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ON WEAKLY ωe^*-CONTINUOUS FUNCTIONS

PINAR ŞAŞMAZ AND MURAD ÖZKOÇ†

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***Abstract*.** The aim of this paper is to introduce and study a new type of weakly continuity called weakly ωe^*-continuity via ωe^*-open sets defined by us. We obtain several characterizations and some fundamental properties of weakly ωe^*-continuous functions. The notion of weakly ωe^*-continuity come across as a notion of weaker than the notion of weakly $\omega\beta$-continuity. Also, we investigate the relationships between weakly ωe^*-continuous functions and other related strong and weak forms of weakly continuity. Furthermore, we introduce the concepts of ωe^*-strongly closed graphs to look into some different properties.

1. Introduction

One of the most studied generalizations of continuity is weak continuity [13] which is firstly defined by Levine in 1961. He introduced the concept of weakly continuity [13] and investigate its some fundamental properties. And then many mathematicians have studied the various types of weakly continuity. For instance, Al-Zoubi and Al-Jarah defined the concept of weakly ω-continuity [5] which is weaker than weakly continuity. Özkoç and Aslım introduced and investigated the concept of weakly e-continuity. Also, Ayhan and Özkoç introduced the concepts of weakly e^*-continuity [8] and weakly a-continuity [8]. Some properties and characterizations of the notion of weakly $\omega\beta$-continuity [4] have been defined and studied by Aljarrah and Noorani.

In this paper, we study and investigate the notion of weakly ωe^*-continuous function [15]. Also, we obtain some fundamental properties and its characterizations. Furthermore, we investigate some properties between graphs and weakly ωe^*-continuity and separation axioms.

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† *Corresponding author.*

2. Preliminaries

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open, clopen) sets of X is denoted $C(X)$ (resp. $O(X)$, $CO(X)$) and the family of all closed (resp. open) sets of X containing a point x of X is denoted by $C(X,x)$ (resp. $O(X,x)$). The co-countable topology on X, τ_{coc}; the topology whose open sets are the empty set and complements of subsets of X which are at most countable and the indiscrete topology on X, τ_{ind}; the usual topology on $\mathbb{R}$, τ_u. We recall the following definitions which will be used throughout this paper.

Definition 2.1. A subset A of a space X is called:

(1) regular open [18] if $A = int(cl(A))$. The complement of a regular open set is called regular closed. A point $x \in X$ is said to be the δ-cluster point [19] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. If $A = \delta$-$cl(A)$, then A is called δ-closed [19], and the complement of a δ-closed set is called δ-open. The set $\{x|(\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by δ-$int(A)$.

(2) β-open [1] if $A \subseteq cl(int(cl(A)))$. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by β-$cl(A)$. The union of all β-open sets of X contained in A is called the β-interior of A and is denoted by β-$int(A)$.

(3) a-open [9] if $A \subseteq int(cl(\delta$-$int(A))$. The complement of an a-open set is called a-closed [9]. The intersection of all a-closed sets containing A is called the a-closure [9] of A and is denoted by a-$cl(A)$. The union of all a-open sets of X contained in A is called the a-interior [9] of A and is denoted by a-$int(A)$.

(4) e^*-open [11] if $A \subseteq cl(int(\delta$-$cl(A))$. The complement of an e^*-open set is called e^*-closed [11]. The intersection of all e^*-closed sets containing A is called the e^*-closure [11] of A and is denoted by e^*-$cl(A)$. The union of all e^*-open sets of X contained in A is called the e^*-interior [11] of A and is denoted by e^*-$int(A)$.

(5) pre-open [14] if $A \subseteq int(cl(A))$. The complement of an pre-open set is called pre-closed [14]. The intersection of all pre-closed sets containing A is called the pre-closure [14] of A and is denoted by $pcl(A)$. The union of all pre-open sets of X contained in A is called the pre-interior [14] of A and is denoted by $pint(A)$.

(6) A subset A of a space X is said to be ω-open [6] (resp. $\omega\beta$-open [3]) if for every $x \in A$ there exists an open (resp. β-open) set U containing x such that $U \setminus A$ is countable. The complement of an ω-open set (resp. $w\beta$-open set) is said to be ω-closed (resp. $\omega\beta$-closed).

