

ON SOME SETS IN DIGITAL TOPOLOGY

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Abstract. The notions of strong and weak forms of open sets and closed sets in the digital line and digital plane have been used in digital image filtering techniques. The purpose of this paper is to characterize such sets in Digital Topology with special reference to b -closed sets and b -open sets.

1. Introduction and preliminaries

Digital topology deals with properties and features of two-dimensional (2D) or three-dimensional (3D) digital images that correspond to topological properties (e.g., connectedness) or topological features (e.g., boundaries) of objects. Concepts and results of digital topology are used to specify and justify important (low-level) image analysis algorithms, including algorithms for thinning, border or surface tracing, counting of components or tunnels, or region-filling. Digital topology was first studied in the late 1960s by the computer image analysis researcher Azriel Rosenfeld, whose publications on the subject played a major role in establishing and developing the field. The term "digital topology" was itself invented by Rosenfeld, who used it in a 1973 publication for the first time.

The information required for a digital picture can be stored by specifying the colour at each pixel. If a digital picture is formed by simple closed curve, one can specify the pixels on the simple closed curves and then specify uniformly the colours for the insides and the outside. This method results in the reduction of computer memory usage significantly. This method employs the celebrated Jordan curve theorem, which states that every simple closed curve in the plane separates the plane into two connected components.

Kong and Kopperman [13] gave a topological approach to digital topology. Kong et al. [14] studied the digital fundamental group and also established that on a strongly normal digital picture space, the discrete and continuous concepts are equivalent. Maki et al. [16] investigated the digital line and operations approaches of $T_{\frac{1}{2}}$ spaces. Devi et al. [5] studied the topological properties of wgp -closed sets in the digital plane. Saha et al. [20] investigated that the basic

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parts of digital geometry can be generalized into sets of convex voxels. Thangavelu [22] discussed the properties of non-empty digital intervals $[a, b] \cap \mathbb{Z}$ and cardinalities of subspace topology on digital intervals are characterized. The interested reader also can refer to the papers [6], [7], [8], [9] and [10] to see the recent research in this regard. In this paper, we characterize some sets in the Khalimsky topology.

Throughout this paper, (X, τ) , (Y, σ) and $(\mathbb{Z}, \kappa)$ (or simply, X , Y and $\mathbb{Z}$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X, τ) , $Cl(A)$, $Int(A)$ and $X \setminus A$ denote the closure of A , the interior of A and the complement of A in X , respectively.

Definition 1.1. A subset A of a space X is said to be

- (i) regular open [21] if $A = Int(Cl(A))$ and regular closed if $A = Cl(Int(A))$.
- (ii) α -open [18] if $A \subseteq Int(Cl(Int(A)))$ and α -closed if $Cl(Int(Cl(A))) \subseteq A$.
- (iii) semi-open [15] if $A \subseteq Cl(Int(A))$ and semi-closed if $Int(Cl(A)) \subseteq A$.
- (iv) preopen [17] if $A \subseteq Int(Cl(A))$ and preclosed if $Cl(Int(A)) \subseteq A$.
- (v) semi-preopen [2] or β -open [1] if $A \subseteq Cl(Int(Cl(A)))$ and semi-preclosed or β -closed if $Int(Cl(Int(A))) \subseteq A$.
- (vi) b -open [3] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ and b -closed if $Cl(Int(A)) \cap Int(Cl(A)) \subseteq A$.
- (vii) $*b$ -open [11] if $A \subseteq Cl(Int(A)) \cap Int(Cl(A))$ and $*b$ -closed if $Cl(Int(A)) \cup Int(Cl(A)) \subseteq A$.
- (viii) a p -set [23] if $Cl(Int(A)) \subseteq Int(Cl(A))$.
- (ix) a q -set [24] if $Int(Cl(A)) \subseteq Cl(Int(A))$.
- (x) $b^\#$ -open [19] if $A = Cl(Int(A)) \cup Int(Cl(A))$ and $b^\#$ -closed if $A = Cl(Int(A)) \cap Int(Cl(A))$.
- (xi) b^{**} -open [4] if $A \subseteq Int(Cl(Int(A))) \cup Cl(Int(Cl(A)))$ and b^{**} -closed if $Int(Cl(Int(A))) \cap Cl(Int(Cl(A))) \subseteq A$.
- (xii) a t -set [25] if $Int(A) = Int(Cl(A))$.
- (xiii) a t^* -set [11] if $Cl(A) = Cl(Int(A))$.

Lemma 1.2. [12] Let A be a subset of $\mathbb{Z}$. Then

- (i) A is open if and only if for every $x \in A$, the following hold:
(x is odd) or (x is even with $x - 1, x + 1 \in A$).
- (ii) A is closed if and only if for every $x \in A$, the following holds:
(x is even) or (x is odd with $x - 1, x + 1 \in A$).

Let $\mathfrak{N}(x)$ denote the smallest neighbourhood of x in $(\mathbb{Z}, \kappa)$. Then

$$\mathfrak{N}(x) = \begin{cases} \{x\} & \text{if } x \text{ is odd} \\ \{x-1, x, x+1\} & \text{if } x \text{ is even.} \end{cases}$$

Let $\mathfrak{C}(x)$ denote the smallest closed set containing x in $(\mathbb{Z}, \kappa)$. Then

$$\mathfrak{C}(x) = \begin{cases} \{x\} & \text{if } x \text{ is even} \\ \{x-1, x, x+1\} & \text{if } x \text{ is odd} \end{cases}$$

The topology satisfying the properties in Lemma 1.2 is exactly the Khalimsky line topology on $\mathbb{Z}$. Based on the concepts of "open" and "closed" in Lemma 1.2, we obtain the topology on $\mathbb{Z}$, which is referred to as the Khalimsky line topology on $\mathbb{Z}$.

