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VAGUE SOFT SEPARATION AXIOMS VIA VAGUE SOFT QUASI-COINCIDENT

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Abstract. Zorlutuna et al introduced the concept of soft points and studied several properties in soft topological spaces. As a generalization, in our previous study we introduced the concept of vague soft points and explored some relationships between vague soft separation axioms that contradicted the results obtained in general topological spaces. In this paper, we introduced the notions of x-vague soft points, vague soft characteristic functions and defined vague soft subspace topology. Also we make an attempt to bring the traditional relationships among vague soft separation axioms as in the case of general topological spaces by introducing the notion of vague soft quasi-coincident.

1. Introduction

Uncertainty is the root cause of hindrance in decision making. Many Mathematical theories like vague set theory [9], fuzzy set theory [30], intuitionistic fuzzy set theory [6] and rough set theory [19] have been developed to approach the problems associated with uncertainty. Inspite of the development of these theories challenges do exist due to the inadequacy in parameters. Molodtsov [18] proposed the soft set theory enough parameters and Maji et al. [15] tested the applications of soft set theory in resolving the lack of certainty. Shabir and Naz [21] introduced the notion of soft topological spaces, that was studied by various authors like W. K. Min [17], S. Hussain et al.[11], I. Zorlutuna et al. [31], O. Tantawy et al. [22], D.N. Georgiou et al.[10], B.P. Varol et al.[23] and G. Senel et al.[20]. Later, researchers showed their interest in working on the combination of these fields like fuzzy soft sets [14, 13], intuitionistic fuzzy soft sets [8] and soft rough fuzzy sets and soft fuzzy rough sets[16].

In 2010, W. Xu et al.[29] developed the notion of vague soft sets to generalize the concept of soft set theory. C. Wang et al. [26] further studied the topological structure

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of vague soft sets defined over an initial universal set and a fixed set of parameters. Authors like Bhargavi et al. [7], K. Alhazaymeh et al. [2, 3, 4, 5], A. Al-Quran et al.[1], C. Wang et al.[27, 24] had contributed to the expansion of application of vague soft set theory.

In this work, we introduced the new concept of x-vague soft points and Vague soft quasi-coincident relation for x-vague soft points. Subsequently we put forward the vague soft subspace in vague soft topological spaces and explore several interesting properties of vague soft $\mathfrak{T}_i (i = 0, 1, 2, 3, 4)$-spaces.

2. Preliminaries

Definition 2.1. [9] A vague set $A = \{(x_i, [t_A(x_i), 1 - f_A(x_i)]) \mid x_i \in X\}$ in the universe $X = \{x_1, x_2, ..., x_n\}$ is characterized by a truth-membership function $t_A : X \to [0, 1]$, and a false-membership function $f_A : X \to [0, 1]$, where $t_A(x_i)$ is a lower bound on the grade of membership of x_i derived from the evidence of x_i, $f_A(x_i)$ is the lower bound on the negation of x_i derived from the evidence against x_i and $0 \leq t_A(x_i) + f_A(x_i) \leq 1$ for any $x_i \in X$. The grade of membership of x_i in the vague set is bounded to a subinterval $[t_A(x_i), 1 - f_A(x_i)]$ of [0,1]. The vague value $[t_A(x_i), 1 - f_A(x_i)]$ indicates that the exact grade of membership $\mu_A(x_i)$ of x_i may be unknown, but it is bounded by $t_A(x_i) \leq \mu_A(x_i) \leq 1 - f_A(x_i)$, where $0 \leq t_A(x_i) + f_A(x_i) \leq 1$.

Notations: Let $I[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$. If $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be two elements of $I[0, 1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly we understand the relations $I_1 \leq I_2$ and $I_1 = I_2$. Clearly the relation $I_1 \geq I_2$ does not necessarily imply that $I_1 \supseteq I_2$ and conversely. Also for any two unequal intervals I_1 and I_2, there is no necessity that either $I_1 \geq I_2$ and $I_1 \leq I_2$ will be true.

Definition 2.2. [29] Let X be an initial universe set, $V(X)$ the set of all vague sets on X, E be a set of parameters, and $A \subseteq E$. A pair (F, A) is called a vague soft set over X, where F is a mapping given by $F : A \to V(X)$. The set of all vague soft sets on X is denoted by $V\tilde{S}(X, E)$, called vague soft classes. The interval $[t_{F(e)}(x), 1 - f_{F(e)}(x)]$ of (F, A) is called the vague soft value of $x \in X$ for the parameter $e \in A$ and is denoted by $V_{F(e)}(x)$.

Definition 2.3. [29] A vague soft set (F, A) over X is said to be a null vague soft set denoted by $\hat{\emptyset}$, if $\forall e \in A$, $t_{F(e)}(x) = 0$, $1 - f_{F(e)}(x) = 0$, $x \in X$.
That is, $V_{F(e)}(x) = [0, 0]$, $\forall e \in A, x \in X$.

Definition 2.4. [29] A vague soft set (F, A) over X is said to be an absolute vague soft set denoted by $\hat{X}$, if $\forall e \in A$, $t_{F(e)}(x) = 1$, $1 - f_{F(e)}(x) = 1$, $x \in X$.
That is, $V_{F(e)}(x) = [1, 1]$, $\forall e \in A, x \in X$.

Definition 2.5. [29] The complement of a vague soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c$= (F^c, A) and is given by $t_{F^c(e)}(x) = f_{F(e)}(x)$, $1 - f_{F^c(e)}(x) = 1 - t_{F(e)}(x)$, for all $e \in A$, $x \in X$.
That is, $V_{F^c(e)}(x) = [f_{F(e)}(x), 1 - t_{F(e)}(x)]$, $\forall e \in A, x \in X$.

Definition 2.6. [26] Let X be an initial universe set, E be the nonempty **fixed set of parameters** and τ be the collection of vague soft sets over X, then τ is said to be a vague soft topology on X if

(1) $\hat{\emptyset}_E, \hat{X}_E$ belongs to τ.
(2) the union of any number of vague soft sets in τ belongs to τ.
(3) the intersection of any two vague soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a vague soft topological space over X.

Definition 2.7. [25] Let $V\tilde{S}(X, E)$ and $V\tilde{S}(Y, K)$ be two vague soft classes, and let $u : X \to Y$ and $p : E \to K$ be mappings. Then a vague soft function $g_{pu} = (u, p) : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ is defined as: for $(F, A) \in V\tilde{S}(X, E)$, the image of (F, A) under g_{pu} denoted by $g_{pu}(F, A) = (u(F), p(A))$, is a vague soft set in $V\tilde{S}(Y, K)$ given by

$$t_{u(F)(\beta)}(y) = \begin{cases} \sup\limits_{\alpha \in p^{-1}(\beta) \cap A,\ x \in u^{-1}(y)} t_{F(\alpha)}(x) & \text{if } u^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and $$1 - f_{u(F)(\beta)}(y) = \begin{cases} \sup\limits_{\alpha \in p^{-1}(\beta) \cap A,\ x \in u^{-1}(y)} 1 - f_{F(\alpha)}(x) & \text{if } u^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

for all $\beta \in p(A)$ and $y \in Y$.

Definition 2.8. [25] Let $V\tilde{S}(X, E)$ and $V\tilde{S}(Y, K)$ be two vague soft classes, and let $g_{pu} = (u, p) : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function and (G, B) be a vague soft set in $V\tilde{S}(Y, K)$. Then the inverse image of (G, B) under g_{pu}, denoted by $g_{pu}^{-1}(G, B) = (u^{-1}(G),\, p^{-1}(B))$ is a vague soft set in $V\tilde{S}(X, E)$ given by

$$t_{u^{-1}(G)(\alpha)}(x) = t_{G(p(\alpha))}(u(x)) \text{ and } 1 - f_{u^{-1}(G)(\alpha)}(x) = 1 - f_{G(p(\alpha))}(u(x))$$

for all $\alpha \in p^{-1}(B)$ and $x \in X$.

Definition 2.9. [25] Let X be an initial universe set and E be the set of parameters. If $\tau = \{\hat{\emptyset}_E, \hat{X}_E\}$, then τ is called vague soft indiscrete topology on X and (X, τ, E) is vague soft indiscrete topological space over X.

Definition 2.10. [25] Let X be an initial universe set and E be the set of parameters. If τ be the collection of all vague soft sets which can be defined over X, then τ is called the vague soft discrete topology on X and (X, τ, E) is said to be a vague soft discrete topological space over X.

Definition 2.11. [25] A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is called a vague soft point, denoted by e_F if for the element $e \in E$, $F(e) \neq \tilde{\emptyset}$ and $F(e') = \tilde{\emptyset}$ for all $e' \in E \setminus \{e\}$.

Theorem 2.12. [25] *Every non-null vague soft set can be written as the union of all its vague soft points.*

Theorem 2.13. [25] *Let $(F, A), (G, B)$ be two vague soft sets in $V\tilde{S}(X, E)$ and $V\tilde{S}(Y, K)$ respectively, and g_{pu}: $V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then*
(1) $(F, A) \subseteq g_{pu}^{-1}(g_{pu}(F, A))$ and if g_{pu} is injective, the equality holds.
(2) $g_{pu}(g_{pu}^{-1}(G, B)) \subseteq (G, B)$ and if g_{pu} is surjective, the equality holds.

Theorem 2.14. [25] *Let (X, τ, E) and (Y, σ, K) be two vague soft topological spaces. The vague soft function g_{pu}: $V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ is called vague soft continuous, if and only if for all $(G, K) \in \sigma$, $g_{pu}^{-1}(G, K) \in \tau$.*

Definition 2.15. [12] A vague soft mapping $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is said to be vague soft open mapping if $g_{pu}(F, E) \in \sigma$ for each $(F, E) \in \tau$.

