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SOFT GAMES ON SOFT TOPOLOGICAL SPACES

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Abstract. In this paper, we give new types of soft games, dual soft games, determined soft games and undetermined soft games. We study the equivalences between types of soft games. In these soft games, players I and II alternately choose points and their soft sets respectively. Player I wins if and only if the moves of II cover the space of game by soft topological space.

1. Introduction

Molodtsov [9] initiated a novel concept of soft set theory, which is a completely new approach for modeling vagueness and uncertainty. He successfully applied the soft set theory to several directions such as smoothness of functions, game theory, Riemann Integration, and theory of measurement. In recent years, development in the fields of soft set theory and its application has been taking place in a rapid pace. This is because of the general nature of parameterization expressed by a soft set. Cagman et al. [2] modified the definition of soft sets which is similar to that of Molodtsov. Soft topological spaces which deal with uncertainties have been studied by some authors as Shabir et al. [10] and Cagman [2] initiated the study of soft topological spaces. They defined separately soft topology and established their several properties. Moldtsov [9] described the person's behavior with the help of the s-function which for any set of strategies indicates the set of ϵ-optimal choices. He asked which s-function has to be chosen for describing the person's behavior under uncertainty. Recently, game theory has been extended to topological spaces [4, 5, 6, 7] and their applications.

Throughout this paper, suppose we have a soft game for Nash equilibrium of two players I and II in a new form $< (F_i, E_i), S_i : i = 1, 2, \ldots, n >$. The soft strategy of this game players I and II alternately choose points and their soft open sets respectively. I wins if and only if the moves of II soft cover the soft topological space. Finally, we give a generalization for the concept of equivalence between soft games.

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2. Preliminaries

Throughout this paper, let X be a nonempty set (universe), E be a set of parameters and $\mathcal{P}(X)$ denote the power set of X.

Definition 2.1. [9] A pair (F, E), denoted by F, is called a soft set over X, where F is a mapping given by $F : E \to \mathcal{P}(X)$. For $e \in E$, $F(e)$ may be considered as the set of $e-$approximate elements of the soft set (F, E) i.e $(F, E)=\{F(e) \mid e \in E,\ F : E \to \mathcal{P}(X)\}$. If $e \notin E$, then $F(e)=\emptyset$. A soft set F is called finite (resp., countable) if $F(e)$ is finite (resp., countable) set for each $e \in E$. The set of all soft sets over X will be denoted by $S_E(X)$.

Definition 2.2. [3] For any two soft sets F, G defined over a common universe X, we have

(1). F is said to be null soft set, denoted by $\tilde{\emptyset}$, if $F(e)=\emptyset$ for all $e \in E$.
(2). F is said to be absolute soft set, denoted by $\tilde{X}$, if $F(e)=X$ for all $e \in E$.
(3). F is soft subset of G, denoted by $F \tilde{\subseteq} G$, if $F(e) \subseteq G(e)$ for all $e \in E$.
(4). F and G are soft equal, denoted by $F=G$, if $F \tilde{\subseteq} G$ and $G \tilde{\subseteq} F$.
(5). The soft union of F and G, denoted by $F \tilde{\cup} G$, is a soft set over X and defined by $F \tilde{\cup} G : E \to \mathcal{P}(X)$ such that $(F \tilde{\cup} G)(e)=F(e) \cup G(e)$ for all $e \in E$.
(6). The soft intersection of F and G, denoted by $F \tilde{\cap} G$, is a soft set over X and defined by $F \tilde{\cap} G : E \to \mathcal{P}(X)$ such that $(F \tilde{\cap} G)(e)=F(e) \cap G(e)$ for all $e \in E$.
(7). The soft complement $(\tilde{X} - F)$ of a soft set F is denoted by $F^{\tilde{c}}$ and defined by $F^{\tilde{c}} : E \to \mathcal{P}(X)$ such that $F^{\tilde{c}}(e)=X \setminus F(e)$ for all $e \in E$.

Definition 2.3. [11] Let Γ be an arbitrary indexed set and $L=\{F_i \mid i \in \Gamma\}$ be a subfamily of $S_E(X)$.

(1). The soft union of L is the soft set H, where $H(e)=\tilde{\cup}_{i \in \Gamma} F_i(e)$ for each $e \in E$. We write $\tilde{\cup}_{i \in \Gamma} F_i=H$.
(2). The soft intersection of L is the soft set M, where $M(e)=\tilde{\cap}_{i \in \Gamma} F_i(e)$ for each $e \in E$. We write $\tilde{\cap}_{i \in \Gamma} F_i=M$.

Definition 2.4. [10] Let F be a soft set over X and $x \in X$. We say that $x \tilde{\in} F$ whenever $x \in F(e)$ for all $e \in E$.

Definition 2.5. [1] Let $S_E(X)$ and $S_K(Y)$ be families of soft sets, $u : X \to Y$ and $p : E \to K$ be functions. Therefore, $f_{pu} : S_E(X) \to S_K(Y)$ is called a soft function.