2. Characterization of sets in Khalimsky topology

We begin with the following:

Proposition 2.1. If x is an odd integer, then the set $\{x\}$ is $b^\#$ -closed, b^{**} -closed, b -clopen, regular open, semi-open, semi-preopen, a q -set and a t^* -set in $(\mathbb{Z}, \kappa)$.

Proof. Suppose x is odd. Then $\text{Int}_k\{x\} = \{x\}$ and $\text{Cl}_k\text{Int}_k\{x\} = \{x-1, x, x+1\}$. Also $\text{Cl}_k\{x\} = \{x-1, x, x+1\}$ and $\text{Int}_k\text{Cl}_k\{x\} = \{x\}$. So $\text{Int}_k\text{Cl}_k\text{Int}_k\{x\} = \{x\}$ and $\text{Cl}_k\text{Int}_k\text{Cl}_k\{x\} = \{x-1, x, x+1\}$. Since $\text{Int}_k\text{Cl}_k\{x\} \cap \text{Cl}_k\text{Int}_k\{x\} = \{x\}$, by Definition 1(x), $\{x\}$ is $b^\#$ -closed and $\text{Int}_k\text{Cl}_k\{x\} \cup \text{Cl}_k\text{Int}_k\{x\} = \{x-1, x, x+1\} \supseteq \{x\}$, then by Definition 1.1(vi), $\{x\}$ is b -open. Since every $b^\#$ -closed set is b -closed, $\{x\}$ is b -closed and hence $\{x\}$ is b -clopen. Now $\text{Int}_k\text{Cl}_k\text{Int}_k\{x\} \cup \text{Cl}_k\text{Int}_k\text{Cl}_k\{x\} = \{x-1, x, x+1\} \supseteq \{x\}$, then by Definition 1 (xi), $\{x\}$ is b^{**} -open. Since $\{x\} = \text{Int}_k\text{Cl}_k\{x\}$ by Definition 1 (i), $\{x\}$ is regular open and $\{x\} \subseteq \text{Cl}_k\text{Int}_k\{x\} = \{x-1, x, x+1\}$ by Definition 1 (iii), $\{x\}$ is semi-open. Also $\{x\} \subseteq \text{Cl}_k\text{Int}_k\text{Cl}_k\{x\}$ by Definition 1 (v), $\{x\}$ is semi-preopen and $\text{Int}_k\text{Cl}_k\{x\} \subseteq \text{Cl}_k\text{Int}_k\{x\}$ by Definition 1 (ix), $\{x\}$ is a q -set. The relation $\text{Cl}_k\{x\} = \text{Cl}_k\text{Int}_k\{x\}$ implies that $\{x\}$ is a t^* -set. $\square$

Proposition 2.2. If x is an even integer, the set $\{x\}$ is b -closed, b^{**} -open, b^{**} -closed, $*b$ -closed, α -open, α -closed, preclosed, semi-closed, a t -set, a t^* -set in $(\mathbb{Z}, \kappa)$.

Proof. Let x be even. Observe that $\text{Int}_k\{x\} = \emptyset$ and $\text{Cl}_k\text{Int}_k\{x\} = \emptyset$. Then $\text{Cl}_k\{x\} = \emptyset$ and $\text{Int}_k\text{Cl}_k\{x\} = \emptyset$. So $\text{Int}_k\text{Cl}_k\text{Int}_k\{x\} = \{x-1, x, x+1\}$ and $\text{Cl}_k\text{Int}_k\text{Cl}_k\{x\} = \emptyset$. Since $\text{Int}_k\text{Cl}_k\{x\} \cap \text{Cl}_k\text{Int}_k\{x\} = \emptyset \subseteq \{x\}$, by Definition 1(vi), $\{x\}$ is b -closed and $\text{Int}_k\text{Cl}_k\{x\} \cup \text{Cl}_k\text{Int}_k\{x\} = \emptyset \subseteq \{x\}$ that implies $\{x\}$ is $*b$ -closed. Since $\text{Int}_k\text{Cl}_k\text{Int}_k\{x\} \cup \text{Cl}_k\text{Int}_k\text{Cl}_k\{x\} = \{x-1, x, x+1\} \supseteq \{x\}$ by Definition 1 (xi), $\{x\}$ is b^{**} -open and $\text{Int}_k\text{Cl}_k\text{Int}_k\{x\} \cap \text{Cl}_k\text{Int}_k\text{Cl}_k\{x\} = \emptyset \subseteq \{x\}$ by Definition 1 (xi), $\{x\}$ is b^{**} -closed. Also $\{x\} \subseteq \text{Int}_k\text{Cl}_k\text{Int}_k\{x\}$ by Definition 1 (ii), $\{x\}$ is α -open and $\text{Cl}_k\text{Int}_k\{x\} \subseteq \{x\}$ by Definition 1 (ii), $\{x\}$ is α -closed. Moreover, $\text{Cl}_k\text{Int}_k\{x\} \subseteq \{x\}$ by Definition 1 (iv), $\{x\}$ is preclosed and $\text{Int}_k\text{Cl}_k\{x\} \subseteq \{x\}$ by Definition 1(iii) implies $\{x\}$ is semi-closed. Also $\text{Int}_k\{x\} = \text{Int}_k\text{Cl}_k\{x\}$ implies $\{x\}$ is a t -set and $\text{Cl}_k\{x\} = \text{Cl}_k\text{Int}_k\{x\}$ implies $\{x\}$ is a t^* -set. $\square$

Corollary 2.3. In $(\mathbb{Z}, \kappa)$, the set $\{x\}$ is b -clopen, b^{**} -open and a t^* -set for every positive integer x .