The family of all regular open (resp. regular closed, β-open, β-closed, a-open, a-closed, e^*-open, e^*-closed, pre-open, pre-closed, ω-open, ω-closed, $\omega\beta$-open, $\omega\beta$-closed) subsets of X is denoted by $RO(X)$ (resp. $RC(X)$, $\beta O(X)$, $\beta C(X)$, $aO(X)$, $aC(X)$, $e^*O(X)$, $e^*C(X)$, $PO(X)$, $PC(X)$, $\omega O(X)$, $\omega C(X)$, $\omega\beta O(X)$, $\omega\beta C(X)$). The family of all regular open (resp. regular closed, β-open, β-closed, a-open, a-closed, e^*-open, e^*-closed, pre-open, pre-closed, ω-open, ω-closed, $\omega\beta$-open, $\omega\beta$-closed) sets of X containing a point x of X is denoted by $RO(X,x)$ (resp. $RC(X,x)$, $\beta O(X,x)$, $\beta C(X,x)$, $aO(X,x)$,

$aC(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $PO(X,x)$, $PC(X,x)$, $\omega O(X,x)$, $\omega C(X,x)$, $\omega\beta O(X,x)$, $\omega\beta C(X,x)$).

Definition 2.2. Let A be a subset of a space X. A is said to be ωe^*-open [15] (resp. ωa-open [15]) if for every $x \in A$, there exists an e^*-open (resp. a-open) set U containing x such that $U \setminus A$ is countable. The complement of an ωe^*-open (resp. ωa-open) set is called an ωe^*-closed (resp. ωa-closed). The family of all ωe^*-open (resp. ωa-open) sets of X will be denoted by $\omega e^*O(X)$ (resp. $\omega aO(X)$). The family of all ωe^*-open (resp. ωa-open) sets of X containing a point x of X will be denoted by $\omega e^*O(X,x)$ (resp. $\omega aO(X,x)$).

$$\begin{array}{ccccccc}
\text{open} & \to & \beta\text{-open} & \to & e^*\text{-open} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\omega\text{-open} & \to & \omega\beta\text{-open} & \to & \omega e^*\text{-open} & \leftarrow & \omega a\text{-open}
\end{array}$$

Lemma 2.3. [15] *Let X be a topological space. Then the following properties hold.*
(1) *The union of any family of ωe^*-open sets is ωe^*-open.*
(2) *The intersection of an ωa-open set and an ωe^*-open set is ωe^*-open.*

Definition 2.4. [15] Let A be a subset of a space X. The union of all ωe^*-open subsets of X contained in A is called the ωe^*-interior of A and is denoted by ωe^*-$int(A)$.

Theorem 2.5. [15] *Let A be a subset of a space X. Then the following properties hold:*
(1) $\omega e^*\text{-}int(A) \subseteq A$,
(2) $\omega e^*\text{-}int(A) \in \omega e^*O(X)$,
(3) $x \in \omega e^*\text{-}int(A)$ *if and only if there exists* $U \in \omega e^*O(X,x)$ *such that* $U \subseteq A$,
(4) $A \subseteq B \Rightarrow \omega e^*\text{-}int(A) \subseteq \omega e^*\text{-}int(B)$,
(5) $\omega e^*\text{-}int(A) \cup \omega e^*\text{-}int(B) \subseteq \omega e^*\text{-}int(A \cup B)$,
(6) $\omega e^*\text{-}int(A \cap B) \subseteq \omega e^*\text{-}int(A) \cap \omega e^*\text{-}int(B)$,
(7) $A \in \omega e^*O(X)$ *if and only if* $A = \omega e^*\text{-}int(A)$,
(8) $\omega e^*\text{-}int(\omega e^*\text{-}int(A)) = \omega e^*\text{-}int(A)$.

Definition 2.6. [15] Let A be a subset of a space X. The intersection of all ωe^*-closed subsets of X containing A is called the ωe^*-closure of A and is denoted by ωe^*-$cl(A)$.

Theorem 2.7. [15] *Let A and B be two subsets of a space X. Then the following properties hold:*
(1) $A \subseteq \omega e^*\text{-}cl(A)$,
(2) $\omega e^*\text{-}cl(A) \in \omega e^*C(X)$,
(3) $x \in \omega e^*\text{-}cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \omega e^*O(X,x)$,
(4) $A \subseteq B \Rightarrow \omega e^*\text{-}cl(A) \subseteq \omega e^*\text{-}cl(B)$,
(5) $\omega e^*\text{-}cl(A) \cup \omega e^*\text{-}cl(B) \subseteq \omega e^*\text{-}cl(A \cup B)$,
(6) $\omega e^*\text{-}cl(A \cap B) \subseteq \omega e^*\text{-}cl(A) \cap \omega e^*\text{-}cl(B)$,
(7) $A \in \omega e^*C(X)$ *if and only if* $A = \omega e^*\text{-}cl(A)$,
(8) $\omega e^*\text{-}cl(\omega e^*\text{-}cl(A)) = \omega e^*\text{-}cl(A)$,
(9) $\omega e^*\text{-}cl(X \setminus A) = X \setminus \omega e^*\text{-}int(A)$.

