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SOME RESULTS ON μ-NEARLY COMPACTNESS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract. The purpose of this paper is to study the μ-nearly compactness in generalized topological spaces. Then, we also study the fundamental properties of μ-nearly compact space and obtain some results. In addition, we also investigate the characterizations of μ-nearly compact subspaces and subsets of generalized topological spaces. Some examples will be considered in order to establish some of their relationships.

1. Introduction

In 2002, Császár [3] was the first mathematician who introduced the idea of Generalized Topological Spaces (briefly GTS). Since then, there were many researchers that published their papers related to this field. For example, in 2012, Thomas and John [13] came up with the concept of μ-compactness on generalized topological space (GTS). A generalized topological space (X, μ) is called μ-compact if every μ-open cover of X contains a finite subcover. The fundamental properties of μ-compact space and its characterizations have been introduced and studied in generalized topological space. In addition, Thomas and John [13] also deal with the μ-compactness in subspaces of a generalized topological space.

In literature, some generalizations of the concept of compact sets have been studied by topologist for different reasons and purposes. Then, in 2012, Sarsak [10] introduced and studied the concept of weakly μ-compact spaces in μ-structure. A μ-structure (X, μ) is said to be weakly μ-compact (briefly $\omega\mu$-compact) if every μ-open cover of X admits of a finite subfamily, the union of μ-closures of whose elements cover the space. Sarsak also defined μ-regular open and μ-regular closed sets. Later, in 2013, Sarsak [11] aims

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to introduce μ-compact sets in μ-spaces. Here, Sarsak assumed that a GTS (X, μ) is a μ-space. Then X is said to be μ-compact if every μ-open cover of X contains a finite subcover. In addition, some properties and mapping properties of μ-compact sets and μ-Lindelöf properties in μ-spaces are studied.

The notion of nearly compact was discovered by Singal and Mathur in 1969 [12]. The concepts of nearly compact space are characterized by using regular open as a cover. In 2013, Salleh and Kılıçman [9] defined a new notion of nearly Lindelöf spaces in bitopological setting which is called pairwise nearly Lindelöf spaces and several of their characterizations are also investigated. Moreover, a new bitopology generated by (i, j)-regular open subsets which is called pairwise semiregularization is introduced in bitopological setting. Several properties are presented in [9]. In addition, they studied and investigated the pairwise nearly Lindelöf subspaces and subsets and their characterizations. Therefore, the concept of pairwise nearly Lindelöf relative to a bitopological space are investigated and several results are obtained.

The main focus of this paper is to define μ-nearly compact spaces on GTS's and study their several fundamental properties. In the next section, we collect all preliminaries and fundamental definitions useful for this paper. According to Kılıçman and Abuage [6] in 2015, they had studied and defined μ-semiregular spaces and its μ-semiregularization in generalized topological spaces (GTS). Furthermore, we coninue study the μ-nearly compact spaces in GTS and the conditions for characterized the μ-nearly compact spaces. Lastly, we investigate the properties of μ-nearly compact subspaces and subsets of a GTS. At the end of this paper, some counterexamples are demonstrate to establish that μ-nearly compact generalized topological space is not satisfying the hereditary property.

2. Preliminaries

Throughout this paper, all spaces X (or (X, μ)) are always mean generalized topological spaces, unless explicitly stated. There are several fundamental definitions and notations of the essential concepts needed throughout this paper. For a nonempty set X, let $\exp(X)$ denote the power set of X. A collection μ of subsets of $\exp(X)$ is called a generalized topology (briefly GT) and (X, μ) is called a generalized topological space (briefly GTS) [3] if $\emptyset \in \mu$ and the union of elements of μ belongs to μ. If a set $X \in \mu$, then a GTS (X, μ) is called μ-space [8].

For a GTS (X, μ), the elements of μ are called μ-open sets and a subset A of (X, μ) is called μ-closed if $X \setminus A$ is μ-open. If A is a subset of a space (X, μ), then the μ-closure of A, denoted by $c_\mu(A)$ is the intersection of all μ-closed sets containing A, that is, the smallest μ-closed set containing A; and the μ-interior of A, $i_\mu(A)$ is the union of all μ-open sets contained in A, that is, the largest μ-open set contained in A [3, 5].

Let (X, μ) be a generalized topological space (GTS). A subset A of space (X, μ) is called μ-regular open if and only if $A = i_\mu(c_\mu(A))$ and the complements of μ-regular open set is called μ-regular closed [11]. Besides that, a subset J of a GTS (X, μ) is said to be μ-clopen if J is both μ-closed and μ-open. If A is a subset of X, then the collection $\mu\mid_A = \{A \cap U : U \in \mu\}$ is a GT on A, called the subspace generalized topology, and $(A, \mu\mid_A)$ is called a subspace of X [7]. A point $x \in X$ is called μ-accumulation point of a subset D of X if $(U \setminus \{x\}) \cap D \neq \emptyset$ for every μ-open set U containing x.

