian J. Pure Appl. Math., 35(2004), no. 2, 121–132.

[14] F. Liu, X.-M. Li and H.-X. Yi, Value distribution of L-functions concerning shared values and certain differential polynomials, Proc. Japan Acad. Ser. A Math. Sci., 93(2017), no. 5, 41–46

[15] F. Liu, X.-M. Li, and H.-X. Yi, Results on L-functions whose certain differential polynomials share one finite nonzero value, to appear.

[16] A. Z. Mohon ko, The Nevanlinna characteristics of certain meromorphic functions, (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. No. 14 (1971), 83–87.

[17] P. Sahoo and S. Seikh, Uniqueness of meromorphic functions sharing a nonzero polynomial with finite weight, Lobachevskii J. Math., 34(2013), no. 1, 106–115.

[18] P. Sahoo and S. Halder, Uniqueness results related to L-functions and certain differential polynomials, Tbilisi Math. J., 11(2018), no. 4, 67–78.

[19] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 367–385, Univ. Salerno, Salerno, 1992.

[20] H. M. Srivastava, H. Ozden, I. N. Cangul, and Y. Simsek, A unified presentation of certain meromorphic functions related to the families of the partial zeta type functions and the L-functions, Appl. Math. Comput., 219(2012), no. 8, 3903–3913.

[21] H. M. Srivastava, T. Kim, and Y. Simsek, q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series, Russ. J. Math. Phys., 12(2005), no. 2, 241–268.

[22] H. M. Srivastava and H. Tsumura, Certain classes of rapidly convergent series representations for $L(2n, \chi)$ and $L(2n+1, \chi)$, Acta Arith., 100(2001), no. 2, 195–201.

[23] J. Steuding, Value-distribution of L-functions, Lecture Notes in Mathematics, 1877, Springer, Berlin, 2007, xiv+317 pp. ISBN: 978-3-540-26526-9; 3-540-26526-0

[24] J. M. Whittaker, The Order of the Derivative of a Meromorphic Function, J. London Math. Soc., 11(1936), no. 2, 82–87.

[25] C.-C. Yang, On deficiencies of differential polynomials, Math. Z., 125(1972), 107-112.

[26] C.-C. Yang and X. Hua, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math., 22(1997), no. 2, 395–406.

[27] C.-C. Yang and H.-X. Yi, Uniqueness theory of meromorphic functions, Mathematics and its Applications, 557, Kluwer Academic Publishers Group, Dordrecht, 2003, viii+569 pp. ISBN: 1-4020-1448-1

[28] L. Yang, Normality for families of meromorphic functions, Sci. Sinica Ser. A, 29(1986), no. 12, 1263–1274.

[29] L. Yang, Value distribution theory, Translated and revised from the 1982 Chinese original, Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993, xii+269 pp. ISBN: 3-540-54379-1.

[30] Q. C. Zhang, Meromorphic functions sharing three values, Indian J. Pure Appl. Math., 30(1999), no. 7, 667–682.

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