Proof. By Proposition 2.1, $\{x\}$ is b -clopen, b^{**} -open and a t^* -set if x is odd and by Proposition 2.2, $\{x\}$ is b -clopen, b^{**} -open and a t^* -set if x is even. This implies $\{x\}$ is b -clopen, b^{**} -open and a t^* -set for every positive integer x . $\square$

Proposition 2.4. In $(\mathbb{Z}, \kappa)$, $\{x, x+1\}$ is b -clopen, b^{**} -open, b^{**} -closed, semi-closed, semi-open, semi-preopen, semi-preclosed, a q -set, a t -set, a t^* -set for every positive integer x .

Proof. Let $A = \{x, x+1\}$. Suppose x is odd. $\text{Int}_k(A) = \{x\}$ and $\text{Cl}_k\text{Int}_k(A) = \{x-1, x, x+1\}$. Observe that $\text{Cl}_k(A) = \{x-1, x, x+1\}$ and $\text{Int}_k\text{Cl}_k(A) = \{x\}$. Then $\text{Int}_k\text{Cl}_k\text{Int}_k(A) = \{x\}$ and $\text{Cl}_k\text{Int}_k\text{Cl}_k(A) = \{x-1, x, x+1\}$. Since $\text{Int}_k\text{Cl}_k(A) \cap \text{Cl}_k\text{Int}_k(A) = \{x\} \subseteq A$ by Definition 1 (vi), A is b -closed and $\text{Int}_k\text{Cl}_k(A) \cup \text{Cl}_k\text{Int}_k(A) = \{x-1, x, x+1\} \supseteq A$ implies A is b -open. Thus A is b -clopen. Now $\text{Int}_k\text{Cl}_k\text{Int}_k(A) \cup \text{Cl}_k\text{Int}_k\text{Cl}_k(A) = \{x-1, x, x+1\} \supseteq A$, by Definition 1 (xi), A is b^{**} -open and $\text{Int}_k\text{Cl}_k\text{Int}_k(A) \cap \text{Cl}_k\text{Int}_k\text{Cl}_k(A) = \{x\} \subseteq A$ implies A is b^{**} -closed. Since $\text{Int}_k\text{Cl}_k(A) = \{x\} \subseteq A$ implies by Definition 1 (iii), A is semi-closed and $A \subseteq \text{Cl}_k\text{Int}_k(A) = \{x-1, x, x+1\}$ implies A is semi-open. Now $A \subseteq \text{Cl}_k\text{Int}_k\text{Cl}_k(A) = \{x-1, x, x+1\}$ by Definition 1(v) implies A is semi-preopen and $\text{Int}_k\text{Cl}_k\text{Int}_k(A) = \{x\} \subseteq A$ implies A is semi-preclosed. Since $\text{Int}_k\text{Cl}_k(A) = \{x\} \subseteq \text{Cl}_k\text{Int}_k(A) = \{x-1, x, x+1\}$ by Definition 1(ix), A is q -set and $\text{Int}_k(A) = \text{Int}_k\text{Cl}_k(A)$ implies A is a t -set. Also $\text{Cl}_k(A) = \text{Cl}_k\text{Int}_k(A)$ implies that A is a t^* -set.

Let x be even. Observe that $\text{Int}_k(A) = \{x+1\}$ and $\text{Cl}_k\text{Int}_k(A) = \{x, x+1, x+2\}$. Also $\text{Cl}_k(A) = \{x, x+1, x+2\}$ and $\text{Int}_k\text{Cl}_k(A) = \{x+1\}$. Then $\text{Int}_k\text{Cl}_k\text{Int}_k(A) = \{x+1\}$ and $\text{Cl}_k\text{Int}_k\text{Cl}_k(A) = \{x, x+1, x+2\}$. Since $\text{Int}_k\text{Cl}_k(A) \cap \text{Cl}_k\text{Int}_k(A) = \{x+1\} \subseteq A$ by Definition 1(vi), A is b -closed and $\text{Int}_k\text{Cl}_k(A) \cup \text{Cl}_k\text{Int}_k(A) = \{x, x+1, x+2\} \supseteq A$ by Definition 1 (xi), implies that A is b^{**} -open and $\text{Int}_k\text{Cl}_k\text{Int}_k(A) \cap \text{Cl}_k\text{Int}_k\text{Cl}_k(A) = \{x+1\} \subseteq A$

that implies that A is b^* -closed. Since $A \subseteq Cl_k Int_k(A) = \{x, x+1, x+2\}$, by Definition 1(iii), A is semi-open and $Int_k Cl_k(A) = \{x+1\} \subseteq A$ implies A is semi-closed. Now by Definition 1(v), $A \subseteq Cl_k Int_k Cl_k(A) = \{x, x+1, x+2\}$ implies that A is semi-preclosed and $Int_k Cl_k Int_k(A) = \{x+1\} \subseteq A$ implies that A is semi-preclosed. Since $Int_k Cl_k(A) = \{x+1\} \subseteq Cl_k Int_k(A) = \{x, x+1, x+2\}$ by Definition 1(ix), A is q -set and $Int_k(A) = Int_k Cl_k(A)$ implies that A is a t -set. Also $Cl_k(A) = Cl_k Int_k(A)$ implies that A is a t^* -set. Therefore for every integer x , A is b -clopen, b^{**} -open, b^{**} -closed, semi-closed, semi-open, semi-preopen, semi-preclosed, a q -set, a t -set, a t^* -set. $\square$