3. μ-Semiregular Spaces

For this section, there are few definitions and theorems regarding μ-semiregularity in generalized topological spaces are discussed.

Definition 3.1. A generalized topological space (X, μ) is said to be μ-regular if for every $x \in X$ and every μ-open set V of X containing x, there exists a μ-open set U such that $x \in U \subseteq c_\mu(U) \subseteq V$.

Definition 3.2. Let (X, μ) be a GTS. Then, X is said to be μ-almost regular if for every $x \in X$ and every μ-regular open set V of X containing x, there exists a μ-regular open set U such that $x \in U \subseteq c_\mu(U) \subseteq V$.

If $\beta \subseteq \exp(X)$ and $\emptyset \in \beta$, then β is called a μ-base for μ if $\left\{\bigcup \beta' : \beta' \subseteq \beta\right\} = \mu$, hence we can say that μ is generated by β [4]. Thus, in a generalized topological space (X, μ), the collection of μ-regular open sets forms a μ-base which is called μ-semiregularization of X, as in the following definition.

Definition 3.3. Let (X, μ) be a GTS. A generalized topology generated by μ-regular open sets of X is denoted by μ_δ and it is called μ-semiregularization of X. In addition, X is said to be μ-semiregular if the collection of μ-regular open sets forms a μ-base for the generalized topology μ or equivalently, $\mu = \mu_\delta$.

Theorem 3.4. *A generalized topological space (X, μ) is μ-semiregular if and only if for every $x \in X$ and every μ-open subset V of X containing x, there exist a μ-open set U such that $x \in U \subseteq i_\mu(c_\mu(U)) \subseteq V$.*

Proof. Let (X, μ) be a μ-semiregular space. Then $\mu = \mu_\delta$, i.e., μ is generated by μ-regular open sets in (X, μ). Assume that $x \in X$, and let V be a μ-open set in (X, μ) containing x. Since the collection of μ-regular open sets in (X, μ) forms a μ-base for μ, there exists a μ-open set U in (X, μ) such that $x \in U \subseteq i_\mu(c_\mu(U)) \subseteq V$. Conversely, assume the condition holds. Generally, we have $\mu_\delta \subseteq \mu$. Assume that $t \in X$ and $F_t \in \mu$ with $t \in F_t$. By hypothesis, there exists a μ-open set U_t in (X, μ) such that $t \in U_t \subseteq i_\mu(c_\mu(U_t)) \subseteq F_t$. Thus, the collection $\{i_\mu(c_\mu(U_t)) : t \in X\}$ forms a μ-base for μ_δ which implies $F_t \in \mu_\delta$. Therefore, $\mu \subseteq \mu_\delta$ and hence (X, μ) is a μ-semiregular. ☐

4. μ-Nearly Compact Spaces

In this section, we study and introduce some basic definitions and properties of μ-nearly compact spaces. So base on these definitions, we give several properties of μ-nearly compact spaces and develop this concept as well as how they relate to the concepts of μ-open and μ-closed sets and its combination.

Definition 4.1. Let a GTS (X, μ) be a μ-space. A collection M of subsets of X is called μ-regular open cover of X if every element of M is μ-regular open set of X and the union of the elements of M is equal to X.

Definition 4.2. [1] A μ-space (X, μ) is said to be μ-nearly compact if every μ-open cover $M = \{U_\alpha : \alpha \in I\}$ of X has a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $X = \bigcup_{i=1}^{n} i_\mu(c_\mu(U_{\alpha_i}))$.

Theorem 4.3. *Let a GTS (X,μ) be a μ-space, then X is μ-nearly compact if and only if every μ-regular open cover of X has a finite subcover.*

Proof. Let X be a μ-nearly compact space and let $M=\{U_\alpha:\alpha\in I\}$ be a μ-regular open cover of X. Then $X=\bigcup_{\alpha\in I}U_\alpha=\bigcup_{\alpha\in I}i_\mu\left(c_\mu\left(U_\alpha\right)\right)$. Hence M is also a μ-open cover of X since $i_\mu\left(c_\mu\left(U_\alpha\right)\right)$ is also μ-open set. So there exists a finite subset $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of I such that $X=\bigcup_{i=1}^{n}i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)=\bigcup_{i=1}^{n}U_{\alpha_i}$. Therefore, M has a finite subcollection $M'=\{U_{\alpha_1},U_{\alpha_2},\ldots,U_{\alpha_n}\}$ which covers X. Conversely, let $M=\{U_\alpha:\alpha\in I\}$ be a μ-open cover of X. Since $U_\alpha\subseteq i_\mu\left(c_\mu\left(U_\alpha\right)\right)$, then $X=\bigcup_{\alpha\in I}U_\alpha\subseteq\bigcup_{\alpha\in I}i_\mu\left(c_\mu\left(U_\alpha\right)\right)$. Hence $\{i_\mu\left(c_\mu\left(U_\alpha\right)\right):\alpha\in I\}$ is a μ-regular open cover of X. So there is a finite subset $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of I such that that $X=\bigcup_{i=1}^{n}i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. Therefore, X is μ-nearly compact. □

Observe that every μ-compact space is μ-nearly compact but the converse is not true in general by the following counterexample and also by Examples 5.10 and 5.11.