Proposition 2.16. [12] Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft bijective function. Then g_{pu} is vague soft open mapping iff g_{pu}^{-1} is vague soft continuous mapping.

3. Vague soft quasi-coincident

In this paper, we are studing the relationship among vague soft separation axioms in vague soft topological spaces. C. Wang et al. [26] formulated vague soft topological spaces only by taking fixed set of parameters. If we take the subset of elementary set, it doesnot satisfy the intersection axiom of vague soft topological spaces. Hence we have considered the entire fixed set of parameters for this current work.

Definition 3.1. Let X be an initial universal set, E be the nonempty set of parameters and $x \in X$ a fixed element in X. A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is called x**-vague soft point**, if for the element $e \in E$,

$$V_{F(e')}(x') = \begin{cases} [\alpha, 1-\beta], & \text{if } e' = e \text{ and } x' = x, \\ [0,0], & \text{Otherwise.} \end{cases}$$

for all $x' \in X$ and $e' \in E$.
Or in other-words, a vague soft point e_F is said to be x-vague soft point, if

$$e_F(x') = [t_{F(e)}(x'), 1 - f_{F(e)}(x')] = \begin{cases} [\alpha, 1-\beta], & \text{if } x' = x, \\ [0,0], & \text{Otherwise.} \end{cases}$$

for all $x' \in X$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1)$ are two fixed real numbers such that $0 \leq \alpha + \beta \leq 1$ with $[\alpha, 1-\beta] \neq [0, 0]$. And it is denoted by $x_{e,[\alpha,1-\beta]}$.

In this case, α denotes the degree of membership of $x_{e,[\alpha,1-\beta]}$, β denotes the degree of non-membership of $x_{e,[\alpha,1-\beta]}$, $x \in X$ is called the support of $x_{e,[\alpha,1-\beta]}$ on e with the non vanishing value $[\alpha, 1-\beta]$ and e is called the expressive parameter of $x_{e,[\alpha,1-\beta]}$. The family of all x-vague soft points over (X, E) is denoted by $V\tilde{S}P_x(X, E)$.

Remark 3.2. Let $x_{e,[\alpha,1-\beta]}$, $x_{e,[\gamma,1-\delta]} \in V\tilde{S}P_x(X, E)$ and $x'_{e',[\sigma,1-\omega]} \in V\tilde{S}P_{x'}(X, E)$.

(1) $x_{e,[\alpha,1-\beta]} \subseteq x_{e,[\gamma,1-\delta]}$ if $[\alpha, 1-\beta] \leq [\gamma, 1-\delta]$. That is $\alpha \leq \gamma$ and $1-\beta \leq 1-\delta$.
(2) $x_{e,[\alpha,1-\beta]} = x'_{e',[\sigma,1-\omega]}$ iff $x = x'$, $e = e'$ and $[\alpha, 1-\beta] = [\sigma, 1-\omega]$.

Result 3.3. If x, y are two distinct elements in X, then the corresponding x-vague soft points and the y-vague soft points are disjoint. That is, if $x \neq y$, then $x_e \neq y_e$ for all $e \in E$.

Definition 3.4. A x-vague soft point $x_{e,[\alpha,1-\beta]}$ is said to be contained in a vague soft set (G,E) if $[\alpha, 1-\beta] \leq V_{G(e)}(x)$ and it is denoted by $x_{e,[\alpha,1-\beta]} \subseteq (G, E)$. The collection of all x-vague soft point of (G, E) is denoted by, $x(\mathcal{G}, \mathcal{E})$.

Definition 3.5. A x-vague soft point $x_{e,[\alpha,1-\beta]}$ is said to be contained in a vague soft point e_F if $[\alpha, 1-\beta] \leq V_{F(e)}(x)$ and it is denoted by $x_{e,[\alpha,1-\beta]} \subseteq e_F$. The collection of all x-vague soft point of e_F is denoted by $x(\mathfrak{e}_{\mathfrak{F}})$.

Definition 3.6. Let (X, τ, E) be a vague soft topological space. A vague soft set (G, E) is called vague soft neighborhood of the x- vague soft point $x_{e,[\alpha,1-\beta]}$ if there exists $(F, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F, E) \subseteq (G, E)$. The set of all vague soft neighborhood of x-vague soft point $x_{e,[\alpha,1-\beta]}$ is called its neighborhood system and is denoted by $\mathcal{N}_\tau(x_{e,[\alpha,1-\beta]})$.

Proposition 3.7. Let $\{(G_i, E) : i \in I\}$ be a family of vague soft sets over X.

(i) $x_{e,[\alpha,1-\beta]} \subseteq \bigcap_{i\in I}(G_i, E)$ iff $x_{e,[\alpha,1-\beta]} \subseteq (G_i, E)$ for each $i \in I$.

(ii) If there exist $i \in I$ such that $x_{e,[\alpha,1-\beta]} \subseteq (G_i, E)$, then $x_{e,[\alpha,1-\beta]} \subseteq \bigcup_{i\in I}(G_i, E)$.

Proof. Straightforward. □

Note 3.8. The result (i) of Proposition 3.7 is true only if we consider the entire fixed point parameter set. For the vague soft subset (F, A) of (F, E), it is not true as shown in the following example.

Example 3.9. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $A = \{e_1, e_3\}$, $B = \{e_1, e_4\}$ be two subsets of the parameter E. If $x_{1_{e_4,[0.1,0.4]}} \in V\tilde{S}P_{x_1}(X, E)$ and
$(F, A) = \left\{\left\langle e_1, \frac{[0.2,0.4]}{x_1}, \frac{[0.4,0.5]}{x_2}, \frac{[0.7,0.8]}{x_3}\right\rangle, \left\langle e_3, \frac{[0,0.3]}{x_1}, \frac{[0.5,0.7]}{x_2}, \frac{[0.2,0.7]}{x_3}\right\rangle\right\}$,
$(G, B) = \left\{\left\langle e_1, \frac{[0.1,0.9]}{x_1}, \frac{[0.2,0.7]}{x_2}, \frac{[0.4,0.8]}{x_3}\right\rangle, \left\langle e_4, \frac{[0.3,0.5]}{x_1}, \frac{[0.5,0.6]}{x_2}, \frac{[0.1,0.6]}{x_3}\right\rangle\right\}$ are two vague soft sets, then $x_{1_{e_4,[0.1,0.4]}} \subseteq (F, A) \cap (G, B) = \left\{\begin{array}{c}\left\langle e_1, \frac{[0.1,0.4]}{x_1}, \frac{[0.2,0.5]}{x_2}, \frac{[0.4,0.8]}{x_3}\right\rangle, \\ \left\langle e_3, \frac{[0,0.3]}{x_1}, \frac{[0.5,0.7]}{x_2}, \frac{[0.2,0.7]}{x_3}\right\rangle, \\ \left\langle e_4, \frac{[0.3,0.5]}{x_1}, \frac{[0.5,0.6]}{x_2}, \frac{[0.1,0.6]}{x_3}\right\rangle\end{array}\right\}$
and $x_{1_{e_4,[0.1,0.4]}} \subseteq (G, B)$, but $x_{1_{e_4,[0.1,0.4]}} \nsubseteq (F, A)$.

Remark 3.10. Converse of part (ii) of the above Proposition 3.7 need not be true as shown in the following example.

Example 3.11. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and let $x_{2_{e_1,[0.4,0.6]}} \in V\tilde{S}P_{x_2}(X, E)$. If
$(F, E) = \left\{\left\langle e_1, \frac{[0,0.9]}{x_1}, \frac{[0.3,0.6]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.2,0.7]}{x_1}, \frac{[1,1]}{x_2}\right\rangle\right\}$,
$(G, E) = \left\{\left\langle e_1, \frac{[0.2,0.5]}{x_1}, \frac{[0.4,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.2,1]}{x_1}, \frac{[0.2,0.8]}{x_2}\right\rangle\right\}$ are two vague soft sets, then
$x_{2_{e_1,[0.4,0.6]}} \subseteq (F, E) \cup (G, E) = \left\{\left\langle e_1, \frac{[0.2,0.9]}{x_1}, \frac{[0.4,0.6]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.2,1]}{x_1}, \frac{[1,1]}{x_2}\right\rangle\right\}$ but
$x_{2_{e_1,[0.4,0.6]}} \nsubseteq (F, E)$ and $x_{2_{e_1,[0.4,0.6]}} \nsubseteq (G, E)$.

Proposition 3.12. Let (G, E) and (H, E) be two vague soft sets over X. Then

1. $(G, E) \subseteq (H, E)$ iff for each $x_{e,[\alpha,1-\beta]} \subseteq (G, E)$ implies $x_{e,[\alpha,1-\beta]} \subseteq (H, E)$.
2. $(G, E) = (H, E)$ if for each $x_{e,[\alpha,1-\beta]}$, $x_{e,[\alpha,1-\beta]} \subseteq (G, E) \Leftrightarrow x_{e,[\alpha,1-\beta]} \subseteq (H, E)$.

Proof. Obvious. □

Proposition 3.13. [28] Let U be an initial universe set. Let $f_A \in S(U, A)$ be a soft set and let $(x_e)_E \in P(U, A)$ be a soft point (in the sense of Ningxin Xie). Then $(x_e)_E \in f_A$ iff $(x_e)_E \notin f_A^c$.

Remark 3.14. The following Example 3.15 illustrates that the above Proposition 3.13 fails to satisfy in vague soft set theory.