Proposition 2.5. In $(\mathbb{Z}, \kappa)$, $\{x-1, x, x+1\}$ is $b^\#$ -open, b -clopen, b^{**} -closed, semi-closed, semi-preclosed, a q -set, a t -set, a t^* -set if x is an odd integer.

Proof. Let $A = \{x-1, x, x+1\}$ and x be odd. Observe that $Int_k(A) = \{x\}$ and $Cl_k Int_k(A) = \{x-1, x, x+1\}$. Also $Int_k Cl_k Int_k(A) = \{x\}$ and $Cl_k(A) = \{x-1, x, x+1\}$. We have $Int_k Cl_k(A) = \{x\}$ and $Cl_k Int_k Cl_k(A) = \{x-1, x, x+1\}$. Since $Int_k Cl_k(A) \cap Cl_k Int_k(A) = \{x\} \subseteq \{x-1, x, x+1\}$ by Definition 1(vi), A is b -closed. Now by Definition 1(x), $Int_k Cl_k(A) \cup Cl_k Int_k(A) = A$ implies that A is $b^\#$ -open and since every $b^\#$ -open set is b -open, A is b -open and therefore b -clopen. Since $Int_k Cl_k Int_k(A) \cap Cl_k Int_k Cl_k(A) \subseteq A$ by Definition 1(xi), A is b^{**} -closed and by Definition 1(iii), $Int_k Cl_k(A) = \{x\} \subseteq A$ implies that A is semi-closed. Now by Definition 1(v), $Int_k Cl_k Int_k(A) = \{x\} \subseteq A$ implies that A is semi-preclosed and by Definition 1(ix), $Int_k Cl_k(A) = \{x\} \subseteq Cl_k Int_k(A) = A$ implies that A is a q -set. Since $Int_k(A) = Int_k Cl_k(A)$ implies that A is a t -set and $Cl_k(A) = Cl_k Int_k(A)$ implies that A is a t^* -set. $\square$

Proposition 2.6. In $(\mathbb{Z}, \kappa)$, the set $\{x-1, x, x+1\}$ is regular open, $b^\#$ -closed, b -clopen, b^{**} -open, semi-open, semi-preopen, a q -set, a t -set, a t^* -set if x is an even integer.

Proof. Let $A = \{x-1, x, x+1\}$ and x be even. $Int_k(A) = A$ and $Cl_k Int_k(A) = \{x-2, x-1, x, x+1, x+2\}$. Observe that $Int_k Cl_k Int_k(A) = A$ and $Cl_k(A) = \{x-2, x-1, x, x+1, x+2\}$. Also $Int_k Cl_k(A) = A$. Now $Cl_k Int_k Cl_k(A) = \{x-2, x-1, x, x+1, x+2\}$. Since $A = Int_k Cl_k(A) = \{x-1, x, x+1\}$, by Definition 1(i), A is regular open. Also by Definition 1(xi), $Int_k Cl_k(A) \cap Cl_k Int_k(A) = A$ implies that A is $b^\#$ -closed and by Definition 1(vi), $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x-2, x-1, x, x+1, x+2\} \supseteq A$ that implies A is b -open. Since every $b^\#$ -closed is b -closed, A is b -closed and therefore b -clopen. Since $Int_k Cl_k Int_k(A) \cup Cl_k Int_k Cl_k(A) = \{x-2, x-1, x, x+1, x+2\} \supseteq A$ by Definition 1(xi), A is b^{**} -open and by Definition 1(iii), $A \subseteq Cl_k Int_k(A) = \{x-2, x-1, x, x+1, x+2\}$ implies that A is semi-open. Now by Definition 1(v), $A \subseteq Cl_k Int_k Cl_k(A) = \{x-2, x-1, x, x+1, x+2\}$ implies that A is semi-preopen and $Int_k Cl_k(A) = A \subseteq Cl_k Int_k(A) = \{x-2, x-1, x, x+1, x+2\}$ by Definition 1(ix) implies that A is a q -set. Also $Int_k(A) = Int_k Cl_k(A)$ implies that A is a t -set and $Cl_k(A) = Cl_k Int_k(A)$ implies that A is a t^* -set. $\square$

Corollary 2.7. In $(\mathbb{Z}, \kappa)$, the set $\{x-1, x, x+1\}$ is b -clopen, a q -set, a t -set, a t^* -set for every positive integer x .

Proof. Let $A = \{x-1, x, x+1\}$. Then by Proposition 2.6, A is b -clopen, q -set, t -set, t^* -set if x is odd and by Proposition 2.6, A is b -clopen, a q -set, a t -set, a t^* -set if x is even. This implies that A is b -clopen, a q -set, a t -set, a t^* -set for every positive integer x . $\square$

Corollary 2.8. In $(\mathbb{Z}, \kappa)$, the set $\{x, x+1, x+2\}$ is regular open, $b^\#$ -closed, b -clopen, b^{**} -open, semi-open, semi-preopen, a q -set, a t -set, a t^* -set if x is an odd integer.