Example 4.4. Let $X=\{1,2,3,\ldots\}$ be an infinite set and $\beta=\{\emptyset\}\cup\{\{1,b\};b\in X,b\neq 1\}$. Assume that the generalized topology $\mu\left(\beta\right)$ is generated on X by the μ-base β. Hence $(X,\mu\left(\beta\right))$ is a $\mu\left(\beta\right)$-space. Therefore, $\emptyset$ and X are the only μ-regular open sets of X since the μ-closure of any subset of X containing 1 is X. So, a generalized topological space $(X,\mu\left(\beta\right))$ is μ-nearly compact but it is not μ-compact since $\{\{1,b\};b\in X,b\neq 1\}$ is a μ-open cover of X with no finite subcollection covers X.

Theorem 4.5. *If a GTS (X,μ) is a μ-space then the following statements are all equivalents:*

(i) *(X,μ) is a μ-nearly compact.*

(ii) *For every collection $\{S_\alpha:\alpha\in I\}$ of μ-regular closed subsets of X such that $\bigcap_{\alpha\in I}S_\alpha=\emptyset$, there is a finite subcollection $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of I such that $\bigcap_{i=1}^{n}S_{\alpha_i}=\emptyset$.*

(iii) *For every collection $\{S_\alpha:\alpha\in I\}$ of μ-regular closed subsets of X with finite intersection property, the intersection $\bigcap_{\alpha\in I}S_\alpha\neq\emptyset$.*

(iv) *For every collection of μ-closed subsets $\{S_\alpha:\alpha\in I\}$ of X such that $\bigcap_{\alpha\in I}S_\alpha=\emptyset$, there is a finite subset $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of I such that $\bigcap_{i=1}^{n}c_\mu\left(i_\mu\left(S_{\alpha_i}\right)\right)=\emptyset$.*

(v) *For every collection $\{S_\alpha:\alpha\in I\}$ of μ-closed subsets of X with finite intersection property, the intersection $\bigcap_{i=1}^{n}c_\mu\left(i_\mu\left(S_{\alpha_i}\right)\right)\neq\emptyset$.*

Proof. $(i)\iff(ii)$: Let $\{S_\alpha:\alpha\in I\}$ be a collection of μ-regular closed subsets of X such that $\bigcap_{\alpha\in I}S_\alpha=\emptyset$. Then $X=X\setminus\bigcap_{\alpha\in I}S_\alpha=\bigcup_{\alpha\in I}\left(X\setminus S_\alpha\right)$, so $\{X\setminus S_\alpha:\alpha\in I\}$ forms a μ-regular open cover of X. By (i), there is a finite subset $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of I such that $X=\bigcup_{i=1}^{n}\left(X\setminus S_{\alpha_i}\right)$. Thus $X\setminus\bigcup_{i=1}^{n}\left(X\setminus S_{\alpha_i}\right)=\emptyset$, i.e., $\bigcap_{i=1}^{n}S_{\alpha_i}=\emptyset$.

Conversely, let $M=\{U_\alpha:\alpha\in I\}$ be a μ-regular open cover of X. Then $X=\bigcup_{\alpha\in I}U_\alpha$ and hence $X\setminus\bigcup_{\alpha\in I}U_\alpha=\emptyset$, i.e., $\bigcap_{\alpha\in I}X\setminus U_\alpha=\emptyset$. By (ii), there is a finite subset $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of I such that $\bigcap_{i=1}^{n}X\setminus U_{\alpha_i}=\emptyset$. So, $X=X\setminus\bigcap_{i=1}^{n}X\setminus U_{\alpha_i}=\bigcup_{i=1}^{n}U_{\alpha_i}$. As a result, X is μ-nearly compact.

$(i) \iff (iv)$: Let $\{S_\alpha : \alpha \in I\}$ be a collection of μ-closed subsets of X such that $\bigcap_{\alpha\in I} S_\alpha = \emptyset$. Then $X = X \setminus \bigcap_{\alpha\in I} S_\alpha = \bigcup_{\alpha\in I} X \setminus S_\alpha$, so $\{X \setminus S_\alpha : \alpha \in I\}$ forms a μ-open cover of X. By (i), there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $X = \bigcup_{i=1}^{n} i_\mu(c_\mu(X \setminus S_{\alpha_i}))$. Therefore $X \setminus \bigcup_{i=1}^{n} i_\mu(c_\mu(X \setminus S_{\alpha_i})) = \emptyset$, i.e., $\bigcap_{i=1}^{n} c_\mu(i_\mu(S_{\alpha_i})) = \emptyset$.