Example 3.15. Let $X=\{x_1, x_2\}$, $E=\{e_1, e_2\}$, let $x_{1_{e_1,[0.2,0.5]}} \in V\tilde{S}P_{x_1}(X, E)$. If (G, E) is a vague soft set defined as follows:

$(G, E) = \left\{\left\langle e_1, \frac{[0.3,0.6]}{x_1}, \frac{[0.4,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.1,0.8]}{x_2}\right\rangle\right\}$

and its complement $(G, E)^c$ is given by

$(G, E)^c = \left\{\left\langle e_1, \frac{[0.4,0.7]}{x_1}, \frac{[0.5,0.6]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.2]}{x_1}, \frac{[0.2,0.9]}{x_2}\right\rangle\right\}$, then we note that $x_{1_{e_1,[0.2,0.5]}} \subseteq (G, E)$ and $x_{1_{e_1,[0.2,0.5]}} \subseteq (G, E)^c$.

Theorem 3.16. *Every vague soft point can be written as the union of all its x-vague soft points.*

Proof. Let X be an initial universal set, E be the set of parameters, and e_F be a vague soft point. Assume that for each $x \in \mathcal{Q} = \{y \in X : V_{F(e)}(y) \neq [0, 0]\}$, $x(\mathfrak{e}_{\mathfrak{F}})$ is a family of all x-vague soft points of e_F. For a fixed $x \in \mathcal{Q}$,

$$V_{F(e)}(x) = \bigcup_{x_{e,[\alpha,1-\beta]} \in x(\mathfrak{e}_{\mathfrak{F}})} x_{e,[\alpha,1-\beta]}$$

we have,

$$e_F = \{(x, V_{F(e)}(x)) : x \in X\} = \bigcup_{x_{e,[\alpha,1-\beta]} \in x(\mathfrak{e}_{\mathfrak{F}}),\ x \in \mathcal{Q}} x_{e,[\alpha,1-\beta]}.$$

□

Theorem 3.17 (**Representation theorem for vague soft sets**). *Every non-null vague soft set can be written as the union of all its x-vague soft points.*

Proof. It is follows from the Theorems 2.12 and 3.16. □

Definition 3.18. A x-vague soft point $x_{e,[\alpha,1-\beta]}$ is said be **vague soft quasi-coincident** with the vague soft set (G, E), denoted by $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (G, E) iff $\alpha > f_{G(e)}(x)$ or $1 - \beta > 1 - t_{G(e)}(x)$. And the negation of $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (G, E) is denoted by $x_{e,[\alpha,1-\beta]} \not\mathbf{q}\ (G, E)$

Example 3.19. Let $X=\{x_1, x_2\}$, $E=\{e_1, e_2\}$, let $x_{1_{e_1,[0.3,0.4]}} \in V\tilde{S}P_{x_1}(X, E)$. If (G, E) is a vague soft set $(G, E) = \left\{\left\langle e_1, \frac{[0.5,0.8]}{x_1}, \frac{[0.4,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.1,0.8]}{x_2}\right\rangle\right\}$, then $x_{1_{e_1,[0.3,0.4]}}$ $\mathbf{q}$ (G, E), since $0.3 > 0.2 = f_{G(e_1)}(x_1)$.

Definition 3.20. Let (G, E) and (H, E) be two vague soft sets over X. Then, (G, E) and (H, E) are said to be vague soft quasi-coincident, denoted by (G, E) $\mathbf{q}$ (H, E), iff there exists a pair of elements $x \in X$ and $e \in E$ such that $t_{G(e)}(x) > f_{H(e)}(x)$ or $1 - f_{G(e)}(x) > 1 - t_{H(e)}(x)$. And the negation of (G, E) $\mathbf{q}$ (H, E) denoted by the symbol $(G, E) \not\mathbf{q}\ (H, E)$.

Example 3.21. Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$. Let

$(G, E) = \left\{\left\langle e_1, \frac{[0.1,0.3]}{x_1}, \frac{[0.2,0.6]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.3,0.8]}{x_1}, \frac{[0.2,0.5]}{x_2}\right\rangle\right\}$ and

$(H, E) = \left\{\left\langle e_1, \frac{[0.2,0.8]}{x_1}, \frac{[0.3,0.7]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.3,0.7]}{x_1}, \frac{[0.2,0.8]}{x_2}\right\rangle\right\}$ be two vague soft sets over X.

Then there exist a pair of elements $x_1 \in X$ and $e_2 \in E$ such that

$0.8 = 1 - f_{G(e_2)}(x_1) > 1 - t_{H(e_2)}(x_1) = 0.7$. Hence, (G, E) $\mathbf{q}$ (H, E).

Proposition 3.22. Let (G, E) and (H, E) be two vague soft sets and a x-vague soft point $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$. Then

1. (G, E) $\mathbf{q}$ (H, E) iff $(G, E) \not\subseteq (H, E)^c$,

2. $(G,E) \not{q} (H,E)$ iff $(G,E) \subseteq (H,E)^c$,
3. $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (G,E) iff $x_{e,[\alpha,1-\beta]} \not\subseteq (G,E)^c$,
4. $x_{e,[\alpha,1-\beta]}) \not{q} (G,E)$ iff $x_{e,[\alpha,1-\beta]} \subseteq (G,E)^c$.

Proof. Straightforward. □

Example 3.23. 1. Consider the Example 3.19 and we note that, $x_{1_{e_1,[0.3,0.4]}}$ $\mathbf{q}$ (G,E) iff $x_{1_{e_1,[0.3,0.4]}} \not\subseteq (G,E)^c = \left\{ \left\langle e_1, \frac{[0.2,0.5]}{x_1}, \frac{[0.5,0.6]}{x_2} \right\rangle, \left\langle e_2, \frac{[0.1,0.2]}{x_1}, \frac{[0.2,0.9]}{x_2} \right\rangle \right\}$.

2. Consider the above Example 3.21 and we note that, (G,E) $\mathbf{q}$ (H,E) iff

$$\left\{ \begin{array}{c} \left\langle e_1, \dfrac{[0.1,0.3]}{x_1}, \dfrac{[0.2,0.6]}{x_2} \right\rangle, \\ \left\langle e_2, \dfrac{[0.3,0.8]}{x_1}, \dfrac{[0.2,0.5]}{x_2} \right\rangle \end{array} \right\} = (G,E) \not\subseteq (H,E)^c = \left\{ \begin{array}{c} \left\langle e_1, \dfrac{[0.2,0.8]}{x_1}, \dfrac{[0.3,0.7]}{x_2} \right\rangle, \\ \left\langle e_2, \dfrac{[0.3,0.7]}{x_1}, \dfrac{[0.2,0.8]}{x_2} \right\rangle \end{array} \right\}.$$

Remark 3.24. Let $(G,E),(H,E) \in V\tilde{S}(X,E)$.
1. $(G,E) \not{q} (G,E)^c$.
2. If (G,E) $\mathbf{q}$ (H,E), then $(G,E) \cap (H,E) \neq \hat{\emptyset}_E$.
3. If $(G,E) \cap (H,E) = \hat{\emptyset}_E$, then $(G,E) \not{q} (H,E)$.

Proposition 3.25. Let $\{(G_i,E) : i \in I\}$ be a family of vague soft sets over X.
1. $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ $\bigcup\limits_{i\in I}(G_i,E)$ iff there exists $i \in I$ such that $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (G_i,E).
2. If $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ $\bigcap\limits_{i\in I}(G_i,E)$, then $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (G_i,E) for each $i \in I$.

Proof. Straightforward. □

Proposition 3.26. Let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X,E)$, $(G,E),(H,E),(S,E) \in V\tilde{S}(X,E)$. Then we have:
1. $x \neq y \Rightarrow x_{e,[\alpha,1-\beta]} \not{q}\, y_{e',[\alpha',1-\beta']}$ for all $e,e' \in E$, $\forall \alpha,\alpha' \in I$ and $\beta,\beta' \in [0,1)$.
2. $x_{e,[\alpha,1-\beta]} \not{q}\, y_{e',[\alpha',1-\beta']} \Leftrightarrow x \neq y$ or $x = y$ with $\alpha \leq \beta'$ & $1-\beta \leq \alpha'$.
3. $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (G,E) and $(G,E) \subseteq (H,E), \Rightarrow x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (H,E).
4. (G,E) $\mathbf{q}$ $(H,E) \Leftrightarrow x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ (H,E) for some $x_{e,[\alpha,1-\beta]} \in x(\mathcal{G},\mathcal{E})$.
5. (G,E) $\mathbf{q}$ (H,E) and $(H,E) \subseteq (S,E)$, then (G,E) $\mathbf{q}$ (S,E).
6. $(G,E) \not{q} (H,E)$ and $(S,E) \subseteq (G,E)$, then $(S,E) \not{q} (H,E)$.
7. $(G,E) \not{q} (H,E)$ and $(S,E) \subseteq (H,E)$, then $(G,E) \not{q} (S,E)$.

Proof. The proof is follows from the Definitions 3.18, 3.20 and Proposition 3.22. □

Proposition 3.27. Let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$, $(G,E),(H,E) \in V\tilde{S}(X,E)$. Then we have:
1. $x_{e,[\alpha,1-\beta]}$ $\mathbf{q}$ $(G,E) \Leftrightarrow (G,E)$ $\mathbf{q}$ $x_{e,[\alpha,1-\beta]}$.
2. $x_{e,[\alpha,1-\beta]} \not{q} (G,E) \Leftrightarrow (G,E) \not{q}\, x_{e,[\alpha,1-\beta]}$.
3. (G,E) $\mathbf{q}$ $(H,E) \Leftrightarrow (H,E)$ $\mathbf{q}$ (G,E).
4. $(G,E) \not{q} (H,E) \Leftrightarrow (H,E) \not{q} (G,E)$.