Proof. If x is odd, then $\{x+1\}$ is even and by Proposition 2.6, $\{x, x+1, x+2\}$ is regular open, $b^\#$ -closed, b -clopen, b^{**} -open, semi-open, semi-preopen, a q -set, a t -set, a t^* -set. $\square$

Corollary 2.9. In $(\mathbb{Z}, \kappa)$, the set $\{x, x+1, x+2\}$ is $b^\#$ -open, b -clopen, b^{**} -closed, semi-closed, semi-preclosed, a q -set, a t -set, a t^* -set if x is an even integer.

Proof. If $\{x\}$ is even, then $\{x+1\}$ is odd and by Proposition 2.6, $\{x, x+1, x+2\}$ is $b^\#$ -open, b -clopen, b^{**} -closed, semi-closed, semi-preclosed, a q -set, a t -set, a t^* -set.

Theorem 2.10. In $(\mathbb{Z}, \kappa)$, any three consecutive integers form either a $b^\#$ -closed set or $b^\#$ -open set.

Proof. Let $A = \{x, x+1, x+2\}$. If x is odd, then from Corollary 2.8, A is $b^\#$ -closed and if x is even, by Corollary 2.9, A is $b^\#$ -open. This proves that any three consecutive integers form either a $b^\#$ -closed set or $b^\#$ -open set. $\square$

Proposition 2.11. In $(\mathbb{Z}, \kappa)$, $\{x, x+1, x+2, x+3\}$ is b -clopen, semi-closed, semi-open, semi-preopen, semi-preclosed, b^{**} -closed, b^{**} -open, a q -set, a t -set, a t^* -set for every positive integer x .

Proof. Let $A = \{x, x+1, x+2, x+3\}$ and x be odd. $Int_k(A) = \{x, x+1, x+2\}$ and $Cl_k Int_k(A) = \{x-1, x, x+1, x+2, x+3\}$. Observe $Int_k Cl_k Int_k(A) = \{x, x+1, x+2\}$ and $Cl_k(A) = \{x-1, x, x+1, x+2, x+3\}$. Also $Int_k Cl_k(A) = \{x, x+1, x+2\}$ and $Cl_k Int_k Cl_k(A) = \{x-1, x, x+1, x+2, x+3\}$. Since $Int_k Cl_k(A) \cap Cl_k Int_k(A) = \{x, x+1, x+2, x+3\} \subseteq A$. Then by Definition 1 (vi), A is b -closed. Moreover, $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x-1, x, x+1, x+2, x+3\} \supseteq A$ implies that A is b -open. Thus A is b -clopen. Also $A \subseteq Cl_k Int_k(A) = \{x-1, x, x+1, x+2, x+3\}$ then by Definition 1(iii), A is semi-open and $Int_k Cl_k(A) = \{x, x+1, x+2\} \subseteq A$ implies that A is semi-closed. Also by Definition 1(v), $A \subseteq Cl_k Int_k Cl_k(A) = \{x-1, x, x+1, x+2, x+3\}$ implies that A is semi-preopen and $Int_k Cl_k Int_k(A) = \{x, x+1, x+2\} \subseteq A$ implies that A is semi-preclosed. Since $Int_k Cl_k Int_k(A) \cup Cl_k Int_k Cl_k(A) = \{x-1, x, x+1, x+2, x+3\} \supseteq A$. Then by Definition 1(xi), A is b^{**} -open and $Int_k Cl_k Int_k(A) \cap Cl_k Int_k Cl_k(A) \subseteq A$ implies that A is b^{**} -closed. Also by Definition 1(ix), $Int_k Cl_k(A) = \{x, x+1, x+2\} \subseteq Cl_k Int_k(A) = \{x-1, x, x+1, x+2, x+3\}$ implies that A is a q -set. Since $Int_k(A) = Int_k Cl_k(A)$ implies that A is a t -set and $Cl_k(A) = Cl_k Int_k(A)$ implies that A is a t^* -set.