Conversely, let $M = \{U_\alpha : \alpha \in I\}$ be a μ-open cover of X. Then $X = \bigcup_{\alpha\in I} U_\alpha$ and hence $X \setminus \bigcup_{\alpha\in I} U_\alpha = \emptyset$, i.e., $\bigcap_{\alpha\in I} X \setminus U_\alpha = \emptyset$. By (iv), there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $\bigcap_{i=1}^{n} c_\mu(i_\mu(X \setminus U_{\alpha_i})) = \emptyset$. So $X = X \setminus \bigcap_{i=1}^{n} c_\mu(i_\mu(X \setminus U_{\alpha_i})) = \bigcup_{i=1}^{n} i_\mu(c_\mu(U_{\alpha_i}))$. Thus X is μ-nearly compact.

The proofs of $(ii) \iff (iii)$ and $(iv) \iff (v)$ are straightforward. □

Theorem 4.6. *A μ-semiregular space (X, μ) is μ-nearly compact if and only if it is μ-compact.*

Proof. Let $M = \{U_\alpha : \alpha \in I\}$ be a μ-open cover of X. For every point $x \in X$, there exists $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. Since X is μ-semiregular and $U_{\alpha_x} \in M$, there is a μ-open set V_{α_x} such that $x \in V_{\alpha_x} \subseteq i_\mu(c_\mu(V_{\alpha_x})) \subseteq U_{\alpha_x}$. Hence, $X = \bigcup_{x\in X} V_{\alpha_x} \subseteq \bigcup_{x\in X} i_\mu(c_\mu(V_{\alpha_x}))$. Thus, $\{i_\mu(c_\mu(V_{\alpha_x})) : x \in X\}$ forms a μ-regular open cover of X. Since X is μ-nearly compact, there exists a finite subset of points $\{x_1, x_2, \ldots, x_n\}$ of X such that $X = \bigcup_{i=1}^{n} i_\mu\left(c_\mu\left(V_{\alpha_{x_i}}\right)\right) \subseteq \bigcup_{i=1}^{n} U_{\alpha_{x_i}}$. As a result, X is μ-compact. The converse is obvious. □

Definition 4.7. [14] Let (X, μ) be a GTS. Then $\mu_x = \{U \in \mu : x \in U\}$.

Theorem 4.8. *Let a GTS (X, μ) be a μ-regular space. Every infinite subset of μ-nearly compact space has at least one μ-accumulation point in X.*

Proof. Assume that X is a μ-nearly compact space. Let D be an infinite subset of X. Suppose that D has no μ-accumulation points in X. Then, for every point $x \in X$, there is a μ-open set $U_x \in \mu_x$ such that $U_x \cap D = \{x\}$ or $\emptyset$, or in easier way we can write $U_x \cap D \subseteq \{x\}$. Since X is μ-regular space, and U_x is μ-open set containing x, there exists a μ-open set V_x such that $x \in V_x \subseteq c_\mu(V_x) \subseteq U_x$. So, $X = \bigcup\limits_{x\in X} V_x \subseteq \bigcup\limits_{x\in X} c_\mu(V_x) \subseteq \bigcup\limits_{x\in X} U_x$. Now $\{V_x : x \in X\}$ is a μ-open cover of X. Since X is μ-nearly compact, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $X = \bigcup_{i=1}^{n} i_\mu(c_\mu(V_{x_i})) \subseteq \bigcup_{i=1}^{n} c_\mu(V_{x_i}) \subseteq \bigcup_{i=1}^{n} U_{x_i}$. So, for any $x \in X$, we will have $x \in U_{x_i}$ for some $i = 1, 2, \ldots, n$. But

$$(U_{x_1} \cap D) \cup (U_{x_2} \cap D) \cup \cdots \cup (U_{x_n} \cap D) \subseteq \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\}.$$

This implies

$$(U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}) \cap D \subseteq \{x_1, x_2, \ldots, x_n\},$$

or

$$D \subseteq [X \setminus (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n})] \cup \{x_1, x_2, \ldots, x_n\}.$$

We want to show that

$$D \subseteq \{x_1, x_2, \ldots, x_n\}.$$

Let $x \in D$, we claim that $x \notin X \setminus (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n})$. Assume that $x \in X \setminus (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n})$, so $x \notin (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n})$, i.e., $x \notin U_{x_i}$ for each $i = 1, 2, \ldots, n$, which is a contradiction. Thus we must have $x \in \{x_1, x_2, \ldots, x_n\}$ showing that $D \subseteq \{x_1, x_2, \ldots, x_n\}$. This contradicts the fact that D is infinite. □

Theorem 4.9. *Let X be a μ-almost regular and μ-nearly compact space. Then for every disjoint μ-regular closed sets C_1 and C_2, there exist two μ-open sets U and V such that $U \cap V = \emptyset$ and $C_1 \subseteq U, C_2 \subseteq V$.*