Proof. The proof is follows from the Definitions 3.18, 3.20 and Proposition 3.22. □

Definition 3.28. Let g_{pu}:$V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ and $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$. Then the image of $x_{e,[\alpha,1-\beta]}$ under g_{pu}, denoted by $g_{pu}(x_{e,[\alpha,1-\beta]})$, is defined by $g_{pu}(x_{e,[\alpha,1-\beta]}) = u(x)_{p(e),[\alpha,1-\beta]}$. It is easy to see that $g_{pu}(x_{e,[\alpha,1-\beta]}) \in V\tilde{S}P_y(Y,K)$, namely $g_{pu}(x_{e,[\alpha,1-\beta]}) = y_{k,[\alpha,1-\beta]}$, where $y = u(x), k = p(e)$.

Example 3.29. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$, Y=$\{y_1, y_2\}$ K=$\{k_1, k_2, k_3\}$ and Let $x_{1_{e_2,[0.2,0.9]}} \in V\tilde{S}P_{x_1}(X,E)$. Consider vague soft function $g_{pu} : (X,\tau,E) \to (Y,\sigma,K)$ where $u : X \to Y$ and $p : E \to K$ are defined as follows:

$$u(x_1) = y_1,\ u(x_2) = y_2;\ p(e_1) = k_1,\ p(e_2) = k_3.$$

Then the image of $x_{1_{e_2,[0.2,0.9]}}$ under g_{pu} is given by $g_{pu}(x_{1_{e_2,[0.2,0.9]}}) = y_{1_{k_3,[0.2,0.9]}}$.

Theorem 3.30. *Let* g_{pu}:$V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ *be a vague soft function and let* $(F,E),(G,E) \in V\tilde{S}(X,E)$. *If* $(F,E)\not\mathbf{q}(G,E)$, *then* $g_{pu}(F,E)\not\mathbf{q}g_{pu}(G,E)$.

Proof. Let $(F,E)\not\mathbf{q}(G,E)$. Then $(F,E) \subseteq (G,E)^c$. Since $(F,E) \subseteq (G,E)^c$, we have $g_{pu}(F,E) \subseteq g_{pu}((G,E)^c) = (g_{pu}(G,E))^c$. Thus $g_{pu}(F,E) \subseteq (g_{pu}(G,E))^c \Rightarrow g_{pu}(F,E)\not\mathbf{q}g_{pu}(G,E)$. □

Theorem 3.31. *Let* g_{pu}:$V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ *be a vague soft function and let* $(G,E) \in V\tilde{S}(X,E), (H,K) \in V\tilde{S}(Y,K)$. *Then,*

(1) $g_{pu}(G,E)\not\mathbf{q}(H,K) \Rightarrow (G,E)\not\mathbf{q}\ g_{pu}^{-1}(H,K)$.

(2) $g_{pu}(G,E)\mathbf{q}(H,K) \Rightarrow (G,E)\mathbf{q}\ g_{pu}^{-1}(H,K)$.

Proof. 1. Let $g_{pu}(G,E)\not\mathbf{q}(H,K)$. Then

$$\begin{aligned} g_{pu}(G,E)\not\mathbf{q}(H,K) &\Rightarrow g_{pu}(G,E) \subseteq (H,K)^c \\ &\Rightarrow (G,E) \subseteq g_{pu}^{-1}(g_{pu}(G,E)) \subseteq g_{pu}^{-1}((H,K)^c) = g_{pu}^{-1}((H,K))^c. \\ &\Rightarrow (G,E) \subseteq g_{pu}^{-1}((H,K))^c \\ &\Rightarrow (G,E)\not\mathbf{q}\ g_{pu}^{-1}(H,K). \end{aligned}$$

2. Let $g_{pu}(G,E)\mathbf{q}(H,K)$. Suppose that $(G,E)\not\mathbf{q}\ g_{pu}^{-1}(H,K)$. Then $(G,E) \subseteq (\ g_{pu}^{-1}(H,K))^c =\ g_{pu}^{-1}((H,K)^c)$. It follows that, $g_{pu}(G,E) \subseteq (g_{pu}(g_{pu}^{-1}((H,K)^c)) \subseteq (H,K)^c$.
Thus $g_{pu}(G,E) \subseteq (H,K)^c \Rightarrow g_{pu}(G,E)\not\mathbf{q}(H,K)$, a contradiction. □

Definition 3.32. Let g_{pu}:$(X,\tau,E) \to (Y,\sigma,K)$ be a vague soft function and $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$. Then g_{pu} is said to be continuous at $x_{e,[\alpha,1-\beta]}$, if for each vague soft neighbourhood $(H,K) \in VS(Y,K)$ of $g_{pu}(x_{e,[\alpha,1-\beta]})$ there exists a vague soft neighbourhood (F,E) of $x_{e,[\alpha,1-\beta]}$ such that $g_{pu}(F,E) \subseteq (H,K)$. If g_{pu} is vague soft continuous function at all $x_{e,[\alpha,1-\beta]}$, then g_{pu} is called soft continuous function.

4. Vague soft subspace topology

Definition 4.1. Let (X,τ,E) be a vague soft topological space. A subfamily $\mathcal{B} \in \tau$ is a vague soft base for τ iff each member of τ can be expressed as the union of members of $\mathcal{B}$.

Example 4.2. Let (X,τ,E) be a vague soft discrete topological space. Then τ has a vague soft base as $\mathcal{B} = \bigcup\limits_{x\in X} V\tilde{S}P_x(X,E)$.

Example 4.3. Let $X = \{x_1, x_2\}, E = \{e_1, e_2\}$. If (X, τ, E) is a vague soft topological space with $\tau = \{\hat{\emptyset}_E, (F_1, E), (F_2, E), (F_3, E), \hat{X}_E\}$, where

$$(F_1, E) = \left\{ \left\langle e_1, \frac{[0.2, 0.9]}{x_1}, \frac{[0.1, 0.5]}{x_2} \right\rangle, \left\langle e_2, \frac{[0.4, 0.9]}{x_1}, \frac{[0.3, 0.6]}{x_2} \right\rangle \right\},$$

$$(F_2, E) = \left\{ \left\langle e_1, \frac{[1, 1]}{x_1}, \frac{[0.1, 0.5]}{x_2} \right\rangle, \left\langle e_2, \frac{[1, 1]}{x_1}, \frac{[1, 1]}{x_2} \right\rangle \right\},$$

$$(F_3, E) = \left\{ \left\langle e_1, \frac{[0.2, 0.9]}{x_1}, \frac{[1, 1]}{x_2} \right\rangle, \left\langle e_2, \frac{[0.4, 0.9]}{x_1}, \frac{[0.3, 0.6]}{x_2} \right\rangle \right\},$$

then $\mathcal{B} = \{\hat{\emptyset}_E, (F_1, E), (F_2, E), (F_3, E)\}$ is a vague soft base for τ.

Result 4.4. Let (X, τ_1, E), (X, τ_2, E) be two vague soft topological spaces over X.
1. If $\mathcal{B}$ is a vague soft basis for τ_1 and τ_2, then $\tau_1 = \tau_2$.
2. If $\mathcal{B}_1$, $\mathcal{B}_2$ are vague soft basis for τ_1 and τ_2 respectively with $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then $\tau_1 \subseteq \tau_2$.

Proposition 4.5. Let $g_{pu}:(X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function and $\mathcal{B}$ be a vague soft base for σ. Then f is vague soft continuous iff for each $(G, K) \in \mathcal{B}$ the inverse image $g_{pu}^{-1}(G, K) \in \tau$.

Proof. Necessary part is obvious.
Conversely, suppose that for each $(G, K) \in \mathcal{B}$ the inverse image $g_{pu}^{-1}(G, K) \in \tau$. Let $(H, K) \in \sigma$. Then there exist $\{(H_i, K) : i \in \Delta\} \subseteq \mathcal{B}$ such that $(H, K) = \bigcup_{i \in \Delta} (H_i, K)$ and $(H_i, K) \in \sigma$, $i \in \Delta$. Thus, $g_{pu}^{-1}(H, K) = g_{pu}^{-1}[\bigcup_{i \in \Delta} (H_i, K)] = \bigcup_{i \in \Delta} [g_{pu}^{-1}(H_i, K)] \in \tau$. Hence g_{pu} is vague soft continuous. □

Definition 4.6. Let $Y \subseteq X$. The vague characteristic set of Y on X is a vague set $\chi_Y = \{(x, [t_{\chi_Y}(x), 1 - f_{\chi_Y}(x)]) \mid x \in X\}$ over X, where
$t_{\chi_Y}(x) = \begin{cases} 1, & \text{if } x \in Y, \\ 0, & \text{otherwise,} \end{cases}$ and $1 - f_{\chi_Y}(x) = \begin{cases} 1, & \text{if } x \in Y, \\ 0, & \text{otherwise.} \end{cases}$ for all $x \in X$.

Definition 4.7. Let X be an initial universe set, E be the set of parameters and let $Y \subseteq X$. The vague soft characteristic set of Y over E is a vague soft set (F_Y, E) which is defined by $F_Y : E \to V(X)$ such that $F_Y(e) = \chi_Y$ for all $e \in E$. Thus, for all $x \in X$,
$V_{\chi_Y(e)}(x) = \begin{cases} [1, 1], & \text{if } x \in Y \\ [0, 0], & \text{otherwise,} \end{cases}$ for all $e \in E$. And it is denoted by (χ_Y, E).

Proposition 4.8. Let (X, τ, E) be a vague soft topological space. Then the collection $\mathfrak{Y} = \{(F, E) \cap (\chi_Y, E) : (F, E) \in \tau\}$ defines a vague soft topology on Y.