Let x be even. Observe that $Int_k(A) = \{x+1, x+2, x+3\}$ and $Cl_k Int_k(A) = \{x, x+1, x+2, x+3, x+4\}$. Also $Int_k Cl_k Int_k(A) = \{x+1, x+2, x+3\}$ and $Cl_k(A) = \{x, x+1, x+2, x+3, x+4\}$. We have $Int_k Cl_k(A) = \{x+1, x+2, x+3\}$ and $Cl_k Int_k Cl_k(A) = \{x, x+1, x+2, x+3, x+4\}$. Since $Int_k Cl_k(A) \cap Cl_k Int_k(A) = \{x+1, x+2, x+3\} \subseteq A$, by Definition 1(vi), A is b -closed and $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x, x+1, x+2, x+3, x+4\} \supseteq A$ implies that A is b -open. Thus A is b -clopen. Now by Definition 1(iii), $A \subseteq Cl_k Int_k(A) = \{x, x+1, x+2, x+3, x+4\}$ implies that A is semi-open and $Int_k Cl_k(A) = \{x+1, x+2, x+3\} \subseteq A$ implies that A is semi-closed. Since $A \subseteq Cl_k Int_k Cl_k(A) = \{x, x+1, x+2, x+3, x+4\}$ by Definition 1 (v), A is semi-preopen and $Int_k Cl_k Int_k(A) = \{x+1, x+2, x+3\} \subseteq A$ implies that A is semi-preclosed. Also by Definition 1(xi), $Int_k Cl_k Int_k(A) \cup Cl_k Int_k Cl_k(A) = \{x, x+1, x+2, x+3, x+4\} \supseteq A$ implies that A is b^{**} -open and $Int_k Cl_k Int_k(A) \cap Cl_k Int_k Cl_k(A) \subseteq A$ implies that A is b^{**} -closed. Since $Int_k Cl_k(A) = \{x+1, x+2, x+3\} \subseteq Cl_k Int_k(A) = \{x, x+1, x+2, x+3, x+4\}$ by Definition 1(ix), A is a q -set. Now $Int_k(A) = Int_k Cl_k(A)$ implies that A is a t -set and $Cl_k(A) = Cl_k Int_k(A)$ implies that A is a t^* -set. Therefore, for every integer x , A is b -clopen, semi-closed, semi-open, semi-preopen, semi-preclosed, b^{**} -open, b^{**} -closed, a q -set, a t -set, a t^* -set. $\square$

Proposition 2.12. In $(\mathbb{Z}, \kappa)$, $\{x, x+1, x+2, x+3\}$ is neither open nor closed.

Proof. Let $A = \{x, x+1, x+2, x+3\}$. Suppose x is odd. Then $\{x+3\}$ is even. Since $\{x+4\} \notin A$, by Lemma 1.2(i), A is not open. Also since $\{x-1\} \notin A$, by Lemma 1.2(ii), A is

not closed. The proof of the case where x is even is analogous. Thus $\{x, x+1, x+2, x+3\}$ is neither open nor closed. $\square$

Proposition 2.13. Let x be an odd integer. Then for every positive integer $n \geq 2$, the set $A = \{x, x+1, x+2, \dots, x+n-1, x+n\}$ is both $b^\#$ -closed and b -clopen in $(\mathbb{Z}, \kappa)$.

Proof. Let x be even. Since x is odd, $x+n$ is odd. Then $Int_k(A) = A = \{x, x+1, x+2, \dots, x+n-1, x+n\}$ and $Cl_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$. Then $Int_k Cl_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n\}$ and $Cl_k Int_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$. Therefore, $Int_k Cl_k(A) \cap Cl_k Int_k(A) = A$. Then by Definition 1(x), A is $b^\#$ -closed. Similarly $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\} \supseteq A$, then by Definition 1(vi), A is b -open. Since every $b^\#$ -closed is b -closed, A is b -closed. Suppose n is odd. Since x is odd, $x+n$ is even. Then $Int_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$ and $Cl_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n\}$. Then $Int_k Cl_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$ and $Cl_k Int_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n\}$. Then $Int_k Cl_k(A) \cap Cl_k Int_k(A) = A$. Therefore by Definition 1(x), A is $b^\#$ -closed. Similarly $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\} \supseteq A$, then by Definition 1(vi), A is b -open. Since every $b^\#$ -closed set is b -closed, A is b -closed and b -clopen. Thus, A is both $b^\#$ -closed and b -clopen for every positive integer $n \geq 2$. $\square$

Proposition 2.14. Let x be even. For every positive integer n , the set $A = \{x, x+1, x+2, \dots, x+n-1, x+n\}$ is b -clopen in $(\mathbb{Z}, \kappa)$.

Proof. Suppose n is even. Since x is even, $x+n$ is odd. Then $Int_k(A) = A = \{x+1, x+2, \dots, x+n-1, x+n\}$ and $Cl_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$. Then $Int_k Cl_k(A) = \{x+1, x+2, \dots, x+n-1, x+n\}$ and $Cl_k Int_k(A) = \{x-1, x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$. Also $Int_k Cl_k(A) \cap Cl_k Int_k(A) = \{x+1, x+2, \dots, x+n-1, x+n\} \subseteq A$. Then by Definition 1(vi), A is b -closed. Similarly, $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\} \supseteq A$, then by Definition 1(vi), A is b -open. Thus A is b -clopen. Suppose n is odd. Since x is even, $x+n$ is even. Then $Int_k(A) = \{x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$ and $Cl_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n\}$. Also $Int_k Cl_k(A) = \{x+1, x+2, \dots, x+n-1, x+n, x+n+1\}$ and $Cl_k Int_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n\}$. Therefore, $Int_k Cl_k(A) \cap Cl_k Int_k(A) = \{x+1, x+2, \dots, x+n-1, x+n\} \subseteq A$. Then by Definition 1(vi), A is b -closed. Similarly $Int_k Cl_k(A) \cup Cl_k Int_k(A) = \{x, x+1, x+2, \dots, x+n-1, x+n, x+n+1\} \supseteq A$. Then by Definition 1(vi), A is b -open. Thus A is b -clopen. $\square$

Proposition 2.15. In $(\mathbb{Z}^2, \kappa^2)$, the set $\{(x, y)\}$ is regular open, preopen, α -open, semi-open, β -open, b -clopen, $b^\#$ -closed, semi-closed, β -closed, q -set, t -set, t^* -set if x and y are odd integers.