Definition 2.8. A function of $f : X \to Y$ is called:

(1) e^*-continuous [11] (resp. ω-continuous [12]) if $f^{-1}[V] \in e^*O(X)$ (resp. $\omega O(X)$) for each open set V of Y.
(2) ωe^*-continuous [16] (resp. $\omega\beta$-continuous [4]) if $f^{-1}[V] \in \omega e^*O(X)$ (resp. $\omega\beta O(X)$) for each open set V of Y.
(3) weakly continuous [13] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open U in X containing x such that $f[U] \subseteq cl(V)$.
(4) weakly ω-continuous [5] (or ω-weakly continuous) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an ω-open U in X containing x such that $f[U] \subseteq cl(V)$.
(5) weakly e^*-continuous [17] (resp. weakly β-continuous [4]) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an e^*-open (resp. β-open) U in X containing x such that $f[U] \subseteq cl(V)$.
(6) weakly $\omega\beta$-continuous [2] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an $\omega\beta$-open U in X containing x such that $f[U] \subseteq cl(V)$.

Definition 2.9. [15] A function $f : X \to Y$ is called:

(1) ωe^*-irresolute if the inverse image of every ωe^*-open set in Y is ωe^*-open in X.
(2) ωe^*-open if $f[V]$ is ωe^*-open in Y for every ωe^*-open set V of X.

Definition 2.10. [15] A space X is said to be ωe^*-connected if X can not be expressed as the disjoint union of two non-empty ωe^*-open sets.

Definition 2.11. [15] A space X is said to be ωe^*-T_2 if for every two distinct points x and y in X, there exist $U \in \omega e^*O(X, x)$ and $V \in \omega e^*O(X, y)$ such that $U \cap V = \emptyset$.

3. Main results

Definition 3.1. A function $f : X \to Y$ is said to be weakly ωe^*-continuous [15] (resp. weakly ωa-continuous) at $x \in X$ if for each open set V of Y containing $f(x)$, there exists an ωe^*-open set (resp. ωa-open) U in X containing x such that $f[U] \subseteq cl(V)$. If for each $x \in X$, f is weakly ωe^*-continuous (resp. weakly ωa-continuous) at $x \in X$, then f is said to be weakly ωe^*-continuous (resp. weakly ωa-continuous).

Theorem 3.2. *Let $f : X \to Y$ be a function. Then the following statements are equivalent:*

(1) *f is weakly ωe^*-continuous at $x \in X$,*
(2) *$x \in \omega e^*$-$int(f^{-1}[cl(V)])$ for each open neighborhood V of $f(x)$.*

Proof. $(1) \Rightarrow (2)$: Let $V \in O(Y, f(x))$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists G \in \omega e^*O(X, x))(f\,[G] \subseteq cl(V))$$
$$\Rightarrow (\exists G \in \omega e^*O(X, x))(G \subseteq f^{-1}[cl(V)])$$
$$\Rightarrow (\exists G \in \omega e^*O(X, x))(G = \omega e^*\text{-}int(G) \subseteq \omega e^*\text{-}int(f^{-1}[cl(V)]))$$
$$\Rightarrow x \in \omega e^*\text{-}int(f^{-1}[cl(V)]).$$

$(2) \Rightarrow (1)$: Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow U := \omega e^*\text{-}int(f^{-1}[cl(V)]) \in \omega e^*O(X, x)$$
$$\Rightarrow (U \in \omega e^*O(X, x))(f[U] = f[\omega e^*\text{-}int(f^{-1}[cl(V)])] \subseteq f[f^{-1}[cl(V)]] \subseteq cl(V)). \qquad \square$$

Theorem 3.3. *Let $f : X \to Y$ be a function. Then the followings are equivalent:*

(1) *f is weakly ωe^*-continuous,*
(2) $\omega e^*\text{-}cl(f^{-1}[int(cl(V))]) \subseteq f^{-1}[cl(V)]$ *for every* $V \subseteq Y$,
(3) $\omega e^*\text{-}cl(f^{-1}[int(F)]) \subseteq f^{-1}[F]$ *for every* $F \in RC(Y)$,
(4) $\omega e^*\text{-}cl(f^{-1}[V]) \subseteq f^{-1}[cl(V)]$ *for every* $V \in O(Y)$,
(5) $f^{-1}[V] \subseteq \omega e^*\text{-}int(f^{-1}[cl(V)])$ *for every* $V \in O(Y)$.