Proof. (1) $\hat{\emptyset}_E,\ \hat{X}_E \in \tau$ implies $\hat{\emptyset}_E,\ \hat{X}_E \cap (\chi_Y, E) = \hat{Y}_E \in \mathfrak{Y}$.
(2) Let $(G_i, E) \in \tau_Y$, $i \in \Delta$. Then for $i \in \Delta$ there exists $(F_i, E) \in \tau$ such that $(G_i, E) = (F_i, E) \cap (\chi_Y, E)$. Since τ is vague soft topology,
$\bigcup_{i \in \Delta} (G_i, E) = \bigcup_{i \in \Delta} [(F_i, E) \cap (\chi_Y, E)] = [\bigcup_{i \in \Delta} (F_i, E)] \cap (\chi_Y, E) \in \mathfrak{Y}$.

(3) Let $(H_1, E), (H_2, E) \in \mathfrak{Y}$. Then there exist $(F_1, E), (F_2, E) \in \tau$ such that $(H_1, E) = (F_1, E) \cap (\chi_Y, E)$ and $(H_2, E) = (F_2, E) \cap (\chi_Y, E)$. Since τ is vague soft topology,
$(H_1, E) \cap (H_2, E) = [(F_1, E) \cap (\chi_Y, E)] \cap [(F_2, E) \cap (\chi_Y, E)]$
$\Rightarrow (H_1, E) \cap (H_2, E) = [(F_1, E) \cap (F_2, E)] \cap (\chi_Y, E) \in \mathfrak{Y}$.

□

Definition 4.9. Let (X, τ, E) be a vague soft topological space. If $Y \subseteq X$ and (χ_Y, E) is the vague soft characteristic function of Y, then the collection $\mathfrak{Y} = \{(F, E) \cap (\chi_Y, E) : (F, E) \in \tau\}$ defines a vague soft topology on Y, called the vague soft subspace topology (or vague soft relative topology). It is denoted by τ_Y and the triplet (Y, τ_Y, E) is called a vague soft subspace topological space of (X, τ, E).

Example 4.10. Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Let us consider the vague soft topology on X as

$$\tau = \left\{ \hat{\emptyset}_E, (F, E) = \left\{ \begin{array}{c} \left\langle e_1, \frac{[0.5, 0.8]}{x_1}, \frac{[0.2, 0.5]}{x_2}, \frac{[0.4, 0.9]}{x_3}, \frac{[0.6, 0.7]}{x_4} \right\rangle, \\ \left\langle e_2, \frac{[0, 0.3]}{x_1}, \frac{[0.4, 0.5]}{x_2}, \frac{[0, 0.6]}{x_3}, \frac{[0.3, 1]}{x_4} \right\rangle \end{array} \right\}, \hat{X}_E \right\}.$$

If $Y = \{x_1, x_3\} \subset X$, then

$$\tau_Y = \left\{ \begin{array}{c} \hat{\emptyset}_E = \left\{ \left\langle e_1, \frac{[0, 0]}{x_1}, \frac{[0, 0]}{x_2}, \frac{[0, 0]}{x_3}, \frac{[0, 0]}{x_4} \right\rangle, \left\langle e_2, \frac{[0, 0]}{x_1}, \frac{[0, 0]}{x_2}, \frac{[0, 0]}{x_3}, \frac{[0, 0]}{x_4} \right\rangle \right\}, \\ (F_Y, E) = \left\{ \begin{array}{c} \left\langle e_1, \frac{[0.5, 0.8]}{x_1}, \frac{[0, 0]}{x_2}, \frac{[0.4, 0.9]}{x_3}, \frac{[0, 0]}{x_4} \right\rangle, \\ \left\langle e_2, \frac{[0, 0.3]}{x_1}, \frac{[0, 0]}{x_2}, \frac{[0, 0.6]}{x_3}, \frac{[0, 0]}{x_4} \right\rangle \end{array} \right\}, \\ \hat{Y}_E = \left\{ \left\langle e_1, \frac{[1, 1]}{x_1}, \frac{[0, 0]}{x_2}, \frac{[1, 1]}{x_3}, \frac{[0, 0]}{x_4} \right\rangle, \left\langle e_2, \frac{[1, 1]}{x_1}, \frac{[0, 0]}{x_2}, \frac{[1, 1]}{x_3}, \frac{[0, 0]}{x_4} \right\rangle \right\} \end{array} \right\}$$

is a vague soft subspace topology on Y and (Y, τ_Y, E) is called a vague soft subspace topological space of (X, τ, E).

Definition 4.11. Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E). The vague soft set (H, E) is called vague soft open in a vague soft subspace topological space (Y, τ_Y, E), if $(H, E) \in \tau_Y$.

Definition 4.12. Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E). The vague soft complement of $(H, E) \in V\tilde{S}(Y, E)$ related to vague soft subspace (Y, τ_Y, E) is denoted by $(H, E)^{\tilde{\complement}_Y} = (H^{\tilde{\complement}_Y}, E)$ and is defined by

$$V_{H^{\tilde{\complement}_Y}(e)}(x) = \begin{cases} V_{H^c(e)}(x), & \text{if } e \in E, x \in Y, \\ [0, 0], & \text{if } e \in E,\ x \notin Y. \end{cases}$$

Clearly, $(H, E)^{\tilde{\complement}_Y} \in V\tilde{S}(Y, E)$.

Definition 4.13. Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E). The vague soft set (H, E) is called vague soft open (resp. vague soft closed) in a vague soft subspace topological space (Y, τ_Y, E), if $(H, E) \in \tau_Y$ (resp. $(H, E) \in \tau_Y^c$).

Result 4.14. Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E) and $(H, E) \in V\tilde{S}(Y, E)$. Then

1. $(H, E)^{\tilde{\complement}_Y} \neq (H, E)^c$, equality hold if $Y = X$.
2. $(H, E)^{\tilde{\complement}_Y} = (H, E)^c \cap (\chi_Y, E)$.
3. $(H, E) \in \tau_Y \Leftrightarrow (H, E)^{\tilde{\complement}_Y} \in \tau_Y^c$.
4. $((H, E)^{\tilde{\complement}_Y})^{\tilde{\complement}_Y} = (H, E)$.

Theorem 4.15. *Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E) and $(H, E) \subseteq \hat{Y}_E$. If $(H, E) \in \tau$, then $(H, E) \in \tau_Y$.*

Proof. Since $\hat{X}_E \in \tau$, we have $(H, E) = (H, E) \cap \hat{X}_E \in \tau_Y$. □

Remark 4.16. A vague soft open set in a vague soft subspace (Y, τ_Y, E) need not be a vague soft open in the vague soft topological space (X, τ, E) which is given in the following example.

Example 4.17. Consider the vague soft subspace (Y, τ_Y, E) defined in the Example 4.10. Here $(F_Y, E) \in \tau_Y$ but $(F_Y, E) \notin \tau$.

Theorem 4.18. *Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E). If $(H, E) \in \tau_Y$ and $(\chi_Y, E) \in \tau$, then $(H, E) \in \tau$.*

Proof. Assume that $(\chi_Y, E) \in \tau$. Let $(H, E) \in \tau_Y$ be any vague soft open set in (Y, τ_Y, E). Then there exists $(F, E) \in \tau$ is such that $(H, E) = (F, E) \cap (\chi_Y, E)$. That is $(H, E) = (F, E) \cap (\chi_Y, E) \in \tau$. Hence, $\tau_Y \subseteq \tau$. □

Definition 4.19. A vague soft subspace (Y, τ_Y, E) of (X, τ, E) is said to be vague soft open (resp., vague soft closed) if $(\chi_Y, E) \in \tau$ (resp., $(\chi_Y, E) \in \tau^c$)).

Proposition 4.20. Let $(Y_1, \tau_{Y_1}, E), (Y_2, \sigma_{Y_2}, K)$ be vague soft subspace topological spaces of (X_1, τ, E), (X_2, σ, K) respectively. Let $\mathcal{B}'$ be a base for σ_{Y_2}. Then g_{pu}:$(Y_1, \tau_{Y_1}, E) \to (Y_2, \sigma_{Y_2}, K)$ is vague soft continuous iff for each $(H, K) \in \mathcal{B}'$ the intersection $g_{pu}^{-1}(H, K) \cap (\chi_{Y_1}, E) \in \tau_{Y_1}$.

Proof. The proof is similar to that of Proposition 4.5. □

Theorem 4.21. *Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E). Then a vague soft set (H, E) is vague soft closed in (Y, τ_Y, E) iff it equals the intersection of vague soft closed set of (X, τ, E) with (χ_Y, E).*

Proof. Assume that (H, E) is vague soft closed in (Y, τ_Y, E). Then $(H, E)^{\tilde{\complement}_Y}$ is vague soft open in (Y, τ_Y, E). By the definition of vague soft subspace topology, there exists $(F, E) \in \tau$ such that $(H, E)^{\tilde{\complement}_Y} = (F, E) \cap (\chi_Y, E)$. Since $(F, E)^c \in \tau^c$ and $(H, E) = (F, E)^c \cap (\chi_Y, E)$, (H, E) equals the intersection of vague soft closed set of (X, τ, E) with (χ_Y, E).