Proof. Let $A = \{(x, y)\} = \{x\} \times \{y\}$. Suppose x and y are odd. Observe that $Int_k(A) = Int_k\{x\} \times Int_k\{y\} = \{x\} \times \{y\} = (x, y)$. Also $Cl_k Int_k(A) = Cl_k Int_k\{x\} \times Cl_k Int_k\{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\} \supseteq (x, y)$. Now $A \subseteq Cl_k Int_k(A)$ implies by Definition 1(iii), A is semi-open. We have $Cl_k(A) = Cl_k\{x\} \times Cl_k\{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\}$. So $Int_k Cl_k(A) = Int_k Cl_k\{x\} \times Int_k Cl_k\{y\} = \{x\} \times \{y\} = (x, y)$. $A = Int_k Cl_k(A)$ implies that A is regular open by Definition 1(i). Also $Int_k Cl_k(A) \subseteq A$ implies that A is semi-closed and $A \subseteq Int_k Cl_k(A)$ implies that A is preopen. Observe that $Int_k Cl_k Int_k(A) = Int_k Cl_k Int_k\{x\} \times Int_k Cl_k Int_k\{y\} = \{x\} \times \{y\} = (x, y)$. Also $A = Int_k Cl_k Int_k(A)$ implies that A is α -open by Definition 1(ii). On the other hand, $Int_k Cl_k Int_k(A) \subseteq A$ implies that A is β -closed by Definition 1(v). Moreover, $Cl_k Int_k Cl_k(A) = Cl_k Int_k Cl_k\{x\} \times Cl_k Int_k Cl_k\{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1)\}$.

1), $(x+1, y-1)$, $(x+1, y)$, $(x+1, y+1)$ $\supseteq (x, y)$. Also $A \subseteq Cl_k Int_k Cl_k(A)$ implies that A is semi-preopen or β -open. Also observe that $Int_k Cl_k(A) \cap Cl_k Int_k(A) = Int_k Cl_k\{x\} \cap Cl_k Int_k\{x\} \times Int_k Cl_k\{y\} \cap Cl_k Int_k\{y\} = \{x\} \times \{y\} = (x, y)$. Thus $A = Int_k Cl_k(A) \cap Cl_k Int_k(A)$ by Definition 1(x), A is $b^\#$ -closed. Also $Int_k Cl_k(A) \cup Cl_k Int_k(A) = Int_k Cl_k\{x\} \cup Cl_k Int_k\{x\} \times Int_k Cl_k\{y\} \cup Cl_k Int_k\{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\} \supseteq (x, y)$. $A \subseteq Int_k Cl_k(A) \cup Cl_k Int_k(A)$ implies that A is b -open by Definition 1(vi). Since every $b^\#$ -closed set is b -closed, A is b -clopen. Since $Int_k Cl_k(A) \subseteq Cl_k Int_k(A)$ by Definition 1(ix), A is a q -set. Also $Cl_k(A) = Cl_k Int_k(A)$ implies that A is a t^* -set and $Int_k(A) = Int_k Cl_k(A)$ implies that A is a t -set. Therefore, $A = (x, y)$ is regular open, preopen, α -open, semi-open, β -open, b -clopen, $b^\#$ -closed, semi-closed, β -closed, q -set, t -set, t^* -set if x and y are odd. $\square$

Proposition 2.16. In $(\mathbb{Z}^2, \kappa^2)$, the set $\{(x, y)\}$ is semi-closed, preclosed, α -open, α -closed, *b -closed, b -closed, a p -set, a q -set, a t -set and a t^* -set if one of the following holds:

- (i) x and y are even.
- (ii) x is even and y is odd.
- (iii) x is odd and y is even.

Proof. Suppose that x and y are even. Let $A = \{(x, y)\} = \{x\} \times \{y\}$. Since $Int_k(A) = Int_k\{x\} \times Int_k\{y\} = \emptyset \times \emptyset = \emptyset$ and also $Cl_k Int_k(A) = Cl_k Int_k\{x\} \times Cl_k Int_k\{y\} = \emptyset \times \emptyset = \emptyset \subseteq A$ implies that A is preclosed. We have $Cl_k(A) = Cl_k\{x\} \times Cl_k\{y\} = \emptyset \times \emptyset = \emptyset$ and $Int_k Cl_k(A) = Int_k Cl_k\{x\} \times Int_k Cl_k\{y\} = \emptyset \times \emptyset = \emptyset \subseteq A$ which implies that A is semi-closed. Also $Int_k Cl_k Int_k(A) = Int_k Cl_k Int_k\{x\} \times Int_k Cl_k Int_k\{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\} \supseteq (x, y)$. Since $A \subseteq Int_k Cl_k Int_k(A)$, so A is α -open by Definition 1(ii). Observe that $Cl_k Int_k Cl_k(A) = Cl_k Int_k Cl_k\{x\} \times Cl_k Int_k Cl_k\{y\} = \emptyset \times \emptyset = \emptyset \subseteq A$ which implies that A is α -closed by Definition 1(ii). To show that A is b -closed, we observe that $Int_k Cl_k(A) \cap Cl_k Int_k(A) = Int_k Cl_k\{x\} \cap Cl_k Int_k\{x\} \times Int_k Cl_k\{y\} \cap Cl_k Int_k\{y\} = \emptyset \times \emptyset = \emptyset \subseteq A$. So A is b -closed by Definition 1(vi). Now we will show that A is *b -closed. We have $Int_k Cl_k(A) \cup Cl_k Int_k(A) = Int_k Cl_k\{x\} \cup Cl_k Int_k\{x\} \times Int_k Cl_k\{y\} \cup Cl_k Int_k\{y\} = \emptyset \times \emptyset = \emptyset \subseteq A$. A is also a q -set since $Int_k Cl_k(A) \subseteq Cl_k Int_k(A)$ by utilizing Definition 1(ix). Moreover, $Cl_k Int_k(A) \subseteq Int_k Cl_k(A)$ implies that A is a p -set by Definition 1(viii). By $Cl_k(A) = Cl_k Int_k(A)$, A is a t^* -set and by $Int_k(A) = Int_k Cl_k(A)$, A is a t -set. Therefore, $A = (x, y)$ is semi-closed, preclosed, α -open, α -closed, *b -closed, b -closed, a p -set, a q -set, a t -set and a t^* -set if x and y are even integers. The proof of (ii) and (iii) can be done similarly. $\square$