Proof. $(1) \Rightarrow (2)$: Let $V \subseteq Y$ and $x \notin f^{-1}[cl(V)]$.

$$\left.\begin{array}{r} x \notin f^{-1}[cl(V)] \Rightarrow f(x) \notin cl(V) \Rightarrow (\exists U \in O(Y, f(x)))(U \cap V = \emptyset) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists W \in \omega e^*O(X, x))(f[W] \cap int(cl(V)) \subseteq cl(U) \cap int(cl(V)) = \emptyset)$$
$$\Rightarrow (\exists W \in \omega e^*O(X, x))(W \cap f^{-1}[int(cl(V))] \subseteq f^{-1}[f[W]] \cap f^{-1}[int(cl(V))] = \emptyset)$$
$$\Rightarrow x \notin \omega e^*\text{-}cl(f^{-1}[int(cl(V))]).$$

$(2) \Rightarrow (3)$: Let $F \in RC(Y)$.

$$\left.\begin{array}{r} F \in RC(Y) \Rightarrow F \in C(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[int(cl(F))]) = \omega e^*\text{-}cl(f^{-1}[int(F)]) \subseteq f^{-1}[cl(F)] = f^{-1}[F].$$

$(3) \Rightarrow (4)$: Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \Rightarrow cl(V) \in RC(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[V]) \subseteq \omega e^*\text{-}cl(f^{-1}[int(cl(V))]) \subseteq f^{-1}[cl(V)].$$

$(4) \Rightarrow (5)$: Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \Rightarrow Y \setminus cl(V) \in O(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[Y \setminus cl(V)]) \subseteq f^{-1}[cl(Y \setminus cl(V))]$$
$$\Rightarrow X \setminus \omega e^*\text{-}int(f^{-1}[cl(V)]) \subseteq f^{-1}[cl(Y \setminus cl(V))] \subseteq X \setminus f^{-1}[V]$$
$$\Rightarrow f^{-1}[V] \subseteq \omega e^*\text{-}int(f^{-1}[cl(V)]).$$

$(5) \Rightarrow (1)$: Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \Rightarrow x \in f^{-1}[V] \\ \text{Hypothesis} \end{array}\right\} \Rightarrow W := \omega e^*\text{-}int(f^{-1}[cl(V)]) \in \omega e^*O(X, x)$$
$$\Rightarrow (W \in \omega e^*O(X, x))(f[W] = f[\omega e^*\text{-}int(f^{-1}[cl(V)])] \subseteq f[f^{-1}[cl(V)]] \subseteq cl(V)).$$ □

Definition 3.4. A function $f : X \to Y$ is said to be ωe^*-continuous [16] if for every at $x \in X$ and if for each open set V of Y containing $f(x)$, there exists an ωe^*-open set U in X containing x such that $f[U] \subseteq V$.

Theorem 3.5. *If a function $f : X \to Y$ is ωe^*-continuous, then f is weakly ωe^*-continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r} (x \in X)(V \in O(Y, f(x))) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists U \in \omega e^*O(X, x))(f[U] \subseteq V \subseteq cl(V)).$$ □

Theorem 3.6. *Let $f : X \to Y$ be a function. Then the followings are equivalent:*

(1) *f is weakly ωe^*-continuous,*

(2) ωe^*-$cl(f^{-1}[int(cl(V))]) \subseteq f^{-1}[cl(V)]$ *for every* $V \in PO(Y)$,
(3) ωe^*-$cl(f^{-1}[V]) \subseteq f^{-1}[cl(V)]$ *for every* $V \in PO(Y)$,
(4) $f^{-1}[V] \subseteq \omega e^*$-$int(f^{-1}[cl(V)])$ *for every* $V \in PO(Y)$.