Conversely, assume that $(H, E) \in V\tilde{S}(Y, E)$ with $(H, E) = (G, E) \cap (\chi_Y, E)$, where $(G, E) \in \tau^c$. Then $(G, E)^c \in \tau$. By the definition of vague soft subspace topology $(G, E)^c \cap (\chi_Y, E) \in \tau_Y$. But $(H, E)^{\tilde{\complement}_Y} = (G, E)^c \cap (\chi_Y, E)$. Hence $(H, E)^{\tilde{\complement}_Y} \in \tau_Y$, so that (H, E) is vague soft closed in (Y, τ_Y, E). □

Theorem 4.22. *Let (Y, τ_Y, E) be a vague soft subspace topological space of (X, τ, E). Let (S, E) be a vague soft set in Y. Then the vague soft closure of (S, E) in Y equals $v\tilde{s}cl(S, E) \cap (\chi_Y, E)$ where $v\tilde{s}cl(S, E)$ denotes the vague soft closure of (S, E) in X.*

Proof. Let (R, E) denotes the vague soft closure of (S, E) in (Y, τ_Y, E). The vague soft set $v\tilde{s}cl(S, E) \in \tau^c$, and so $v\tilde{s}cl(S, E) \cap (\chi_Y, E) \in \tau_Y^c$ (by Theorem 4.21). Since $v\tilde{s}cl(S, E) \cap (\chi_Y, E) \in \tau_Y^c$ contains (S, E) and since by definition of (R, E) equals the vague soft intersection of all vague soft closed sets of (Y, τ_Y, E) containing (S, E) we have $(R, E) \subseteq (v\tilde{s}cl(S, E) \cap (\chi_Y, E))$.

On the other hand, we know that $(R, E) \in \tau_Y^c$. Hence by Theorem 4.21 we have, $(R, E) = (G, E) \cap (\chi_Y, E)$ for some $(G, E) \in \tau^c$. Then (G, E) is a vague soft closed set containing (S, E); because $v\tilde{s}cl(S, E)$ is the vague soft intersection of such vague soft closed sets, we have $v\tilde{s}cl(S, E) \subseteq (G, E)$. Thus, $v\tilde{s}cl(S, E) \cap (\chi_Y, E) \subseteq (G, E) \cap (\chi_Y, E) = (R, E)$ and then $(R, E) = v\tilde{s}cl(S, E) \cap (\chi_Y, E)$. Hence the vague soft closure of (S, E) in Y equals $v\tilde{s}cl(S, E) \cap (\chi_Y, E)$. □

5. Vague soft $\mathfrak{T}_i (i = 0, 1, 2, 3, 4)$-spaces

Definition 5.1. A vague soft topological space (X, τ, E) is said to be

(1) Vague soft $\mathfrak{T}_0$ (in short, $V\tilde{S}\mathfrak{T}_0$) iff for every $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X, E)$, with $x_{e,[\alpha,1-\beta]} \not q\, y_{e',[\alpha',1-\beta']}$ there exist $(F, E), (G, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F, E)$ and $y_{e',[\alpha',1-\beta']} \not q\, (F, E)$ or $y_{e',[\alpha',1-\beta']} \subseteq (G, E)$ and $x_{e,[\alpha,1-\beta]} \not q\, (G, E)$.

(2) Vague soft $\mathfrak{T}_1$ (in short, $V\tilde{S}\mathfrak{T}_1$) iff for every $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X, E)$, with $x_{e,[\alpha,1-\beta]} \not q\, y_{e',[\alpha',1-\beta']}$ there exist $(F, E), (G, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F, E)$, $y_{e',[\alpha',1-\beta']} \not q\, (F, E)$ and $y_{e',[\alpha',1-\beta']} \subseteq (G, E)$, $x_{e,[\alpha,1-\beta]} \not q\, (G, E)$.

(3) Vague soft $\mathfrak{T}_2$ (in short, $V\tilde{S}\mathfrak{T}_2$) iff for every $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X, E)$, with $x_{e,[\alpha,1-\beta]} \not q\, y_{e',[\alpha',1-\beta']}$ there exist $(F, E), (G, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F, E)$, $y_{e',[\alpha',1-\beta']} \subseteq (G, E)$ and $(F, E) \not q\, (G, E)$.

Theorem 5.2. *If every x-vague soft point $x_{e,[\alpha,1-\beta]}$ is closed, then (X, τ, E) is a $V\tilde{S}\mathfrak{T}_1$-space.*

Proof. Assume that every x-vague soft point in $V\tilde{S}P(X, E)$ is closed. Now let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X, E)$, with $x_{e,[\alpha,1-\beta]} \not q\, y_{e',[\alpha',1-\beta']}$, then $x_{e,[\alpha,1-\beta]} \subseteq (y_{e',[\alpha',1-\beta']})^c \in \tau$ and $y_{e',[\alpha',1-\beta']} \subseteq (x_{e,[\alpha,1-\beta]})^c \in \tau$. Clearly, there exist vague soft open sets $(F, E) = (y_{e',[\alpha',1-\beta']})^c$ and $(G, E) = (x_{e,[\alpha,1-\beta]})^c$ such that, $y_{e',[\alpha',1-\beta']} \not q\, (F, E)$ and $x_{e,[\alpha,1-\beta]} \not q\, (G, E)$. Hence (X, τ, E) is a $V\tilde{S}\mathfrak{T}_1$-space. □

Remark 5.3. The converse of the above Theorem 5.2 need not be true as seen from the following Example.

Example 5.4. Let $X = \{x_1, x_2\}, E = \{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_i, E), \hat{X}_E : i \in I\}$, where

$$(F_i, E) = \left\{ \left\langle e_1, \frac{[\gamma, 1-\delta]}{x_1}, \frac{[\mu, 1-\nu]}{x_2} \right\rangle, \left\langle e_2, \frac{[\gamma, 1-\delta]}{x_1}, \frac{[\mu, 1-\nu]}{x_2} \right\rangle \right\}$$

Then the vague soft topological space (X, τ, E) is a $V\tilde{S}\mathfrak{T}_1$-space. Now for the x-vague soft point $x1_{e_1,[1,1]}$ we note that

$$x1_{e_1,[1,1]} \neq v\tilde{s}cl(x1_{e_1,[1,1]}) = \left\{ \left\langle e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} \right\rangle, \left\langle e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} \right\rangle \right\}.$$

Hence the x-vague soft point $x1_{e_1,[1,1]}$ is not a vague soft closed set.

Proposition 5.5. Every $V\tilde{S}\mathfrak{T}_i$-space is $V\tilde{S}\mathfrak{T}_{i-1}$-space for $i = 1, 2$.

Proof. The proof follows from the Definition 5.1, part 6, 7 of Proposition 3.26 and part 2 of Proposition 3.27. $\square$

Definition 5.6. A property of vague soft topological space (X, τ, E) is said to be **vague soft hereditary** if the property is possessed by every vague soft topological subspace of it.

Theorem 5.7. *A vague soft subspace (Y, τ_Y, E) of a $V\tilde{S}\mathfrak{T}_i$-space (X, τ, E) is $V\tilde{S}\mathfrak{T}_i$, $i = 0, 1, 2$.*

Proof. We merely prove the Theorem when $i = 2$. Let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(Y, E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(Y, E)$ with $x_{e,[\alpha,1-\beta]} \not{q}\, y_{e',[\alpha',1-\beta']}$. Then $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ and $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X, E)$ with $x_{e,[\alpha,1-\beta]} \not{q}\, y_{e',[\alpha',1-\beta']}$ implies there exist $(F, E), (G, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F, E)$, $y_{e',[\alpha',1-\beta']} \subseteq (G, E)$ and $(F, E) \not{q}\, (G, E)$. Thus $(F, E) \cap (\chi_Y, E) \in \tau_Y$ and $(G, E) \cap (\chi_Y, E) \in \tau_Y$. Now, let us take $(F, E) \cap (\chi_Y, E) = (F_Y, E)$ and $(G, E) \cap (\chi_Y, E) = (G_Y, E)$. Then $x_{e,[\alpha,1-\beta]} \subseteq (F_Y, E)$, $y_{e',[\alpha',1-\beta']} \subseteq (G_Y, E)$ and $(F_Y, E) \not{q}\, (G_Y, E)$. Hence the proof. $\square$

Definition 5.8. Let (X, τ, E) and (Y, σ, K) be any vague soft topological spaces. If the bijective function $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ and its inverse function are vague soft continuous functions, then g_{pu} is said to be a vague soft homeomorphism. Moreover, (X, τ, E) and (Y, σ, K) are said to be vague soft homeomorphic spaces.

Definition 5.9. A property of vague soft topological space (X, τ, E) is said to be **Vague Soft Topological Property** if it is preserved under vague soft homeomorphism.

Theorem 5.10. *The property of being $V\tilde{S}\mathfrak{T}_i$-space ($i = 0, 1, 2$) is a vague soft topological property.*

Proof. We prove the Theorem for $i = 2$, the other cases are similar.
Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft homeomorphism and (X, τ, E) be a $V\tilde{S}\mathfrak{T}_2$-space. We have to prove (Y, σ, K) is a $V\tilde{S}\mathfrak{T}_2$-space. So, let $y_{k,[\alpha,1-\beta]} \in V\tilde{S}P_y(Y, K)$, $y'_{k',[\alpha',1-\beta']} \in V\tilde{S}P_{y'}(Y, K)$ with $y_{k,[\alpha,1-\beta]} \not{q}\ y'_{k',[\alpha',1-\beta']}$. Since g_{pu} is vague soft bijective mapping, there exist $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$, $x'_{e',[\alpha',1-\beta']} \in V\tilde{S}P_{x'}(X, E)$ such that $g_{pu}(x_{e,[\alpha,1-\beta]}) = y_{k,[\alpha,1-\beta]}$, $g_{pu}(x'_{e',[\alpha,1-\beta]}) = y'_{k',[\alpha,1-\beta]}$ and $x_{e,[\alpha,1-\beta]} \not{q}\, x'_{e',[\alpha',1-\beta']}$. Since (X, τ, E) is a $V\tilde{S}\mathfrak{T}_2$-space, there exist $(F, E), (G, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F, E)$, $x'_{e',[\alpha',1-\beta']} \subseteq (G, E)$ and $(F, E) \not{q}\, (G, E)$. Thus, $g_{pu}(x_{e,[\alpha,1-\beta]}) = y_{k,[\alpha,1-\beta]} \subseteq g_{pu}(F, E), g_{pu}(x'_{e',[\alpha,1-\beta]}) = y'_{k',[\alpha,1-\beta]} \subseteq g_{pu}(G, E)$ and $g_{pu}(F, E) \not{q}\, g_{pu}(G, E)$. Since $(F, E), (G, E) \in \tau$ and g_{pu} is vague soft open mapping (being a vague soft homeomorphism), we have $g_{pu}(F, E), g_{pu}(G, E) \in \sigma$. Hence, (Y, σ, K) is a $V\tilde{S}\mathfrak{T}_2$-space. $\square$

Definition 5.11. Suppose that x-vague soft points $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$ are closed in (X,τ,E). Then (X,τ,E) is said to be a

(1) Vague soft $\mathfrak{p}$-regular-space if for each $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$ and a vague soft closed set (G,E) in X with $x_{e,[\alpha,1-\beta]} \not q (G,E)$, there exist vague soft open sets (F_1,E) and (F_2,E) such that $x_{e,[\alpha,1-\beta]} \subseteq (F_1,E)$, $(G,E) \subseteq (F_2,E)$ and $(F_1,E) \not q (F_2,E)$.