(1). If $F \in S_E(X)$, then the image of F under f_{pu}, written as $f_{pu}(F)$, is a soft set in $S_K(Y)$ such that

$$f_{pu}(F)(k) = \begin{cases} \bigcup_{e \in p^{-1}(k)} u(F(e)), & p^{-1}(k) \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

for each $k \in Y$.

(2). If $G \in S_K(Y)$, then the inverse image of G under f_{pu}, written as $f^{-1}_{pu}(G)$, is a soft set in $S_E(X)$ such that

$$f^{-1}_{pu}(G)(e) = \begin{cases} u^{-1}(G(p(e))), & p(e) \in Y \\ \emptyset, & \text{otherwise.} \end{cases}$$

for each $e \in E$.

The soft function f_{pu} is called surjective if p and u are surjective, also is said to be injective if p and u are injective.

Several properties and characteristics for f_{pu} and f_{pu}^{-1} are reported in detail in [1].

Definition 2.6. [10] Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then $\tau \tilde{\subseteq} S_E(X)$ is called a soft topology on X if

(1). $\tilde{X}, \tilde{\emptyset} \in \tau$,
(2). The soft union of any number of soft sets in τ belongs to τ,
(3). The soft intersection of any two soft sets in τ belongs to τ.

(X, τ, E) is called a soft topological space over X. A soft set F over X is said to be open soft set in X if $F \tilde{\in} \tau$, and it is called closed soft set in X, if its relative complement $F^{\tilde{c}}$ is open soft set.

Definition 2.7. [10] Let (X, τ, E) be a soft topological space over X and $F \tilde{\in} S_E(X)$. Then, the soft interior, soft closure and soft boundary of F, denoted by $int(F)$, $cl(F)$ and $bd(F)$ respectively, are defined as:

$$\begin{aligned}
int(F) &= \tilde{\cup}\big\{G \mid G \text{ is open soft set and } G \tilde{\subseteq} F\big\}, \\
cl(F) &= \tilde{\cap}\big\{H \mid H \text{ is closed soft set and } F \tilde{\subseteq} H\big\}, \\
bd(F) &= cl(F) \tilde{\cap} cl(\tilde{X} - F).
\end{aligned}$$

3. Soft games on soft open sets

Definition 3.1. For a soft topological space (X, τ, E), a point $p \in X$ is called soft limit point of a soft set A of X if every open soft set containing p contains some points of A. The set of all soft limit points of A is called derived soft set of A and is denoted by $sd(A)$. Points from $A - sd(A)$ are called soft isolated points of the set A. A subset A of X is called soft dense in itself (or has no soft isolated points) if $A \subseteq sd(A)$.

Definition 3.2. A soft game with perfect information $G(X, \tau, E)$ on a soft topological space (X, τ, E) of two players I and II will define as follows: Player I choose the sequence of points $< x_1, x_2, x_3, \cdots >$ and Player II choose the sequence of soft open sets $< (F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots >$. Player I wins the play $G(X, OS(X, \tau, E))$ if and only if he covers the space X by the soft cover $\{(F_n, E_n) : n \in \mathbb{N}\}$. Otherwise Player II wins.

Remark 3.3. In the soft game $G(X, OS(X, \tau, E))$ player II has a chance better than his choice in the game of Galvin [8] and El Atik [4].

Definition 3.4. A soft strategy s for player I is a soft function whose domain is the set of finite sequences of nonempty soft sets and has the property that if $< (F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots, (F_k, E_k) >$ is a finite sequence, then $s < (F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots, (F_k, E_k) >$ is a subset of X. s is said to be a soft winning strategy if each play $(x_1, (F_1, E_1), x_2, (F_2, E_2), \cdots)$ of $G(X, \tau, E)$ for which $x_k = s < (F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots, (F_k, E_k) >$ for each positive integer k is won by I.

The notions of soft strategy and soft winning a strategy for player II are defined analogously.

The dual game $G^*(X, \tau, E)$ (II opens the game) will define as follows:

Player I: $\mathcal{U}_1 \qquad \mathcal{U}_2 \qquad \mathcal{U}_3 \cdots$

Player II: $\qquad (F_1, E_1) \qquad (F_2, E_2) \qquad (F_3, E_3) \cdots$

where $\mathcal{U}_n$ is a soft cover of X by soft sets and $(F_n, E_n) \in \mathcal{U}_n$ for each $n \in \mathbb{N}$. I wins the play $(\mathcal{U}_1, (F_1, E_1), \mathcal{U}_2, (F_2, E_2), \cdots)$ if I covers the space by soft sets, otherwise, II wins.

Definition 3.5. If the player I or II has a soft winning strategy, then the soft game $G(X, \tau, E)$ is said to be a determined game. Otherwise, the game is said to be undetermined.

Definition 3.6. Any two soft games between players I and II are equivalent if one of the following holds:

(1) Player I has soft winning strategies in both soft games.
(2) Player II has soft winning strategies in both soft games.
(3) Both soft games are undetermined.