Proof. $(1) \Rightarrow (2)$: Let $V \in PO(Y)$.

$$\left.\begin{array}{r} V \in PO(Y) \Rightarrow cl(V) \in RC(Y) \\ \text{Hypothesis} \end{array}\right\} \overset{\text{Theorem 3.3(3)}}{\Rightarrow} \omega e^*\text{-}cl(f^{-1}[int(cl(V))]) \subseteq f^{-1}[cl(V)].$$

$(2) \Rightarrow (3)$: Let $V \in PO(Y)$

$$\left.\begin{array}{r} V \in PO(Y) \Rightarrow V \subseteq int(cl(V)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[V]) \subseteq \omega e^*\text{-}cl(f^{-1}[int(cl(V))]) \subseteq f^{-1}[cl(V)].$$

$(3) \Rightarrow (4)$: Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \Rightarrow Y \setminus cl(V) \in O(Y) \Rightarrow Y \setminus cl(V) \in PO(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \omega e^*\text{-}cl(f^{-1}[Y \setminus cl(V)]) \subseteq f^{-1}[cl(Y \setminus cl(V))]$$
$$\Rightarrow X \setminus \omega e^*\text{-}int(f^{-1}[cl(V)]) \subseteq f^{-1}[cl(Y \setminus cl(V))] = X \setminus f^{-1}[int(cl(V))]$$
$$\Rightarrow f^{-1}[V] \subseteq f^{-1}[int(cl(V))] \subseteq \omega e^*\text{-}int(f^{-1}[cl(V)]).$$

$(4) \Rightarrow (1)$: Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r} \left.\begin{array}{r} f(x) \in V \Rightarrow x \in f^{-1}[V] \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f^{-1}[V] \subseteq \omega e^*\text{-}int(f^{-1}[cl(V)]) \\ W := \omega e^*\text{-}int(f^{-1}[cl(V)]) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (W \in \omega e^*O(X, x))(f[W] = f[\omega e^*\text{-}int(f^{-1}[cl(V)])] \subseteq f[f^{-1}[cl(V)]] \subseteq cl(V)). \qquad \square$$

Theorem 3.7. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions. If* f *is weakly* ωe^**-continuous and* g *is continuous, then* $g \circ f$ *is weakly* ωe^**-continuous.*

Proof. Let $x \in X$ and $V \in O(Z, g(f(x)))$.

$$\left.\begin{array}{c} \left.\begin{array}{r} (x \in X)(V \in O(Z, g(f(x)))) \\ g \text{ is continuous} \end{array}\right\} \Rightarrow g^{-1}[V] \in O(Y, f(x)) \\ f \text{ is weakly } \omega e^*\text{-continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists W \in \omega e^*O(X, x))(f[W] \subseteq cl(g^{-1}[V]))$$
$$\Rightarrow (\exists W \in \omega e^*O(X, x))((g \circ f)[W] = g[f[W]] \subseteq g[cl(g^{-1}[V])] \subseteq g[g^{-1}[cl(V)]] \subseteq cl(V)). \qquad \square$$

Theorem 3.8. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions. If* f *is* ωe^**-irresolute and* g *is weakly* ωe^**-continuous, then* $g \circ f$ *is weakly* ωe^**-continuous.*

Proof. Let $x \in X$ and $V \in O(Z, g(f(x)))$.

$$\left.\begin{array}{r} \left.\begin{array}{r} (x \in X)(V \in O(Z, g(f(x)))) \\ g \text{ is weakly } \omega e^*\text{-continuous} \end{array}\right\} \Rightarrow (\exists W \in \omega e^*O(Y, f(x)))(g[W] \subseteq cl(V)) \\ f \text{ is } \omega e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U := f^{-1}[W] \in \omega e^*O(X, x))((g \circ f)[U] = g[f[f^{-1}[W]]] \subseteq g[W] \subseteq cl(V)). \qquad \square$$

Theorem 3.9. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions. If* $g \circ f$ *is weakly* ωe^**-continuous and* f *is* ωe^**-open surjection, then* g *is weakly* ωe^**-continuous.*

Proof. Let $y \in Y$ and $V \in O(Z, g(y))$.

$$\left.\begin{array}{r}(y \in Y)(V \in O(Z, g(y)))\\ f \text{ is surjection}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists x \in X)(y = f(x))(V \in O(Z, g(f(x))))\\ g \circ f \text{ is weakly } \omega e^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists U \in \omega e^*O(X, x))((g \circ f)[U] \subseteq cl(V))\\ f \text{ is } \omega e^*\text{-open}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (H := f[U] \in \omega e^*O(Y, y))(g[H] \subseteq cl(V)).$$
□

Theorem 3.10. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If f is weakly wa-continuous, then g is weakly we*-continuous.*