Proof. Let $x \in X$ and $x \notin C_2$ so that $x \in X \setminus C_2$. Since X is μ-almost regular, for each $x \in X \setminus C_2$, there exists a μ-regular open neighbourhood U_x of x such that $x \in U_x \subseteq c_\mu(U_x) \subseteq X \setminus C_2$. Then $c_\mu(U_x) \cap C_2 = \emptyset$ and hence the family $\{U_x : x \in C_1\} \cup \{X \setminus C_1\}$ forms a μ-regular open cover of X. Since X is μ-nearly compact, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $X = \left(\bigcup_{i=1}^{n} U_{x_i}\right) \cup (X \setminus C_1)$. It follows that $C_1 \subseteq \bigcup_{i=1}^{n} U_{x_i}$ and $c_\mu(U_{x_i}) \cap C_2 = \emptyset$ for each $i = 1, 2, \ldots, n$. Analogously, there exists a finite subfamily $\{V_{y_i} : i = 1, 2, \ldots, n\}$ of μ-regular open subsets of X such that $C_2 \subseteq \bigcup_{i=1}^{n} V_{y_i}$ and $c_\mu(V_{y_i}) \cap C_1 = \emptyset$ for each $i = 1, 2, \ldots, n$. Let

$$G_i = U_{x_i} \setminus \left(\bigcup_{k=1}^{i} c_\mu(V_{y_k})\right) \text{ and } H_i = V_{y_i} \setminus \left(\bigcup_{k=1}^{i} c_\mu(U_{x_k})\right).$$

Since $G_i \cap c_\mu(H_m) = \emptyset$ for $m \leq i$, it follows that $G_i \cap H_m = \emptyset$ for $m \leq i$. Similarly, $H_m \cap c_\mu(G_i) = \emptyset$ for $i \leq m$, it follows that $H_m \cap G_i = \emptyset$ for $i \leq m$. Thus $G_i \cap H_m = \emptyset$ for all m and i, and consequently $U = \bigcup_{i=1}^{n} G_i$ is disjoint from $V = \bigcup_{i=1}^{n} H_i$. Finally, $c_\mu(V_{y_k}) \cap C_1 = \emptyset$ and $c_\mu(U_{x_k}) \cap C_2 = \emptyset$ for all $k = 1, 2, \ldots, n$ and hence $C_1 \subseteq U$ whilst $C_2 \subseteq V$. Since U_{x_i} is μ-regular open set, then it is also μ-open in X. This implies that G_i is μ-open set in X because $\bigcup_{k=1}^{i} c_\mu(H_k)$ is μ-closed in X and the complement of μ-closed set is μ-open. Hence U is μ-open set in X. While V is also μ-open set in X and can be obtained by the similar argument. The proof is complete. □

Lemma 4.10. *Let $\{A_\alpha : \alpha \in I\}$ be a collection of μ-preopen sets in a GTS (X, μ). Then $\bigcup_{\alpha \in I} A_\alpha$ is μ-preopen set in X.*

Proof. For each $\alpha \in I$, we have $A_\alpha \subseteq i_\mu(c_\mu(A_\alpha))$. Then $\bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} i_\mu(c_\mu(A_\alpha)) \subseteq i_\mu\left(\bigcup_{\alpha \in I} c_\mu(A_\alpha)\right) \subseteq i_\mu\left(c_\mu\left(\bigcup_{\alpha \in \Delta} A_\alpha\right)\right)$. Therefore $\bigcup_{\alpha \in I} A_\alpha$ is μ-preopen set in X. □

Proposition 4.11. Let X be a μ-almost regular and μ-nearly compact space. Then for every disjoint μ-regular closed set C_1 and C_2, there exist two μ-preopen sets U and V such that $U \cap V = \emptyset$ and $C_1 \subseteq U, C_2 \subseteq V$.

Proof. Follow the proof of Theorem 4.9 and since G_i and H_i are also μ-preopen sets in X, then U and V are μ-preopen sets by Lemma 4.10. □

5. μ-Nearly Compact Subspaces and Subsets

In this section, we investigate some of the characterizations of μ-nearly compact subspaces and subsets in generalized topological spaces. A subset B of a GTS (X, μ) is said to be μ-nearly compact if B is μ-nearly compact as a subspace of X, i.e., B is nearly compact with respect to the generalized topology $\mu \mid_B$. But first, we need to define the concept of μ-nearly compact relative to the generalized topological spaces.

Definition 5.1. Let a GTS (X, μ) be a μ-space. A subset B of X is called μ-nearly compact relative to X if every cover $M = \{U_\alpha : \alpha \in I\}$ of B by μ-open subsets of X contains a finite subcollection $M' = \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $B \subseteq \bigcup_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. If B is a μ-nearly compact relative to X and $B = X$, then the space B is called μ-nearly compact.

