Poincare Journal of Analysis & Applications
Vol. 8, No. 1(I) (2021), Special issue, 201-207

DOI: 10.33786/pjaa.2021.v08i01(i).019

STRICTLY S-MENGER BOUNDED GROUPS

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Date of Receiving : 09. 08. 2020
Date of Acceptance : 09. 09. 2020

Abstract. In this paper, we define and investigate s-Menger-bounded groups. Also, we define strictly s-Menger-bounded groups using infinite topological game between two players. It is shown that strictly s-Menger-bounded group structure is preserved under irresolute epimorphism and possess open hereditary property.

1. Introduction

Throughout in this paper, a space mean an infinite topological space, e mean identity of the group G and along with the generally accepted symbols, we use our own notations and those from [13] and [19]. Let $\mathcal{U}$ and $\mathcal{V}$ are families of set X. $\mathcal{S}_{fin}(\mathcal{U},\mathcal{V})$ (resp. $\mathrm{S}_1(\mathcal{U},\mathcal{V})$) represents the selection hypothesis: For each $(\mathcal{U}_i)_{i\in N} \in \mathcal{U}$ there is $(\mathcal{V}_i)_{i\in N}$ (resp. $(b_i)_{i\in N}$) such that for each $i \in N$, a finite set $\mathcal{V}_i \subset \mathcal{U}_i$ (resp. $b_i \in \mathcal{U}_i$), and $\bigcup_{i\in N} \mathcal{V}_i$ (resp. $\{b_i\}_{i\in N}$) $\in \mathcal{V}$ (see [23]).
The Menger (M) (resp. Rothberger (R)) covering property of a space X is given as $\mathcal{S}_{fin}(\mathcal{O},\mathcal{O})$ (resp. $S_1(\mathcal{O},\mathcal{O})$) and s-Menger (sM) (resp. s-Rothberger (sR)) covering property [13, 19], is given as $\mathcal{S}_{fin}(s\mathcal{O},s\mathcal{O})$ (resp. $\mathrm{S}_1(s\mathcal{O},s\mathcal{O})$). The family of all open and semi open covers of a space X is represented as $\mathcal{O}$ and $s\mathcal{O}$ respectively.
In 1924 M property was demonstrated by Karl Menger with name of M basis property [16] and next year W. Hurewicz [9] verified an if and only if relation between M basis property and M covering property for a metric space. For more data about selection principles (SP) theory, we allude the peruser to see [14, 21, 22, 25].
For the SP $\mathcal{S}_{fin}(\mathcal{U},\mathcal{V})$ the infinite game, represented as $G_{fin}(\mathcal{U},\mathcal{V})$ [24]. In this game Tom and Jerry, play an inning per positive integer. In the ith turn Tom chooses $U_i \in \mathcal{U}$, and then Jerry counters with a finite set $V_i \subseteq U_i$. Thusly they set up a play U_1, V_1,..., U_i, V_i, ... such a play is won by Jerry if $\bigcup\{V_i : i \in N\} \in \mathcal{V}$; else, Tom wins. When $\mathcal{U}$ and $\mathcal{V}$ are open covers of the space or topological group (TG) then the game is known as M-game.
For the SP $\mathcal{S}_1(\mathcal{U},\mathcal{V})$ the infinite game, represented as $G_1(\mathcal{U},\mathcal{V})$ [5]. In this game Tom

2010 *Mathematics Subject Classification.* 91A44, 54C08, 54D20, 54H10.
Key words and phrases. irresolute mapping, irresolute topological group, s-Menger-bounded group, strictly s-Menger-bounded group.
Communicated by. M. L. Thivagar and S. Jafari

and Jerry, play an inning per positive integer. In the ith turn Tom chooses $U_i \in \mathcal{U}$, then Jerry counters with a $V_i \in U_i$. Thusly they set up a play U_1, V_1,..., U_i, V_i, ... such a play is won by Jerry if $\{V_i : i \in N\} \in \mathcal{V}$; else, Tom wins. When $\mathcal{U}$ and $\mathcal{V}$ are open covers of the space or TG then the game is known as R-game.

Definition 1.1. A TG $(G, *, \tau)$ is known as M-bounded (resp. R-bounded) [12] if there is for each progression $(U_i)_{i\in N}$ of neighborhoods (nbhds) of $e \in G$, a progression $(V_i)_{i\in N}$ (resp. $(x_i)_{i\in N}$) of finite subset (resp. elements) of G such that $G = \bigcup_{i\in N} V_i * U_i$ (resp. $G = \bigcup_{i\in N} x_i * U_i$).