Theorem 3.7. *Let (X, τ, E) be a soft topological space, then the soft game $G(X, \tau, E)$ is equivalent to the game $G^*(X, \tau, E)$.*

Proof. Let s be a soft winning strategy for Player I in $G^*(X, \tau, E)$. Let $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \cdots)$ be an infinite sequence of soft covers by soft sets of X. Since s is a soft winning strategy, then $s(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \cdots) \tilde{\in} \mathcal{U}_n$, for every n and $\bigcup\limits_{n\,\tilde{}\,1}^{\tilde{\infty}} = s(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \cdots) \tilde{=} X$. Player I can use s as a soft winning strategy in $G(X, \tau, E)$ by the following procedure. First I chooses a point $x_1 \in X$, so that every soft set containing x_1 is of the form $s(\mathcal{U})$ for some soft cover by soft sets of X (there must be such a point since the collection of all soft sets not of the form $s(\mathcal{U})$ cannot soft cover X). Let II chooses any soft set U_1 containing x_1. Then $(F_1, E_1) \tilde{=} s(\mathcal{U}_1)$ for some soft cover $\mathcal{U}_1$ by soft sets of X. Now, I chooses $x_2 \in X$ so that every soft set containing x_2 is of the form $s(\mathcal{U}_1, \mathcal{U})$ for some $\mathcal{U}$. II chooses a soft set (F_1, E_1) containing x_2 and $(F_2, E_2) \tilde{=} s(\mathcal{U}_1, \mathcal{U}_2)$. Continuing in this manner , we get a play $(x_1, (F_1, E_1), x_2, (F_1, E_1), \cdots)$ of $G(X, \tau, E)$ and the play $(\mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \cdots)$ of $G^*(X, \tau, E)$. Since s is a soft winning strategy in $G^*(X, \tau, E)$, then $\bigcup\limits_{n=1}^{\tilde{\infty}} s(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \cdots) \tilde{=} X$. Therefore I uses s as a soft winning strategy in $G(X, \tau, E)$. $\square$

We also state the following soft topological game for technical reasons.

Let $\mathbb{N}$ be the set of positive integers, Y be a soft subspace of a topological space (X, τ, E) and L be a soft base for τ. We define the game $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$ as follows:

I: $f_{p_1 u} \qquad f_{p_2 u} \qquad f_{p_3 u} \cdots$

II: $\qquad i_1 \qquad i_2 \qquad i_3 \cdots$

where $f_{p_n u} : \mathbb{N} \longrightarrow L$ and $i_n \in \mathbb{N}$ for each $n \in \mathbb{N}$. If $Y \tilde{\nsubseteq} \bigcup\limits_{i=1}^{\tilde{\infty}} f_{p_n u}(i)$ for each n, then I wins the play $(f_{p_1 u}, i_1, f_{p_2 u}, i_2, \cdots)$. Otherwise II wins.

Lemma 3.8. *Let (Y, τ_Y, E_Y) be a soft subspace of a space (X, τ, E). If Y is a soft Lindlöf space, then the soft game $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$ is equivalent to $G^*(Y, \tau_Y, E_Y)$ and hence to $G(Y, \tau_Y, E_Y)$.*

Proof. Let s be a soft winning strategy for I in $G^*((X,\tau,E),(Y,\tau_Y,E_Y),L)$. We define a soft winning strategy t for him in $G^*(Y,\tau_Y,E_Y)$. Let $(f_{p_1u},\ f_{p_2u},\ f_{p_3u},\cdots)$ be an infinite sequence of soft functions such that $f_{p_nu}:\mathbb{N}\longrightarrow L$. First II chooses a soft cover $\mathcal{U}_1$ by soft sets of Y. I chooses i_1 of $\mathbb{N}$ such that $i_1\tilde{=}s(f_{p_1u})$. Set $Y\tilde{\cap}f_{p_1u}(i_1)\tilde{=}(F_1,E_1)$ and $t(\mathcal{U}_1)\tilde{=}(F_1,E_1)$. Since $f_{p_1u}(i_1)\tilde{\in}L$ and Y is a soft subspace of X, then (F_1,E_1) is a soft set in Y. Continuing in this manner, we get the play $(f_{p_1u},i_1,f_{p_2u},i_2,\cdots)$ of $G^*((X,\tau,E),(Y,\tau_Y,E_Y),L)$ and the play $(\mathcal{U}_1,(F_1,E_1),\mathcal{U}_2,(F_2,E_2),\cdots)$ of $G^*(Y,\tau_Y,E_Y)$. Using the soft winning strategy s for I and soft Lindlöfness of Y,

$$Y\tilde{\subset}\bigcup_{n=1}^{\tilde{\infty}}f_{p_nu}(i_n)\tilde{\subset}Y\tilde{\cap}\bigcup_{n=1}^{\tilde{\infty}}f_{p_nu}(i_n)\tilde{=}\bigcup_{n=1}^{\tilde{\infty}}(Y\tilde{\cap}f_{p_nu}(i_n))\tilde{=}\bigcup_{n\tilde{=}1}^{\infty}(F_n,E_n)$$

. Then $Y\tilde{=}\bigcup_{n=1}^{\tilde{\infty}}(F_n,E_n)$ and so t is a soft winning strategy. □

4. Determination of soft games

Our goal of this section is to give the necessary conditions for the soft open soft game to be determine or not.