Theorem 5.2. *Let a GTS (X, μ) be a μ-space and $A \subseteq X$, then the following are equivalent:*

(i) *A is μ-nearly compact relative to X;*

(ii) *Every collection of μ-regular open subsets of X that cover A has a finite subcollection covering A;*

(iii) *For every collection $\{S_\alpha : \alpha \in I\}$ of μ-regular closed subsets of X such that $\left(\bigcap_{\alpha \in I} S_\alpha\right) \cap A = \emptyset$, there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $\left(\bigcap_{i=1}^{n} S_{\alpha_i}\right) \cap A = \emptyset$;*

(iv) *For every collection $\{S_\alpha : \alpha \in I\}$ of μ-regular closed subsets of X with finite subcollection $\{S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_n}\}$ such that $\left(\bigcap_{i=1}^{n} S_{\alpha_i}\right) \cap A \neq \emptyset$, the intersection $\left(\bigcap_{\alpha \in I} S_\alpha\right) \cap A \neq \emptyset$;*

(v) *For every collection $\{S_\alpha : \alpha \in I\}$ of μ-closed subsets of X such that $\left(\bigcap_{\alpha \in I} S_\alpha\right) \cap A = \emptyset$, there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $\left(\bigcap_{i=1}^{n} c_\mu\left(i_\mu\left(S_{\alpha_i}\right)\right)\right) \cap A = \emptyset$;*

(vi) *For every collection $\{S_\alpha : \alpha \in I\}$ of μ-closed subsets of X with finite subcollection $\{S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_n}\}$ such that $\left(\bigcap_{i=1}^{n} S_{\alpha_i}\right) \cap A \neq \emptyset$, the intersection $\left(\bigcap_{\alpha \in I} c_\mu\left(i_\mu\left(S_\alpha\right)\right)\right) \cap A \neq \emptyset$.*

Proof. (i) $\iff$ (ii) : Let $M = \{U_\alpha : \alpha \in I\}$ be a collection of μ-regular open subsets of X that cover A. Then $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ but M is also a collection of μ-open subsets of X that cover A. By (i), there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I such that $A \subseteq \bigcup_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right) = \bigcup_{i=1}^{n} U_{\alpha_i}$. Coversely, let $\{U_\alpha : \alpha \in I\}$ be a collection of μ-open subsets of X that cover A. Since $U_\alpha \subseteq i_\mu\left(c_\mu\left(U_\alpha\right)\right)$, then $A \subseteq \bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} i_\mu\left(c_\mu\left(U_\alpha\right)\right)$. Thus, $\{i_\mu\left(c_\mu\left(U_\alpha\right)\right) : \alpha \in I\}$ forms a collection of μ-regular open subsets of X covering A. By (ii), there is a finite subcollection $\{i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right) : i = 1, 2, \ldots, n\}$ such that $A \subseteq \bigcup_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. Therefore, A is μ-nearly compact relative to X.

$(ii) \implies (iii)$: Let $\{S_\alpha : \alpha \in I\}$ be a collection of μ-regular closed subsets of X such that $\left(\bigcap_{\alpha\in I} S_\alpha\right) \cap A = \emptyset$. Then, $A \subseteq X \setminus \bigcap_{\alpha\in I} S_\alpha = \bigcup_{\alpha\in I} X \setminus S_\alpha$, so $\{X \setminus S_\alpha : \alpha \in I\}$ forms a collection of μ-regular open subsets of X covering A. By hypothesis, there is a finite subcollection $\{X \setminus S_{\alpha_i} : i = 1, 2, \ldots, n\}$ such that $A \subseteq \bigcup_{i=1}^{n} X \setminus S_{\alpha_i}$. Thus $\left(X \setminus \bigcup_{i=1}^{n} X \setminus S_{\alpha_i}\right) \cap A = \emptyset$, i.e., $\left(\bigcap_{i=1}^{n} S_{\alpha_i}\right) \cap A = \emptyset$.

$(iii) \implies (i)$: Let $\{U_\alpha : \alpha \in I\}$ be a collection of μ-open sets in X such that $A \subseteq \bigcup_{\alpha\in I} U_\alpha$. Then $A \subseteq \bigcup_{\alpha\in I} U_\alpha \subseteq \bigcup_{\alpha\in I} i_\mu\left(c_\mu\left(U_\alpha\right)\right)$, and hence

$$\left(X \setminus \bigcup_{\alpha\in I} i_\mu\left(c_\mu\left(U_\alpha\right)\right)\right) \cap A = \emptyset,$$

i.e., $\left(\bigcap_{\alpha\in I} X \setminus i_\mu\left(c_\mu\left(U_\alpha\right)\right)\right) \cap A = \emptyset$. Since $\{X \setminus i_\mu\left(c_\mu\left(U_\alpha\right)\right) : \alpha \in I\}$ is a collection of μ-regular closed subsets of X, by hypothesis, there is a finite subcollection $\{X \setminus i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right) : i = 1, 2, \ldots, n\}$ such that $\left(\bigcap_{i=1}^{n} X \setminus i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)\right) \cap A = \emptyset$. Therefore, $\left(X \setminus \bigcup_{\alpha\in I} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)\right) \cap A = \emptyset$, i.e., $A \subseteq \bigcup_{\alpha\in I} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. As a consequence, A is μ-nearly compact relative to X.