Definition 1.2. A TG $(G, *, \tau)$ is:

(1) $\aleph_0$-bounded [7] if there is for every nbhd U of $e \in G$, a countable set $K \subset G$ such that $K * U = G$.
(2) o-bounded [8] if there are for every U_0, U_1, ... of nbhds of $e \in G$, finite sets F_0, F_1, ... such that $\bigcup_{i\in\mathbb{N}} F_i * U_i = G$.
It is known in [1] that o-bounded TG is equivalent to M-bounded TG.

Definition 1.3. [8] Suppose that $(G, *, \tau)$ is a TG and that we have two players, Tom and Jerry. Tom chooses an open nbhd U_0 of e in G. Then, Jerry chooses a finite set $F_0 \subset G$. In the second turn, Tom chooses another nbhd U_1 of e in G and Jerry chooses a finite set $F_1 \subset G$. The game continues in the same way until we have U_0, U_1, ... of nbhds of the neutral element and F_0, F_1, ... of finite subsets of G. Jerry wins if $\bigcup_{i\in N} F_i * U_i = G$. Otherwise, Tom wins. Then TG $(G, *, \tau)$ is strictly M-bounded group if there is a winning strategy for Jerry.

Definition 1.4. A mapping $\alpha : (X, \tau) \to (Y, \tau)$ is:

(1) irresolute [3] (resp. pre-semi open) if inverse image (resp. image) of semi open set in Y (resp. X) is semi open in X (resp. Y).
(2) semi homeomorphism [15] if it is bijective, irresolute and pre-semi open.

Lemma 1.5. *A space X is extremely disconnected* [10] *(resp. semi P-space* [20]*) if intersection of two (resp. countable) semi open subsets of X is semi open.*

Lemma 1.6. [18] *Let Y be a subspace of X and $P \in SO(Y)$, then $P = Q \cap Y$ for some $Q \in SO(X)$.*

Definition 1.7. [11] A triplet $(G, *, \tau)$ is termed as irresolute topological group (ITG) if for each $p, q \in G$ and each semi open nbhd R of $p * q^{-1}$ in G there is a semi open nbhds P containing p and Q containing q such that $P * Q^{-1} \subset R$.
We note that, if $(G, *, \tau)$ is ITG, then a mapping $\mu : G \times G \to G$ is an irresolute mapping but, if $\mu : G \times G \to G$ is an irresolute mapping, then $(G, *, \tau)$ may not be an ITG.

Lemma 1.8. [11] *For an ITG $(G, *, \tau)$:*

(1) Left translation is semi homeomorphism and the inverse mapping is homeomorphism.
(2) $P \in SO(G)$ if and only if $P^{-1} \in SO(G)$.
(3) If $P \in SO(G)$ and $Q \subset G$, then $P * Q$ and $Q * P$ are both in $SO(G)$.

Lemma 1.9. [2] *Let $(G, *, \tau)$ be an ITG, and ϵ_e be the collection of all semi open nbhds of e, then:*

(1) there is for each $P \in \epsilon_e$, a $Q \in \epsilon_e$ such that $Q^{-1} \subseteq P$.
(2) there is for each $P \in \epsilon_e$, a $Q \in \epsilon_e$ such that $Q^2 \subseteq P$.
(3) there is for each $P \in \epsilon_e$ and every $x \in P$, a $Q \in \epsilon_e$ such that $x * Q \subseteq P$.
(4) there is for each $P \in \epsilon_e$, a $Q \in \epsilon_e$ such that $Q^{-1} * Q \subseteq P$.

Lemma 1.10. [17] *Let $\{Y_\alpha : \alpha \in \nabla\}$ be any family of topological spaces, $Y = \Pi_{\alpha\in\nabla} Y_\alpha$ the product space and $P = \Pi_{j=1}^{n} P_{\alpha j} \times \Pi_{\alpha\neq\alpha_j} Y_\alpha$ a non empty subset of Y, where n is a positive integer then $P_{\alpha j} \in SO(X_{\alpha j})$; $\forall 1 \leq j \leq n$ if and only if $P \in SO(X)$.*

Definition 1.11. A space X is termed as semi compact [4] (resp. semi-Lindelöf [6]) if every cover of X by semi open sets have a finite (resp. countable) subcover.