Proposition 4.1. Suppose k and λ are infinite cardinals with $k>\lambda$. If X is the soft union of k disjoint nonempty soft sets, each of which is the soft intersection of λ soft sets, then I has no soft winning strategy in $G(X,\tau,E)$.

Proof. Assume that there is a family $\{(F_{\xi,\eta},A_{\xi,\eta}):\xi<k\ \text{ and }\ \eta<\lambda\}$ of soft sets in (X,τ,E) such that k and λ are infinite cardinals with $k>\lambda$ and $X\tilde{=}\tilde{\bigcup}_{\xi<k}X_\xi$, where the soft union is formed by pairwise disjoint nonempty soft sets $X_\xi\tilde{=}\tilde{\bigcap}_{\eta<\lambda}(F_{\xi,\eta},A_{\xi,\eta})$ for $\xi<k$. Let s be a fixed soft strategy of I, we prove that s is not soft winning. Assume that II chooses soft sets of the form $(F_{\xi,\eta},A_{\xi,\eta})$ only. Let $I\tilde{=}\{\mu:\mu<k\}$ such that there is a partial play $(x_1,(F_1,E_1),x_2,(F_2,\ E_2),\cdots,x_n,\ (F_n,E_n))$ such that $x_{m+1}\tilde{=}s((F_1,E_1),(F_2,E_2),\cdots,\ (F_m,E_m))$ for $m<n$, $s((F_1,E_1),\ (F_2,E_2),\cdots,(F_n,E_n))\ \tilde{\in}X_\mu$ and $(F_m,E_m)\tilde{\in}\ \{(F_{\xi,\eta},A_{\xi,\eta}):x_m\tilde{\in}X_\xi\ \text{ and }\ \eta<\lambda\}$ for each $m\leq n$. By a simple calculation the cardinality of $I\leq\lambda$. Hence there exists $\mu_0<k$ such that $\mu_0\notin I$. Let us choosing a point $x_0\in X_\mu$, but $s((F_1,E_1),\ (F_2,E_2),\cdots,(F_n,E_n))\in X_\mu$. Thus II can omit the point x_0 on each his moves. Therefore $\bigcup_{n=1}^{\tilde{\infty}}(F_n,E_n)\tilde{\neq}X$ means that s is not a soft winning strategy for I in $G(X,\tau,E)$. □

Proposition 4.2. Let k be the least cardinal such that the real line is the soft union of k soft nowhere dense soft sets. If X is a soft Lindelöf space of cardinality $<k$, then II has no soft winning strategy in $G(X,\tau,E)$.

Proof. Let L be the collection of all soft sets of (X,τ,E). By Lemma 3.8, it is suffices to prove that II has no soft winning strategy in $G^*((X,\tau,E),(Y,\tau_Y,E_Y),L)$. Let s be any soft strategy for II in $G^*((X,\tau,E),(Y,\tau_Y,E_Y),L)$. For $i_1,i_2,\cdots,i_r\in\mathbb{N}$, let $f_{p_{i_1,i_2,\cdots,i_r}u}\tilde{=}s(i_1,i_2,\cdots,i_r)$. Assume that $\bigcup_{n\tilde{=}1}^{\infty}f_{p_{i_1,i_2,\cdots,i_r}u}(n)\tilde{=}X$ for all $i_1,i_2,\cdots,i_r\in$

$\mathbb{N}$, otherwise, s is not a soft winning strategy. Given any $\sigma \in \mathbb{N}^{\mathbb{N}}$, there is a uniquely determined play $(f_{p_1 u}, i_1, f_{p_2 u}, i_2, \cdots)$ in which II uses the soft strategy and I plays $i_1 \tilde{=} \sigma(1)$, $i_2 \tilde{=} \sigma(2)$, $\cdots$. Let $D_\delta \tilde{=} \tilde{\bigcup}_{n=1}^{\infty} f_{p_n u}(i_n)$. For each $x\delta \tilde{\in} X$, let $F_x \tilde{=} \{\sigma \tilde{\in} \mathbb{N}^{\mathbb{N}} : x \tilde{\notin} D_\delta\}$. For $i_1, \cdots i_r \in \mathbb{N}$, let $[i_1, i_2, \cdots i_r] \tilde{=} \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma(1) \tilde{=} i_1, \cdots \sigma(r) \tilde{=} i_r\}$. Given any $x \tilde{\in} X$ and any $i_1, i_2, \cdots i_r \in \mathbb{N}$, since $\tilde{\bigcup}_{n=1}^{\infty} f_{p_{i_1, i_2, \cdots, i_r} u}(n) \tilde{=} X$, we choose $i_{r+1} \tilde{\in} \mathbb{N}$ so that $x \tilde{\in} f_{p_{i_1, i_2, \cdots, i_r} u}(i_{r+1})$, then $[i_1, i_2, \cdots i_r, i_{r+1}] \tilde{\cap} F_x \tilde{=} \tilde{\phi}$. This shows that the soft sets F_x are soft nowhere dense in $\mathbb{N}^{\mathbb{N}}$. Since the cardinality of $X < k$, this shows that $\tilde{\bigcup}_{x \in X} F_x \tilde{\neq} \mathbb{N}^{\mathbb{N}}$. Choose $\sigma \in \mathbb{N}^{\mathbb{N}} - \tilde{\bigcup}_{x \in X} F_x$. Then $D_\delta \tilde{=} X$, and so s is not soft winning strategy. $\square$