The proofs of $(iii) \iff (iv) \iff (v) \iff (vi)$ are straightforward. □

Theorem 5.3. *Let B be a subspace of a GTS (X, μ), then:*

(a) *$H \subseteq B$ is $\mu\,|_B$-open in B if and only if $H = G \cap B$ where G is μ-open in X.*

(b) *$F \subseteq B$ is $\mu\,|_B$-closed in B if and only if $F = K \cap B$ where K is μ-closed in X.*

(c) *if $A \subseteq B$, then $c_{\mu|_B}(A) = B \cap c_\mu(A)$.*

(d) *if $A \subseteq B$, then $i_{\mu|_B}(A) \supseteq B \cap i_\mu(A)$.*

Proof. (a) This just the definition of the subspace generalized topology on B.

(b) Clear from part (a).

(c) Since $A \subseteq B$ and $c_{\mu|_B}(A)$ is $\mu\,|_B$-closed set in B, then $c_{\mu|_B}(A) = B \cap K$ where K is a μ-closed set in X containing A by part (b). Since $c_{\mu|_B}(A)$ is the smallest $\mu\,|_B$-closed set containing A, then K must be the smallest μ-closed set containing A, so we must choose $K = c_\mu(A)$. Therefore $c_{\mu|_B}(A) = B \cap c_\mu(A)$.

(d) Since $i_\mu(A)$ is μ-open in X, then $B \cap i_\mu(A)$ is $\mu\,|_B$-open in B contained in A by part (a). Hence $B \cap i_\mu(A) \subseteq i_{\mu|_B}(A)$ since $i_{\mu|_B}(A)$ is the largest $\mu\,|_B$-open set contained in A. □

From part (d), the $i_{\mu|_B}(A)$ is not necessarily equal to the intersection of B with $i_\mu(A)$ as the following counterexample shows.

Example 5.4. Let X be the plane $\left(= \mathbb{R}^2\right)$ with usual generalized topology μ, i.e., generalized topology generated by open rectangle $(a, b) \times (c, d)$, the product of open intervals in $\mathbb{R}$. If $A = B = x$-axis, then $i_{\mu|_B}(A) = A$ while $i_\mu(A) = \emptyset$, so that the first cannot be obtained by intersecting the second with B.

Theorem 5.5. *Let a GTS (X, μ) be a μ-space and B be a subset of X. If B is μ-nearly compact, then it is μ-nearly compact relative to X.*

Proof. Let $M = \{U_\alpha : \alpha \in I\}$ be a cover of B by μ-open subsets of X. For every α, we can find a μ-open set V_α of B with $U_\alpha \cap B = V_\alpha$. Therefore, $\{V_\alpha : \alpha \in I\}$ is a cover of

B by μ-open subsets of B. Since B is μ-nearly compact, there is a finite subcollection $\{V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$ such that

$$\begin{aligned} B &= \bigcup_{i=1}^{n} i_{\mu|_B}\left(c_{\mu|_B}\left(V_{\alpha_i}\right)\right) \\ &= \bigcup_{i=1}^{n} i_{\mu|_B}\left(B \cap c_{\mu}\left(V_{\alpha_i}\right)\right) \\ &\subseteq \bigcup_{i=1}^{n} i_{\mu}\left(c_{\mu}\left(V_{\alpha_i}\right)\right) \\ &\subseteq \bigcup_{i=1}^{n} i_{\mu}\left(c_{\mu}\left(U_{\alpha_i}\right)\right). \end{aligned}$$

by Theorem 5.3. Thus, B is μ-nearly compact relative to X. □

Theorem 5.6. *Let a GTS (X, μ) be a μ-space and assume that B is a μ-open subset of X. Then B is μ-nearly compact if and only if it is μ-nearly compact relative to X.*

Proof. Necessity. This can be proven by Theorem 5.5.

Sufficiency. Let $M = \{U_\alpha : \alpha \in I\}$ be a μ-open cover of B. Since μ-open subsets of a μ-open subspace of X are μ-open sets in X, then the cover of B is the collection M of μ-open subsets of X. Since B is μ-nearly compact relative to X, there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $B \subseteq \bigcup_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. Then,

$$\begin{aligned} B &= B \cap \left(\bigcup_{i=1}^{n} i_{\mu}\left(c_{\mu}\left(U_{\alpha_i}\right)\right)\right) \\ &= \bigcup_{i=1}^{n} B \cap i_{\mu}\left(c_{\mu}\left(U_{\alpha_i}\right)\right) \\ &= \bigcup_{i=1}^{n} i_{\mu|_B}\left(B \cap c_{\mu}\left(U_{\alpha_i}\right)\right) \\ &= \bigcup_{i=1}^{n} i_{\mu|_B}\left(c_{\mu|_B}\left(U_{\alpha_i}\right)\right). \end{aligned}$$

Hence, B is μ-nearly compact. □

Theorem 5.7. *Every μ-regular closed subset of a μ-nearly compact space X is μ-nearly compact relative to X.*