3. Conclusion

In this paper, some nearly open and closed sets including $b^\#$ -closed sets are characterized in digital topology. In future, we study the properties of continuous functions of the same set.

References

- [1] M. E. Abd. El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β -open sets and β -continuous mappings, *Bull. Fac. Sci. Assiut Univ.* 12, 1983, pp. 77-90.
- [2] D. Anrijivic, Semi-preopen sets, *Mat. Vesnik*, 38, 1986, pp. 24-32.
- [3] D. Anrijivic, On b -open sets, *Mat. Vesnik*, 48, 1996, pp. 59-64.
- [4] S. Bharathi, K. Bhuvaneshwari, N. Chandramathi, On locally b^{**} -closed sets, *International Journal of Mathematical Sciences and Applications*, 1, 2011, pp. 636-641.
- [5] R. Devi, K. Bhuvaneshwari and H. Maki, Weak form of gp -closed sets, where $p \in \{\alpha, \alpha^*, \alpha^{**}\}$ and digital planes, *Kochi. Univ.* 25, 2004, pp. 37-54.

- [6] V.A. Chatyrko, S.-E. Han, Y. Hattori, Some remarks concerning semi- $T_{\frac{1}{2}}$ spaces, *Filomat*, 28(1), 2014, pp. 21-25.
- [7] S.-E. Han, Hereditary properties of semi-separation axioms and their applications, *Filomat*, 32(13), 2018, pp. 4689-4700.
- [8] S.-E. Han, Study on topological spaces with the semi- $T_{\frac{1}{2}}$, *Honam Mathematical Jour.*, 35(4), 2013, pp. 707-716.
- [9] S.-E. Han, Semi-separation axioms of the infinite Khalimsky topological sphere, *Topology Appl.*, 275, 2020, 107006, 15 pp.
- [10] S.-E. Han, S. Jafari, J. M. Kang, Topologies on $\mathbb{Z}^n$ that Are Not Homeomorphic to the n -Dimensional Khalimsky Topological Space, *Mathematics*, 7, 2019, 1072, 12 pp.
- [11] T. Indira and K. Rekha, On locally $**b$ -closed sets, *Proceedings of the Heber International Conference on Applications of Mathematics and Statistics (HICAMS)*, 2012.
- [12] E. D. Khalimsky, R. Kopperman and P. R. Meyer, Computer graphics and connected topologies in finite ordered sets, *Topology Appl.* 36, 1990, pp. 1-17.
- [13] T. Y. Kong and R. Kopperman, A topological approach to digital topology, *Amer. Math. Monthly*, 98, 1991, pp. 901-917.
- [14] T. Y. Kong, A. W. Roscoe and A. Rosenfeld, Concepts of digital topology, *Topology Appl.*, 46, 1992, pp. 219-262.
- [15] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70, 1963, pp. 36-41.
- [16] H. Maki, H. Ogata, K. Balachandran, P. Sundaram and R. Devi, The digital lines and operation approaches of $T_{\frac{1}{2}}$ spaces, *Scientiae Math.*, 3, 2000, pp. 345-352.
- [17] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt.*, 53, 1994, pp. 51-63.
- [18] O. Njstad, On some classes of nearly open sets, *Pcif. J. Math.*, 15, 1965, pp. 961-970.
- [19] U. R. Parameswari and P. Thangavelu, On $b^{\#}$ -open sets, *International Journal of Mathematics Trends and Technology*, 3, 2014, pp. 202-218.
- [20] P. K. Saha and A. E. Rosenfeld, The digital topology of sets of convex voxels, *Graphical Models*, 62, 2000, pp. 343-352.
- [21] M. Stone, Applications of the Theory of Boolean Rings to General Topology, *Trans. Amer. Math. Soc.*, 41, 1937, pp. 374-481.
- [22] P. Thangavelu, On the subspace topologies of the Khalimsky topology, *Proceedings of the Second International Conference on Mathematics: trends and developments, The Egyptian Math. Soc.* 3, 2001, pp. 157-168.
- [23] P. Thangavelu and K. C. Rao, p -sets in topological spaces, *Bull. Pure and Appl. Sci.*, 21, 2002, pp. 341-345.
- [24] P. Thangavelu and K. C. Rao, q -sets in topological spaces, *Prog. Math.*, 36, 2002, pp. 159-165.
- [25] J. Tong, On decomposition of continuity in Topological spaces, *Acta Math. Hungar.*, 54, 1989, pp. 51-55.

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