Theorem 4.3. *Let (X, τ, E) be a soft separable metric with no soft isolated points, and that neither (X, τ, E) nor the real line is the soft union of $\leqslant 2^{\aleph_0}$ soft nowhere dense soft sets. Then (X, τ, E) has a soft subspace (Y, τ_Y, E_Y) of cardinality $2^{\aleph_0}$ such that $G(X, \tau, E)$ is undetermined.*

Proof. Let B be a countable soft dense subset of (X, τ, E) and L be a countable soft base for (X, τ, E). By Proposition 4.1 and Lemma 3.8, it is sufficient to find a soft subspace (Y, τ_Y, E_Y) of cardinality $2^{\aleph_0}$ such that II has no soft winning strategy in $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$. The set of all II's soft strategies in $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$ not depends on the soft subspace (Y, τ_Y, E_Y) and has cardinality $2^{\aleph_0}$. Put $S \tilde{=} \{s_\mu : \mu < 2^{\aleph_0}\}$. Define inductively soft set soft dense sets $(F_\mu, A_\mu) \tilde{\subset} X$ and points $x_\mu \tilde{\in} X$ for $\mu < 2^{\aleph_0}$. Let $\alpha < 2^{\aleph_0}$ and suppose that (F_λ, A_λ) and x_λ have already been defined for every $\lambda < \mu$. Put $Y_\mu \tilde{=} B \tilde{\cup} \{x_\lambda : \lambda < \mu\}$. By Proposition 4.2 and Lemma 3.8, II has no soft winning strategy in $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$. In particular, s_μ is not soft winning strategy in that soft game. Let $(f_{p_1 u}, i_1, f_{p_2 u}, i_2, \cdots)$ be some play in which II uses the soft strategy s_μ and I wins the soft game $G^*(X, \tau, E), (Y, \tau_Y, E_Y), L)$. If $Y_\mu \tilde{\nsubseteq} \tilde{\bigcup}_{i=1}^{\infty} f_{p_n u}(i)$ for each n, we put $(F_\mu, A_\mu) \tilde{=} X$, otherwise, we put $(F_\mu, A_\mu) \tilde{=} \tilde{\bigcup}_{n=1}^{\infty} f_{p_n u}(i_n) \tilde{\supset} Y_\mu$. In either cases, (F_μ, A_μ) is soft set soft dense subset of X and $(f_{p_1 u}, i_1, f_{p_2 u}, i_2, \cdots)$ is a soft winning play for I against s_μ in every soft game $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$ where $Y_\mu \tilde{\subset} Y \tilde{\subset} (F_\mu, A_\mu)$. Since X has no soft isolated points and it is not the soft union of $< 2^{\aleph_0}$ soft nowhere dense sets. Choose a point $x_\mu \tilde{\in} (\tilde{\bigcap}_{\lambda < \mu} (F\lambda, A_\lambda) - Y_\mu$. This completes the definition of the (F_μ, A_μ)'s and x_μ's. Finally, put $Y \tilde{=} B \cup \{x_\mu : \mu < 2^{\aleph_0}\}$. Since $Y_\mu \tilde{\subset} Y \tilde{\subset} (F_\mu, A_\mu)$ for every $\mu < 2^{\aleph_0}$, II has no soft winning strategy in $G^*((X, \tau, E), (Y, \tau_Y, E_Y), L)$. $\square$

5. Equivalences between some soft games by winning stratigies

In this section, each soft topological space is assumed to be soft completely regular. K denotes a class of soft spaces such that K contains all soft singletons and soft hereditary with respect to soft closed soft subspaces. K refers to soft spaces induced by a soft operator. We define the soft topological game $G(K, \tau, \mathcal{E})$,where $\mathcal{E}$ is the set of operators