Proof. Let $x \in X$ and $W \in O(X \times Y, g(x))$.

$$\left.\begin{array}{r}W \in O(X \times Y, g(x)) \Rightarrow (\exists U_1 \in O(X, x))(\exists V \in O(Y, f(x)))(g(x) \in U_1 \times V \subseteq W)\\ f \text{ is weakly } \omega a\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists U_1 \in \omega e^*O(X, x))(\exists U_2 \in \omega aO(X, x))(f[U_2] \subseteq cl(V))\\ H := U_1 \cap U_2\end{array}\right\} \Rightarrow$$
$$\begin{array}{rcl}\Rightarrow (H \in \omega e^*O(X, x))(g[H] = g[U_1 \cap U_2] \subseteq U_1 \times f[U_2] & \subseteq & cl(U_1) \times cl(V)\\ & = & cl(U_1 \times V)\\ & \subseteq & cl(W)).\end{array}$$
□

Remark 3.11. From Definitions 2.8, 3.1, 3.4 and Theorem 3.5, we have the following diagram. The converses of these implications need not be true in general as shown by the following example. Examples regarding other implications can be found related papers.

$$\begin{array}{ccccc}\text{weakly continuous} & \to & \text{weakly } \beta\text{-continuous} & \to & \text{weakly } e^*\text{-continuous}\\ \downarrow & & \downarrow & & \downarrow\\ \text{weakly } \omega\text{-continuous} & \to & \text{weakly } \omega\beta\text{-continuous} & \to & \text{weakly } \omega e^*\text{-continuous}\\ \uparrow & & \uparrow & & \uparrow\\ \omega\text{-continuous} & \to & \omega\beta\text{-continuous} & \to & \omega e^*\text{-continuous}\end{array}$$

Example 3.12. [4] Let $X := \{1, 2, 3\}$ with the topologies $\tau := \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$ and $\sigma := \{\emptyset, X, \{1\}, \{2, 3\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the function defined by

$$f(x) := \begin{cases} 1 & , \quad x = 3\\ 2 & , \quad x = 1\\ 3 & , \quad x = 2\end{cases}.$$

Then f is weakly ωe^*-continuous, but f is not weakly e^*-continuous.

Example 3.13. Let $Y := \{1, 2, 3\}$ with the topology $\tau := \{\emptyset, Y, \{1\}, \{2, 3\}\}$. Let $f : (\mathbb{R}, \tau_{coc}) \to (Y, \tau)$ be the function defined by

$$f(x) := \begin{cases} 3 & , \quad x \in \mathbb{Q}\\ 1 & , \quad x \in \mathbb{I}\end{cases}.$$

Then f is weakly ωe^*-continuous, but f is not weakly $\omega\beta$-continuous.

Lemma 3.14. [16] *Let (A, τ_A) be a subspace of (X, τ), $B \subseteq A$ and A is e^*-open sets in (X, τ). Then B is ωe^*-open in (X, τ) if and only if B is ωe^*-open in (A, τ_A).*

Theorem 3.15. *Let $\{U_i | i \in I\}$ be any e^*-open cover of a space X. A function $f : (X, \tau) \to (Y, \sigma)$ weakly ωe^*-continuous if the restriction $f|_{U_i} : (U_i, \tau_{U_i}) \to (Y, \sigma)$ is weakly ωe^*-continuous for each $i \in I$.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in O(Y, f(x))) \\ f|_{U_i} \text{ is weakly } \omega e^*\text{-continuous}\end{array}\right\} \Rightarrow (\exists U \in \omega e^* O(U_i, x))(f|_{U_i}[U] \subseteq cl(V))$$
$$\overset{\text{Lemma 3.14}}{\Rightarrow} (U_i \in e^* O(X, x))(U \in \omega e^* O(X, x))(f[U] \subseteq cl(V)).$$ □

Theorem 3.16. *If $f : X \to Y$ is weakly ωe^*-continuous and Y is Hausdorff, then for each $(x, y) \notin G(f)$, there exists an ωe^*-open set $U \subseteq X$ and an open set $V \subseteq Y$ containing x and y, respectively, such that $f[U] \cap int(cl(V)) = \emptyset$.*

Proof. Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r}(x, y) \notin G(f) \Rightarrow y \neq f(x) \\ (Y, \sigma) \text{ is Hausdorff}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists V \in O(Y, y))(\exists W \in O(Y, f(x)))(V \cap W = \emptyset) \\ f \text{ is weakly } \omega e^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x))(f[U] \cap int(cl(V)) \subseteq cl(W) \cap int(cl(V)) \subseteq V \cap W = \emptyset)$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x))(f[U] \cap int(cl(V)) = \emptyset).$$ □

Definition 3.17. A function $f : X \to Y$ is said to be contra we^*-closed graph [15] if for each $(x, y) \notin G(f)$, there exist an ωe^*-open subset U of X and a closed subset V of Y such that $(U \times V) \cap G(f) = \emptyset$.