Proof. Let K be a μ-regular closed subset of X. If $M = \{U_\alpha : \alpha \in I\}$ is a cover of K by μ-open subsets of X, then $X = \left(\bigcup_{\alpha \in I} U_\alpha\right) \cup (X \setminus K)$. Thus, the cover of X is the collection $\{U_\alpha : \alpha \in I\} \cup \{X \setminus K\}$ of μ-open subsets of X. Since X is μ-nearly compact, there is a finite subcollection $\{X \setminus K, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that

$$\begin{aligned} X &= \bigcup_{i=1}^{n} i_{\mu}\left(c_{\mu}\left(U_{\alpha_i}\right)\right) \cup i_{\mu}\left(c_{\mu}\left(X \setminus K\right)\right) \\ &= \bigcup_{i=1}^{n} i_{\mu}\left(c_{\mu}\left(U_{\alpha_i}\right)\right) \cup \left(X \setminus K\right). \end{aligned}$$

Hence $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ is a finite subcollection of μ-open subsets of X such that $K \subseteq \bigcup_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. Therefore, K is μ-nearly compact relative to X. □

Theorem 5.8. *A μ-clopen subset of a μ-nearly compact space X is μ-nearly compact.*

Proof. Let J be a μ-clopen subset of X. Since J is a μ-open subset of X, by Theorem 5.6, it is sufficient to prove that J is μ-nearly compact relative to X. Suppose $M = \{U_\alpha : \alpha \in I\}$ is a collection of μ-open subsets of X covering J. Then

$\{U_\alpha : \alpha \in I\} \cup \{X \setminus J\}$ forms a μ-open cover of X. Since X is μ-nearly compact, there is a finite subcollection $\{X \setminus J, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that

$$\begin{aligned} X &= \bigcup\nolimits_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right) \cup i_\mu\left(c_\mu\left(X \setminus J\right)\right) \\ &= \bigcup\nolimits_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right) \cup \left(X \setminus J\right). \end{aligned}$$

Thus $J \subseteq \bigcup\nolimits_{i=1}^{n} i_\mu\left(c_\mu\left(U_{\alpha_i}\right)\right)$. This shows that J is μ-nearly compact relative to X. $\square$

Definition 5.9. *A subset E of a GTS (X, μ) is said to be μ-dense in $F \subseteq X$ if $F \subseteq c_\mu(E)$. In particular, E is μ-dense in X or is a μ-dense subset of X if $c_\mu(E) = X$.*

Observe that the conditions in Theorem 5.7 that a subset should be μ-regular closed and in Theorem 5.8 that a subset should be μ-clopen is necessary and it is not sufficient to be only μ-closed or μ-open as examples below show. In general, arbitrary subspaces of μ-nearly compact spaces need not be μ-nearly compact. A μ-closed subset of a μ-nearly compact space is not μ-nearly compact as the following examples show.

Example 5.10. Consider the space from Example 4.4. So the generalized topological space $(X, \mu(\beta))$ is a μ-nearly compact. Let $B = \{5, 10, 15, 20, \ldots\}$, then it is clear that B is a proper infinite $\mu(\beta)$-closed subset of X. It can be verified that B is not μ-nearly compact since $\{\{e\} : e \in B\}$ is a $\mu(\beta)\mid_B$-open cover of B but we cannot find a finite subcollection such that the $\mu\mid_B$-interiors of $\mu\mid_B$-closures of whose members covers B.

Example 5.11. Let $X = \{1, 2, 3, \ldots\}$ be an infinite set with generalized topology μ consist of $\emptyset$ and all supersets of $\{1\}$. Then (X, μ) is a μ-space. Thus every nonempty μ-open subset of X is μ-dense in X. Therefore (X, μ) is μ-nearly compact but not μ-compact since we cannot find a finite subcollection of μ-open sets that cover X. Next we let $E = \{10, 20, 30, \ldots\}$, then it is clear that E is a proper infinite μ-closed subset of X. It can be verified that E is not μ-nearly compact since $\{\{a\} : a \in E\}$ is a $\mu\mid_E$-open cover of E but we cannot find a finite subcollection such that the $\mu\mid_E$-interiors of $\mu\mid_E$-closures of whose members covers E.

A μ-open subset of a μ-nearly compact space is not μ-nearly compact as illustrate by the following example.

Example 5.12. Let $X = [0, 1]$. Define a neighborhood basis of all points of X except 0 in the usual fashion. Define a neighborhood basis of 0 to be the family of sets of the form $[0, \varepsilon[$, from which all points af the form $\frac{1}{n}, n \in \mathbb{N}$ have been removed. We define generalized topology μ by this neighborhood basis, so (X, μ) is a μ-space. It is clear that (X, μ) is μ-compact and hence μ-nearly compact. Let a μ-open set, $D = \{[\alpha, \beta[: 0 < \alpha < \beta < 1\}$. It can be verified that D is not μ-nearly compact.

6. Conclusion

In our work we have studied the properties of μ-nearly compactness in generalized topological spaces. Moreover, μ-nearly compact subspaces and subsets in generalized topological spaces are studied and some of their characterizations are also investigated. From the last three examples, we can conclude that, the μ-nearly compactness is not hereditary property. In the future works, we will try to define or expand some other new notions for the μ-compactness in generalized topological spaces.

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