on K, as follows: if $(F_0, E_0)\tilde{=}(X,\tau,\mathcal{E})$
I: $(F_1, E_1) \quad (F_3, E_3) \quad (F_5, E_5) \cdots$
II: $\quad (F_2, E_2) \quad (F_4, E_4) \quad (F_6, E_6) \cdots$
where $(F_{2n+1}, E_{2n+1}) \tilde{\in} C(X,\tau,\mathcal{E})\tilde{\cap}K$, $(F_{2n+2}, E_{2n+2}) \tilde{\in} C(X,\tau,\mathcal{E})$, $(F_{2n+1}, E_{2n+1})\tilde{\cup}(F_{2n+2}, E_{2n+2})\tilde{\subset} (F_{2n}, E_{2n})$ and $(F_{2n+1}, E_{2n+1}) \tilde{\cap}(F_{2n+2}, E_{2n+2}) \tilde{=}\tilde{\phi}$ for each $n \geq 0$. I wins the play $((F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots)$ if $\tilde{\bigcap\limits_{n=0}^{\infty}} (F_{2n}, E_{2n})\tilde{=} \tilde{\phi}$, otherwise, II wins.

Definition 5.1. A soft function s is a soft strategy for player I in $G(K,\tau,\mathcal{E})$ if the domain consists of finite sequences $((F_0, E_0), (F_1, E_1), \cdots, (F_{2n}, E_{2n}))$ if $s((F_0, E_0), (F_1, E_1), \cdots, (F_{2n}, E_{2n}))\tilde{\subset} C(X,\tau,\mathcal{E})$ and $((F_0, E_0), (F_1, E_1), \cdots, (F_{2n}, E_{2n}), (F_{2n+}, E_{2n+}))$ satisfies the conditions of the soft game $G(K,\tau,\mathcal{E})$ for each $n \geq 0$. If I wins the play $((F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots)$, then s is called a soft winning strategy for player I.

The game $G^{'}(K,\tau,\mathcal{E})$ define as follows:
I: $\quad (F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots$
II: $\quad (F_1, U_1), (F_2, U_2), (F_3, U_3), \cdots$
where $(F_n, E_n)\tilde{\in}C(X,\tau,\mathcal{E})\tilde{\cap}K$ and (F_n, U_n) is a soft set such that $(F_n, U_n)\tilde{\supset} (F_n, E_n)$ for each $n \in \mathbb{N}$. I wins the play $((F_1, E_1), (F_1, U_1), (F_2, E_2), (F_2, U_2), \cdots)$ of $G^{'}(K,\tau,\mathcal{E})$ if $\{(F_n, U_n) : n \in \mathbb{N}\}$ soft covers X, otherwise, II wins.

Remark 5.2. If $K\tilde{=}\{0\}\tilde{\cup}\{\{x\} : x\tilde{\in}X\}$, where $\{0\}$ is null soft space. It is clear that $G^{'}(K,\tau,\mathcal{E})$ coincides the game $G(K,\tau,\mathcal{E})$.

Remark 5.3. The soft winning strategies in the game $G^{'}(K,\tau,\mathcal{E})$ dependent on the opponent's moves only, that means a soft strategy s of I defined for $\tilde{\phi}$ and $((F_1, U_1), (F_2, U_2), (F_3, U_3), \cdots, (F_n, U_n))$, while a soft strategy t of II is defined for $((F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots, (F_n, E_n))$.

Theorem 5.4. *The soft games $G^{'}(K,\tau,\mathcal{E})$ and $G(K,\tau,\mathcal{E})$ are equivalent.*

Proof. Let s be a soft winning strategy for I in $G(K,\tau,\mathcal{E})$ and a winning strategy t for him in $G^{'}(K,\tau,\mathcal{E})$. Let $(F_0, E_0)\tilde{=}(F_0, E_0)^{'}\tilde{=}(X,\tau,\mathcal{E})$, $(F_1, E_1)\tilde{=}s((F_0, E_0))$, $(F_1, E_1)\tilde{=}(F_1, E_1)^{'}$ and $t((F_0, E_0)^{'})\tilde{=}(F_1, E_1)^{'}$. Since $(F_1, E_1)\tilde{\in}C(X,\tau,\mathcal{E})\tilde{\cap}K$, then $(F_1, E_1)^{'}$ is so. Let us consider II chooses a soft set (F_1, U_1) and $(F_1, U_1)\tilde{\supset}(F_1, E_1)^{'}$ and $(F_2, E_2)\tilde{=}(X,\tau,\mathcal{E}) - (F_1, U_1)$, $(F_3, E_3)\tilde{=}s((F_0, E_0), (F_1, E_1), (F_2, E_2))$, $(F_2, E_2)^{'}\tilde{=}(F_3, E_3)$ and $t((F_1, U_1))\tilde{=}(F_2, E_2)^{'}$. Since $(F_3, E_3)\tilde{\in}C(X,\tau,\mathcal{E})\tilde{\cap}K$, then $(F_2, E_2)^{'}$ is so. Continuing in the same manner, we get the play $((F_1, E_1), (F_2, E_2), (F_3, E_3), \cdots)$ of $G(K,\tau,\mathcal{E})$ and the play $((F_1, E_1)^{'}, (F_1, U_1), (F_2, E_2)^{'}, (F_2, U_2), (F_3, E_3)^{'}, \cdots)$ of $G^{'}(K,\tau,\mathcal{E})$. Since s is a soft winning strategy of $G(K,\tau,\mathcal{E})$, then $\tilde{\bigcup\limits_{n=1}^{\infty}} ((X,\tau,\mathcal{E}) - (F_{2n}, E_{2n}))\tilde{=}(X,\tau,\mathcal{E})$, by hypothesis, $(X,\tau,\mathcal{E}) - (F_{2n}, E_{2n})\tilde{=}(F_{2n}, U_{2n})$ for each $n \in \mathbb{N}$. Therefore t is a soft winning strategy. □