Definition 3.18. A function $f : X \to Y$ is said to be ωe^*-strongly closed graph if for each $(x, y) \notin G(f)$, there exist an ωe^*-open subset U of X and an open subset V of Y such that $(x, y) \in U \times V$ and $(U \times cl(V)) \cap G(f) = \emptyset$.

Theorem 3.19. *If $f : X \to Y$ is weakly ωe^*-continuous and Y is Uryshon, then $G(f)$ is ωe^*-strongly closed.*

Proof. Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r}(x, y) \notin G(f) \Rightarrow y \neq f(x) \\ (Y, \sigma) \text{ is Uryshon}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists V \in O(Y, y))(\exists W \in O(Y, f(x)))(cl(V) \cap cl(W) = \emptyset) \\ f \text{ is weakly } \omega e^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x))(f[U] \cap cl(V) \subseteq cl(W) \cap cl(V) = \emptyset)$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x))((U \times cl(V)) \cap G(f) = \emptyset).$$ □

Theorem 3.20. *If $f : X \to Y$ is weakly ωe^*-continuous injection and $G(f)$ is ωe^*-strongly closed, then X is ωe^*-T_2.*

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x_1, x_2 \in X)(x_1 \neq x_2) \\ f \text{ is injection}\end{array}\right\} \Rightarrow f(x_1) \neq f(x_2) \Rightarrow (x_1, f(x_2)) \notin G(f) \\ G(f) \text{ is } \omega e^*\text{-strongly closed}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists U \in \omega e^* O(X, x_1))(\exists V \in O(Y, f(x_2)))((U \times cl(V)) \cap G(f) = \emptyset) \\ f \text{ is weakly } \omega e^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x_1))(\exists W \in we^* O(X, x_2))(f[U] \cap f[W] \subseteq f[U] \cap cl(V) = \emptyset)$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x_1))(\exists W \in we^* O(X, x_2))(f[U \cap W] = \emptyset)$$
$$\Rightarrow (\exists U \in \omega e^* O(X, x_1))(\exists W \in we^* O(X, x_2))(U \cap W = \emptyset).$$ □

Theorem 3.21. *If $f : X \to Y$ is surjection and $G(f)$ is ωe^*-strongly closed, then Y is T_2.*

Proof. Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$.

$$\left.\begin{array}{r}(y_1, y_2 \in Y)(y_1 \neq y_2)\\ f \text{ is surjection}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists x_1 \in X)(y_2 \neq y_1 = f(x_1)) \Rightarrow (x_1, y_2) \notin G(f)\\ G(f) \text{ is } \omega e^*\text{-strongly closed}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in \omega e^*O(X, x_1))(\exists V \in O(Y, y_2))(f[U] \cap cl(V) = \emptyset)$$
$$\Rightarrow (\exists V \in O(Y, y_2))(f(x_1) = y_1 \notin cl(V))$$
$$\Rightarrow (\exists V \in O(Y, y_2))(\exists W \in O(Y, y_1))(W \cap V = \emptyset).$$
□

Corollary 3.22. If $f : X \to Y$ is weakly ωe^*-continuous bijective and $G(f)$ is ωe^*-strongly closed, then X and Y are ωe^*-T_2.

Proof. It is clear from the fact that every T_2 space is ωe^*-T_2 space and Theorem 3.20 and Theorem 3.21. □

Theorem 3.23. *If $f : X \to Y$ is weakly ωe^*-continuous surjection and X is ωe^*-connected, then Y is connected.*

Proof. Suppose that Y is not connected.