(2) Vague soft $\mathfrak{p}$-normal-space if for every vague soft closed sets (G_1,E) and (G_2,E) in X with $(G_1,E) \not q (G_2,E)$, there exist vague soft open sets (F_1,E) and (F_2,E) such that $(G_1,E) \subseteq (F_1,E)$, $(G_2,E) \subseteq (F_2,E)$ and $(F_1,E) \not q (F_2,E)$.

(3) Vague soft $\mathfrak{T}_3$ (in short, $V\tilde{S}\mathfrak{T}_3$)-space if it is vague soft $\mathfrak{p}$-regular and $V\tilde{S}\mathfrak{T}_1$-space.

(4) Vague soft $\mathfrak{T}_4$-(in short, $V\tilde{S}\mathfrak{T}_4$)space if it is vague soft $\mathfrak{p}$-normal and $V\tilde{S}\mathfrak{T}_1$-space.

Theorem 5.12. *Let (X,τ,E) be a vague soft topological space, then (X,τ,E) is a vague soft $\mathfrak{p}$-regular-space iff for each x-vague soft point $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$ and $(G,E) \in \mathcal{N}_\tau(x_{e,[\alpha,1-\beta]})$ with $(G,E) \in \tau$ there exists $(F,E) \in \mathcal{N}_\tau(x_{e,[\alpha,1-\beta]})$ such that $v\tilde{s}cl(F,E) \subseteq (G,E)$.*

Proof. Let (X,τ,E) be a vague soft $\mathfrak{p}$-regular-space. Now, let us consider, $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$ and $(G,E) \in \mathcal{N}_\tau(x_{e,[\alpha,1-\beta]})$ with $(G,E) \in \tau$. Then $(G,E)^c \in \tau^c$. Since $(G,E) \not q (G,E)^c$, we have $(x_{e,[\alpha,1-\beta]}) \not q (G,E)^c$. By hypothesis, there exist $(F,E),(F_1,E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F,E)$, $(G,E)^c \subseteq (F_1,E)$ and $(F,E) \not q (F_1,E)$. Thus, $(F,E) \subseteq (F_1,E)^c \Rightarrow v\tilde{s}cl(F,E) \subseteq v\tilde{s}cl((F_1,E)^c) = (F_1,E)^c$. But $(F_1,E)^c \subseteq (G,E)$. Hence $v\tilde{s}cl(F,E) \subseteq (G,E)$.

Conversely, let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$ and $(H,E) \in \tau^c$ with $x_{e,[\alpha,1-\beta]} \not q (H,E)$. Then $x_{e,[\alpha,1-\beta]} \subseteq (H,E)^c \in \tau$. By hypothesis, there exists $(G_1,E) \in \mathcal{N}_\tau(x_{e,[\alpha,1-\beta]})$ such that $v\tilde{s}cl(G_1,E) \subseteq (H,E)^c$. Thus $(H,E) \subseteq (v\tilde{s}cl(G_1,E))^c = (G_2,E)$ (say). Since $v\tilde{s}cl(G_1,E) \not q (v\tilde{s}cl(G_1,E))^c = (G_2,E) \in \tau$ we have $v\tilde{s}cl(G_1,E) \not q (G_2,E)$. Then $(G_1,E) \not q (G_2,E)$. Hence (X,τ,E) is a vague soft $\mathfrak{p}$-regular-space. □

Theorem 5.13. *Let (X,τ,E) be a vague soft topological space, then (X,τ,E) is a vague soft $\mathfrak{p}$-normal-space iff for every $(H,E) \in \tau^c$ and $(G,E) \in \mathcal{N}_\tau(H,E)$ with $(G,E) \in \tau$ there exists $(F,E) \in \mathcal{N}_\tau(H,E)$ such that $v\tilde{s}cl(F,E) \subseteq (G,E)$.*

Proof. It is analogous to that of Theorem 5.12. □

Proposition 5.14. 1. Every vague soft $\mathfrak{p}$-normal $(VS\mathfrak{T}_4)$-space is vague soft $\mathfrak{p}$-regular $(VS\mathfrak{T}_3)$.

2. Every vague soft $\mathfrak{p}$-regular $(V\tilde{S}\mathfrak{T}_3)$ -space is vague soft $\mathfrak{T}_2$.

Proof. 1. Let (X,τ,E) be a vague soft $\mathfrak{p}$-normal-space. Let $x_{e,[\alpha,1-\beta]} \subseteq V\tilde{S}P_x(X,E)$ and $(F,E) \in \tau^c$ with $x_{e,[\alpha,1-\beta]} \not q (F,E)$. Since $x_{e,[\alpha,1-\beta]}$, $(F,E) \in \tau^c$, there exist $(F_1,E),(F_2,E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \in (F_1,E)$, $(F,E) \subseteq (F_2,E)$ and $(F_1,E) \not q (F_2,E)$. Hence (X,τ,E) is a vague soft $\mathfrak{p}$-regular space.

2. Let (X,τ,E) be a vague soft $\mathfrak{p}$-regular-space and let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X,E)$, $y_{e',[\alpha',1-\beta']} \in V\tilde{S}P_y(X,E)$, with $x_{e,[\alpha,1-\beta]} \not q y_{e',[\alpha',1-\beta']}$. Since $y_{e',[\alpha',1-\beta']} \in \tau^c$, and by our hypothesis, there exist $(G_1,E),(G_2,E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (G_1,E)$, $y_{e',[\alpha',1-\beta']} \subseteq (G_2,E)$ and $(G_1,E) \not q (G_2,E)$. Hence (X,τ,E) is vague soft $\mathfrak{T}_2$. □

Theorem 5.15. *A vague soft subspace (Y, τ_Y, E) of a vague soft $\mathfrak{p}$-regular $(V\tilde{S}\mathfrak{T}_3)$-space (X, τ, E) is vague soft $\mathfrak{p}$-regular $(V\tilde{S}\mathfrak{T}_3)$.*

Proof. Let (Y, τ_Y, E) be a vague soft subspace of vague $\mathfrak{p}$-regular-space (X, τ, E). Then x-vague soft points $x_{e,[\alpha,1-\beta]}$ of $V\tilde{S}(Y, K)$ are closed in (Y, τ_Y, E) and hence (Y, τ_Y, E) is Vague soft $\mathfrak{T}_1$ (by Theorem 5.2). Let $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(Y, E)$ and $(G, E) \in \tau_Y^c$ with $x_{e,[\alpha,1-\beta]} \not{q} (G, E)$. Since $(G, E) \in \tau_Y^c$, take $v\tilde{s}cl(G, E) \cap (\chi_Y, E) = (G, E)$ where $v\tilde{s}cl(G, E)$ denotes the vague soft closure of (G, E) in (X, τ, E). Then $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ and $v\tilde{s}cl(G, E) \in \tau^c$ with $x_{e,[\alpha,1-\beta]} \not{q} v\tilde{s}cl(G, E)$. By hypothesis, there exist $(F_1, E), (F_2, E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F_1, E)$, $v\tilde{s}cl(G, E) \subseteq (F_2, E)$ and $(F_1, E) \not{q} (F_2, E)$. Thus $(F_1, E) \cap (\chi_Y, E) \in \tau_Y$ and $(F_2, E) \cap (\chi_Y, E) \in \tau_Y$. Now, let us take $(F_1, E) \cap (\chi_Y, E) = (F_{1_Y}, E)$ and $(F_2, E) \cap (\chi_Y, E) = (F_{2_Y}, E)$. Then $x_{e,[\alpha,1-\beta]} = x_{e,[\alpha,1-\beta]} \cap (\chi_Y, E) \subseteq (F_1, E) \cap (\chi_Y, E) = (F_{1_Y}, E)$, $(G, E) \subseteq v\tilde{s}cl(G, E) \cap (\chi_Y, E) \subseteq (F_2, E) \cap (\chi_Y, E) = (F_{2_Y}, E)$ and $(F_{1_Y}, E) \not{q} (F_{2_Y}, E)$. Hence (Y, τ_Y, E) is vague soft $\mathfrak{p}$-regular. □

Theorem 5.16. *Every vague soft closed subspace of a vague soft $\mathfrak{p}$-normal $(V\tilde{S}\mathfrak{T}_4)$-space is vague soft $\mathfrak{p}$-normal$(V\tilde{S}\mathfrak{T}_4)$.*