Definition 5.5. A family $\mathcal{U}$ of soft sets in a soft topological space $(X,\tau,\mathcal{E})$ is said to be K-soft cover if for each a soft closed set (F, E) of $(X,\tau,\mathcal{E})$ with $(F, E)\tilde{\in}K$, there is $(F, U)\tilde{\in}\mathcal{U}$ so that $(F, E)\tilde{\subset}(F, U)$. In other words, the set of all $C(X,\tau,\mathcal{E})\tilde{\cap}K$ is a soft refinement $\mathcal{U}$.

Now, we define the dual soft game $G^*(K,\tau,\mathcal{E})$ as follows:
I: $\mathcal{U}_1$ $\quad\mathcal{U}_2$ $\quad\mathcal{U}_3\ \cdots$
II: (F_1,U_1) $\quad(F_2,U_2)$ $\quad(F_3,U_3)\ \cdots$ where (F_n,U_n) is a K-soft cover of (X,τ,E) and $(F_n,U_n)\in\mathcal{U}_n$ for each $n\in\mathbb{N}$. I wins the play $(\mathcal{U}_1,(F_1,U_1),\mathcal{U}_2,(F_2,U_2),\cdots)$ of $G^*(K,\tau,\mathcal{E})$ if $\{(F_n,U_n):n\in\mathbb{N}\}$ soft covers $(X,\tau,\mathcal{E})$, otherwise, II wins.

Proposition 5.6. Let $(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3,\cdots)$ be a sequence of K-soft covers of $(X,\tau,\mathcal{E})$ and let s be a soft strategy for I in $G^*(K,\tau,\mathcal{E})$. Then there is a soft closed subset (F,E) of $(X,\tau,\mathcal{E})$ with $(F,E)\tilde{\in}K$ such that for each soft set (F,U) containing (F,E), there is a soft cover $\mathcal{U}$ of $(X,\tau,\mathcal{E})$ such that $(F,U)\tilde{=}s(\mathcal{U}_1,\mathcal{U}_2,\cdots,\mathcal{U}_n,\mathcal{U})$.

Proof. Let $\mathcal{V}$ be the family of all soft sets in $(X,\tau,\mathcal{E})$ that are not of the form $s(\mathcal{U}_1,\mathcal{U}_2,\cdots,\mathcal{U}_n,\mathcal{U})$ for some K-soft cover of X. It follows that $\mathcal{V}$ is not K_α-cover of $(X,\tau,\mathcal{E})$, because otherwise $s(\mathcal{U}_1,\mathcal{U}_2,\cdots,\mathcal{U}_n,\mathcal{U})\tilde{\in}\mathcal{V}$ which contradicts the definition of $\mathcal{V}$. Now, suppose each soft closed subset (F,E) of $(X,\tau,\mathcal{E})$ with $(F,E)\tilde{\in}K$ has a soft set $(F,V)\in\mathcal{V}$. Then, again $\mathcal{V}$ is a K-soft cover of $(X,\tau,\mathcal{E})$ and this is a contradiction. □

Theorem 5.7. *The soft games $G^*(K,\tau,\mathcal{E})$ and $G^{'}(K,\tau,\mathcal{E})$ are equivalent.*