$$\left.\begin{array}{r}Y \text{ is not connected } \Rightarrow (\exists U, V \in O(Y) \setminus \{\emptyset\})(U \cap V = \emptyset)(U \cup V = Y)\\ f \text{ is weakly } \omega e^*\text{-continuous surjection}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f^{-1}[U], f^{-1}[V] \in we^*O(X) \setminus \{\emptyset\})(f^{-1}[U \cap V] = f^{-1}[\emptyset])(f^{-1}[U \cup V] = f^{-1}[Y])$$
$$\Rightarrow (f^{-1}[U], f^{-1}[V] \in we^*O(X) \setminus \{\emptyset\})(f^{-1}[U] \cap f^{-1}[V] = \emptyset)(f^{-1}[U] \cup f^{-1}[V] = X)$$

This means that X is not we^*-connected which is a contradiction. □

4. Conclusion

Without a doubt one of the most studied generalizations of continuity is weak continuity. In this study, we have defined and studied the notion of weakly ωe^*-continuity which is weaker than the notion of weakly continuity. Also, we have examined the relationship among many types of important continuity given in the literature. Moreover, we have obtained many characterizations related to the notion of weakly ωe^*-continuity. Furthermore, we have discussed graph properties as well. We believe that this study will be light for further studies.

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References

[1] M.E. Abd El-Monsef, S.N. El-Deeb, R.A. Mahmoud, β-open sets and β-continuous mappings, Bull Fac Sci Assiut Univ A, 12(1) (1983), 77-90.

[2] H.H. Aljarrah, M.S.M. Noorani, T. Noiri, $\omega\beta$-continuous functions, Eur. J. Pure Appl. Math., 5 (2012), 129-140.

[3] H.H. Aljarrah, M.S.M. Noorani and T. Noiri, On $w\beta$-open sets, submitted.

[4] H.H. Aljarrah, M.S.M. Noorani and T. Noiri, Weakly $\omega\beta$-continuous functions, Inter. Conf. on Math. Sci. and Stat. (ICMSS2013) 396-400, doi:10.1063/1.4823943.

[5] K.Y. Al-Zoubi, H. Al-Jarah. Weakly ω-continuous functions, Acta Math. Univ. Comenianae, (2) (2010), 253-264.
[6] K.Y. Al-Zoubi, B. Al-Nashef, The topology of ω-open subsets, Al-Manarah Journal, 9 (2003), 169-179.
[7] B.S. Ayhan, M. Özkoç, Almost e^*-continuous functions and their characterizations, J Nonlinear Sci Appl, 9(12) (2016), 6408-6423.
[8] B.S. Ayhan, M. Özkoç, On characterizations of weakly e^*-continuity, Divulg. Mat., 21(1-2) (2020), 9-20.
[9] E. Ekici, On a-open sets, $\mathcal{A}^*$-sets and decompositions of continuity and super-continuity, Ann Univ Sci Budapest Eötvös Sect Math., 51 (2008), 39-51.
[10] E. Ekici, On e-open sets, $\mathcal{DP}^*$-sets and $\mathcal{DP}\epsilon^*$-sets and decompositions of continuity, Arab J Sci Eng Sect A Sci, 33(2A) (2008), 269-282.
[11] E. Ekici, e^*-open sets and $(\mathcal{D},\mathcal{S})^*$-sets, Math. Morav., 13(1) (2009), 29-36.
[12] H.Z. Hdeib, ω-continuous functions, Dirasat Journal, 16(2) (1989), 136-153.
[13] N. Levine, A decomposition of continuity in topological spaces, Am. Math. Mon., 68 (1961), 44-46.
[14] A. S. Mashhour, M.E. Abd. El-Monsef and S.N. El-deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.
[15] M. Özkoç, P. Şaşmaz, On contra ωe^*-continuous functions, (Accepted in PJAA).
[16] M. Özkoç, P. Şaşmaz, On ωe^*-continuous functions, (Submitted).
[17] M. Özkoç, G. Aslım, On weakly e-continuous functions, Hacet. J. Math. Stat. 40(6) (2011), 781-791.
[18] M.H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 375-381.
[19] N.V. Veličko, H-closed topological spaces, Amer. Math. Soc. Transl. 78(2) (1968), 103-118.

(Pınar ŞAŞMAZ) Muğla Sitki Koçman University, Graduate School of Natural and Applied Sciences, Mathematics, 48000, Menteşe-Muğla/TURKEY
E-mail address: pinarsasmaz@posta.mu.edu.tr; pinar_sasmaz@hotmail.com

(Murad ÖZKOÇ) Muğla Sitki Koçman University, Faculty of Science, Department of Mathematics, 48000, Menteşe-Muğla/TURKEY
E-mail address, Corresponding author: murad.ozkoc@mu.edu.tr; murad.ozkoc@gmail.com