Proof. Let (Y, τ_Y, E) be a vague soft closed subspace of vague soft $\mathfrak{p}$-normal-space (X, τ, E). Then $(\chi_Y, E) \in \tau^c$ and x-vague soft points $x_{e,[\alpha,1-\beta]}$ of $V\tilde{S}(Y, K)$ are closed in (Y, τ_Y, E). Let $(G_1, E)^*, (G_2, E)^*$ be two vague soft closed sets in Y with $(G_1, E)^* \not{q} (G_2, E)^*$. Then there exist $(G_1, E), (G_2, E) \in \tau^c$ such that $(G_1, E)^* = (G_1, E) \cap (\chi_Y, E)$, $(G_2, E)^* = (G_2, E) \cap (\chi_Y, E)$ by Theorem 4.21. Since $(\chi_Y, E) \in \tau^c$ and $(G_1, E), (G_2, E) \in \tau^c$, we have $(G_1, E) \cap (\chi_Y, E)$, $(G_2, E) \cap (\chi_Y, E) \in \tau^c$. Thus, $(G_1, E)^*, (G_2, E)^* \in \tau^c$. Now, $(G_1, E)^*, (G_2, E)^*$ are two vague soft closed sets in X with $(G_1, E)^* \not{q} (G_2, E)^*$, but (X, τ, E) is a vague soft $\mathfrak{p}$-normal space, there exist $(F_1, E), (F_2, E) \in \tau$ such that $(G_1, E)^* \subseteq (F_1, E)$, $(G_2, E)^* \subseteq (F_2, E)$ and $(F_1, E) \not{q} (F_2, E)$. Thus, there exist $(F_1, E) \cap (\chi_Y, E) = (F_1, E)^* \in \tau_Y^c$ and $(F_2, E) \cap (\chi_Y, E) = (F_2, E)^* \in \tau_Y^c$ such that $(G_1, E)^* = (G_1, E)^* \cap (\chi_Y, E) \subseteq (F_1, E) \cap (\chi_Y, E) = (F_1, E)^*$, $(G_2, E)^* = (G_2, E)^* \cap (\chi_Y, E) \subseteq (F_2, E) \cap (\chi_Y, E) = (F_2, E)^*$ and $(F_1, E)^* \not{q} (F_2, E)^*$. Hence (Y, τ_Y, E) is vague soft $\mathfrak{p}$-normal. □

Theorem 5.17. *The property of being $V\tilde{S}\mathfrak{T}_i$-space $(i = 3, 4)$ is a vague soft topological property.*

Proof. We prove the Theorem for $i = 3$, the other cases are similar.
Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft homeomorphism and (X, τ, E) be a $V\tilde{S}\mathfrak{T}_3$-space. We have to prove (Y, σ, K) is a $V\tilde{S}\mathfrak{T}_3$-space. Since the property of being vague soft $\mathfrak{T}_1$-space is a vague soft topological property and every y-vague soft points in $V\tilde{S}P_y(Y, K)$ are closed in (Y, σ, K) (since g_{pu} is vague soft homeomorphism), it is enough to prove that the property of vague soft $\mathfrak{p}$-regularity is a vague soft topological property.

Let $(G, K) \in \sigma^c$, $y_{k,[\alpha,1-\beta]} \in V\tilde{S}P_y(Y, E)$ with $y_{k,[\alpha,1-\beta]} \not{q} (G, K)$. Since g_{pu} is vague soft bijective continuous mapping, there exist $x_{e,[\alpha,1-\beta]} \in V\tilde{S}P_x(X, E)$ such that

$g_{pu}(x_{e,[\alpha,1-\beta]}) = y_{k,[\alpha,1-\beta]}$, $g_{pu}^{-1}(G,K) \in \tau^c$ and $x_{e,[\alpha,1-\beta]} \not{q}\, g_{pu}^{-1}(G,K)$. Since (X,τ,E) is a $V\tilde{S}\mathfrak{T}_3$-space, there exist $(F_1,E),(F_2,E) \in \tau$ such that $x_{e,[\alpha,1-\beta]} \subseteq (F_1,E), g_{pu}^{-1}(G,K) \subseteq (F_2,E)$ and $(F_1,E) \not{q} (F_2,E)$. Thus, $g_{pu}(x_{e,[\alpha,1-\beta]}) = y_{k,[\alpha,1-\beta]} \subseteq g_{pu}(F_1,E), g_{pu}(g_{pu}^{-1}(G,K)) = (G,K) \subseteq g_{pu}(F_2,E)$ and $g_{pu}(F_1,E) \not{q}\, g_{pu}(F_2,E)$. Since $(F_1,E),(F_2,E) \in \tau$ and g_{pu} is vague soft open mapping (being a vague soft homeomorphism), we have $g_{pu}(F_1,E), g_{pu}(F_2,E) \in \sigma$. Hence, (Y,σ,K) is a $V\tilde{S}\mathfrak{T}_3$-space. □

Conclusion

It is observed that the notion of vague soft characteristic function played an important role in defining the vague soft subspace topology. The introduction of vague soft quasi-coincident had significantly influenced the properties of vague soft separation axioms that were obtained in our previous study [25]. Also, we discussed the vague soft hereditary and topological properties of vague soft $\mathfrak{T}_i (i = 0,1,2,3,4)$-spaces. In future, the authors intend to develop an algorithm to identify the portions of affected organs by cancer cells using the above defined vague soft concepts.

Compliance with ethical standards:

Conflict of Interest: The authors declare that they do not have any conflict of Interest.
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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References

[1] A. Al-Quran, N. Hassan, Nautrosophic vague soft sets and its applications, *Malaysian Journal of Mathematical Sciences*, 11, (2017) 141–163.

[2] K. Alhazaymeh, N. Hassan, Possibility vague soft sets and its application in decision making, *International Journal of Pure and Applied Mathematics*, 77, (2012) 549–563.

[3] K. Alhazaymeh, N. Hassan, Generalized interval-valued vague soft set, *Applied Mathematical Sciences*, 7, (2013) 6983–6988.

[4] K. Alhazaymeh, N. Hassan, Generalized vague soft expert set, *International Journal of Pure and Applied Mathematics*, 93, (2014) 351–360.

[5] K. Alhazaymeh, N. Hassan, Vague soft expert sets and its application in decision making, *Malaysian Journal of Mathematical Sciences*, 11, (2017) 23–39.

[6] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, (1986) 87–96.

[7] Y. Bhargavi, T. Eswarla, Application of vague set in medical diagnosis,, *International Journal of Chemical Sciences*, 14, (2016) 842–846.

[8] N. Cagman, S. Karatas, Intuitionistic fuzzy soft set theory and its decision making, *Journal of Intelligent & Fuzzy Systems*, 24, (2013) 829–836.

[9] W. Gau, D. Buehrer, Vague sets, *IEEE Transactions on Systems Man and Cybernetics*, 23, (1993) 610–614.
[10] D. Georgiou, A. Megaritis, V. Petropoulos, On soft topological spaces, *Applied Mathematics and Information Sciences*, 7, (2013) 1889–1901.
[11] S. Hussain, B. Ahmad, Some properties of soft topological spaces, *Computers and Mathematics with Applications*, 62, (2011) 4058–4067.
[12] V. Inthumathi, M. Pavithra, On vague α-soft continuous functions, *Malaya Journal of Matematik (MJM)*, 8, (2020) 135–143.
[13] A. Kharal, B. Ahmad, Mappings on fuzzy soft classes, *Advances in Fuzzy Systems*, 2009, (2009) 1–6.
[14] P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft set theory, *Journal of Fuzzy Mathematics*, 9, (2001) 589–602.
[15] P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Computers and Mathematics with Applications*, 45, (2003) 555–562.
[16] D. Meng, X. Zhang, K. Qin, Soft rough fuzzy sets and soft fuzzy rough sets, *Computers & mathematics with applications*, 62, (2011) 4635–4645.
[17] W. K. Min, A note on soft topological spaces, *Computers and Mathematics with Applications*, 62, (2011) 3524–3528.
[18] D. Molodtsov, Soft set theory - first results, *Computers and Mathematics with Applications*, 37, (1999) 19–31.
[19] Z. Pawlak, Rough sets, *International journal of computer & information sciences*, 11, (1982) 341–356.
[20] G. Senel, N. Cagman, Soft topological subspaces, *Annals of Fuzzy Mathematics and Informatics*, 10, (2015) 525–535.
[21] M. Shabir, M. Naz, On soft topological spaces, *Computers & Mathematics with Applications*, 61, (2011) 1786–1799.
[22] O. Tantawy, S. El-Sheikh, S. Hamde, Separation axioms on soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, 11, (2016) 511–525.
[23] B. P. Varol, A. Shostak, H. Aygun, A new approach to soft topology, *Hacettepe Journal of Mathematics and Statistics*, 41, (2012) 731–741.
[24] C. Wang, Vague parameterized vague soft set theory and its decision making, *Journal of Intelligent and Fuzzy Systems*, 33, (2017) 2341–2350.
[25] C. Wang, V. Inthumathi, M. Pavithra, Vague soft separation axioms and vague soft continuous functions in vague soft topological spaces, Accepted in Italian Journal of Pure and Applied Mathematics, Vol. 45.
[26] C. Wang, Y. Li, Topological structure of vague soft sets, *Abstract and Applied Analysis*, Volume 2014, (2014) 8 pages.
[27] C. Wang, A. Qu, The applications of vague soft sets and generalized vague soft sets, *Acta Mathematicae Applicatae Sinica*, 31, (2015) 977–990.
[28] N. Xie, Soft points and the structure of soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, 10, (2015) 309–322.
[29] W. Xu, J. Ma, S. Wang, G. Hao, Vague soft set and their properties, *Computers and Mathematics with Application*, 59, (2010) 787–794.
[30] L. Zadeh, Fuzzy sets, *Information and Control*, 8, (1965) 338–353.
[31] I. Zorlutuna, M. Akdag, W. K. Min, S. Atmaca, Remarks on soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, 3, (2012) 171–185.

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