Proof. Let s be a soft winning strategy of I in $G^*(K,\tau,\mathcal{E})$. A soft winning strategy for him in $G^{'}(K,\tau,\mathcal{E})$ is defined as follows: We set $t(\tilde{\phi})\tilde{=}(F_1,E_1)$ where (F_1,E_1) is any soft closed subset of $(X,\tau,\mathcal{E})$ with $(F_1,E_1)\tilde{\in}K$ and with the property each soft set containing (F_1,E_1) has the form $s(\mathcal{U})$ for some K-soft cover $\mathcal{U}$ of $(X,\tau,\mathcal{E})$. Proposition 5.6 assures the existence of (F_1,E_1). Let (F_1,U_1) be any soft set in $(X,\tau,\mathcal{E})$ containing (F_1,E_1). Then there is K-soft cover $\mathcal{U}_1$ of $(X,\tau,\mathcal{E})$ so that $(F_1,U_1)\tilde{=}s(\mathcal{U}_1)$. Again by Proposition 5.6, there is a soft closed subset (F_2,E_2) in $(X,\tau,\mathcal{E})$ with $(F_2,E_2)\tilde{\in}K$ and with property, each soft set containing (F_2,E_2) has the form $s(\mathcal{U}_1,\mathcal{U})$ for some K-soft cover $\mathcal{U}$ of $(X,\tau,\mathcal{E})$. We set $t(\mathcal{U}_1)\tilde{=}(F_2,E_2)$, and so on. Finally, we get the paly $((F_1,E_1),\mathcal{U}_1,(F_2,E_2),\mathcal{U}_2,\cdots)$ of $G^{'}(K,\tau,\mathcal{E})$ and the play $(\mathcal{U}_1,(F_1,U_1),\mathcal{U}_2,(F_2,U_2),\cdots)$ of $G^*(K,\tau,\mathcal{E})$ where $(F_n,U_n)\tilde{=}s(\mathcal{U}_1,\mathcal{U}_2,\cdots,\mathcal{U}_n)$ for each $n\in\mathbb{N}$. Thus t is a soft winning strategy for I in $G^{'}(K,\tau,\mathcal{E})$. Conversely, let s be a soft winning strategy for I in $G^{'}(K,\tau,\mathcal{E})$. We define a soft winning strategy t of him in $G^*(K,\tau,\mathcal{E})$ as follows: let $\mathcal{U}_1$ be a K-soft cover of X. Then we set $t(\mathcal{U}_1)\tilde{=}U_1$, where U_1 is any set from $\mathcal{U}_1$ containing $s(\phi)$. Let $\mathcal{U}_2$ be a K-soft cover of $(X,\tau,\mathcal{E})$. Then we set $t(\mathcal{U}_1,\mathcal{U}_2)\tilde{=}(F_2,U_2)$ where (F_2,U_2) is any soft set from $\mathcal{U}_2$ containing $s((F_1,U_1))$ and so on. Finally, we get the play $(\mathcal{U}_1,(F_1,U_1),\mathcal{U}_2,(F_2,U_2),\cdots)$ of $G^*(K,\tau,\mathcal{E})$, where $s((F_1,U_1),(F_2,U_2),\cdots,(F_n,U_n))\tilde{\in}\mathcal{U}_n$. Thus t is a soft winning strategy for I in $G^{'}(K,\tau,\mathcal{E})$. We can also prove the theorem for II: let s be a soft winning strategy for II in $G^*(K,\tau,\mathcal{E})$. Then a soft winning strategy t for him in $G^{'}(K,\tau,\mathcal{E})$ is defined as follows: let (F_1,E_1) be a soft closed subset of $(X,\tau,\mathcal{E})$ with $(F_1,E_1)\tilde{\in}K$. Then a K-soft cover $s(\tilde{\phi})$ of $(X,\tau,\mathcal{E})$ contains a soft set (F_1,U_1) containing (F_1,E_1). We set $s((F_1,E_1))\tilde{=}(F_1,U_1)$. Let (F_2,E_2) be a soft closed subset of $(X,\tau,\mathcal{E})$ with $(F_2,E_2)\tilde{\in}K$. Then a K- soft cover $s((F_1,U_1))$ contains a soft set (F_2,U_2) containing (F_2,E_2). We set $t((F_1,E_1),(F_2,E_2))\tilde{=}(F_2,U_2)$, and so on. Since $\{(F_n,U_n):n\in\mathbb{N}\}$ not soft covers $(X,\tau,\mathcal{E})$, it follows that t is a soft winning strategy for II in $G^{'}(K,\tau,\mathcal{E})$. We define a soft winning strategy t of him in $G^*(K,\tau,\mathcal{E})$ as follows: we set $\mathcal{U}_1\tilde{=}\{s((F,E)):(F,E)\tilde{\in}C(X,\tau,\mathcal{E})\tilde{\cap}K\}$ and $t(\tilde{\phi})\tilde{=}\mathcal{U}_1$. Let $(F_1,U_1)\tilde{\in}\mathcal{U}_1$. Then there is a soft closed subset (F_1, E_1) of $(X, \tau, \mathcal{E})$ with $(F_1, E_1)\tilde{\in}K$ and $s((F_1, E_1))\tilde{=}(F_1, U_1)$. Now, we set $\mathcal{U}_2\tilde{=}\{s((F_1, E_1), (F, E)) : (F, E)\tilde{\in}C(X, \tau, \mathcal{E})\tilde{\cap}K\}$ and $t((F_1, U_1))\tilde{=}\mathcal{U}_2$, and so on. We get the play $(\mathcal{U}_1, (F_1, U_1), \mathcal{U}_2, (F_2, U_2), \cdots)$ of $G^*(K, \tau, \mathcal{E})$. since $\{(F_n, U_n) : n \in \mathbb{N}\}$ not soft covers X. Then t is a soft winning strategy of II in $G^*(K, \tau, \mathcal{E})$. □

Conclusion

In terms of the notion of a soft set, we define some notions of soft topological games with perfect information. We investigate some properties of it and its connection with some of each other. We use the definition of soft winning strategies to give a generalization for the concept of equivalence between these soft games.

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