2. s-Menger-bounded groups

In this section we define and explore s-Menger-bounded and s-Rothberger-bounded groups.

Definition 2.1. An ITG $(G, *, \tau)$ is:

(1) sM-bounded if there is for each progression $(U_i)_{i\in N}$ of semi open nbhds of $e \in G$, a progression of finite set $(V_i)_{i\in N} \subset G$ such that $G = \bigcup_{i\in N} V_i * U_i$.
(2) sR-bounded if there is for each progression $(U_i)_{i\in N}$ of semi open nbhds of $e \in G$, a progression $(x_i)_{i\in N} \in G$ such that $G = \bigcup_{i\in N} x_i * U_i$.

There is an M-bounded TG which is not sM-bounded ITG.

Example 2.2. We note that real line $(R, +, \tau)$ with usual topology is an ITG. It is given in [19], that it is not s-Menger space but Menger space. Since $(R, +, \tau)$ is TG so it is M-bounded group but not sM-bounded group because it is not s-Menger space.

Theorem 2.3. *Every open subgroup $(H, *)$ of an sM-bounded group $(G, *, \tau)$ is sM-bounded.*

Proof. The proof is easy. □

Theorem 2.4. *Let (G, τ) be a semi P-space. Then semi-Lindelöf, ITG $(G, *, \tau)$ is sM-bounded.*

Proof. Consider that a progression $(P_i)_{i\in N}$ of semi open nbhds of e in G. As G is semi P-space so $U = \bigcap_{i\in N} P_i$ is also semi open nbhd in G containing e_G. Due to semi-Lindelöfness of G, we have countable subset $M = \{x_1, x_2, ...\}$ of G such that $\bigcup_{i\in N} x_i * U = G$. Let $Q_i = \{x_i\}$ for each $i \in N$, then $\bigcup_{i\in N} Q_i * P_i = G$. Therefore, G is sM-bounded. □

Theorem 2.5. *Let G and H be ITGs and $\alpha : (G, \tau) \to (H, \tau)$ be an irresolute epimorphism. If G is an sM-bounded group, then H is sM-bounded.*

Proof. The proof is easy. □

Definition 2.6. An ITG $(G, *, \tau)$ is n_0-bounded if there is for every semi open nbhd P of $e \in G$, a countable subset Q of G such that $Q * P = G$.

Theorem 2.7. *An n_0-bounded group $(G, *, \tau)$ is sM-bounded if and only if all second countable irresolute homomorphic images of G are sM-bounded.*

Proof. The necessity of the condition follows from Theorem 2.5. Consider that G is an n_0-bounded group whose second countable, irresolute, homomorphic images are sM-bounded. Let $(U_i)_{i\in\mathbb{N}}$ be semi open nbhds of e in G. Then by Lemma 1.9(2), for each $i \in \mathbb{N}$ there is a semi open nbhd V_i of e such that $V_i^2 \subseteq U_i$.
As G is an n_0-bounded group, for every V_i one can prove an irresolute homomorphism $\psi_i : G \to H_i$ onto a second countable group H_i and semi open nbhd V_i^* of e in H_i such that: $\psi_i^{-1}(V_i^*) \subseteq V_i$. Let $\varphi = \Delta_{i\in\mathbb{N}}\psi_i : G \to \Pi_{i\in\mathbb{N}}H_i$ be the diagonal product of the homomorphism ψ_i. The group $K = \varphi(G)$ is second countable and, by assumption sM-bounded. Consider the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\psi_i"'] & K \subseteq \prod_{i\in\mathbb{N}} H_i \arrow[d, "f_i"] \\ & H_i \end{tikzcd}$$

where f_i is the restriction of the projection $\pi_i : \Pi_{i\in\mathbb{N}}H_j \longrightarrow H_i$. For every $i \in \mathbb{N}$, define $W_i = f^{-1}(V_i^*)$. Since K is sM-bounded, there is $\{E_i : i \in \mathbb{N}\}$ of finite subsets of K such that $K = \bigcup_{i\in\mathbb{N}}E_i * W_i$. For every $x \in E_i$, choose $g_x \in \varphi^{-1}(x)$ and define $F_i = \{g_x : x \in E_i\}$. Let us show that $G = \bigcup_{i\in\mathbb{N}}F_i * U_i$. For every $g \in G$, there exist $i \in \mathbb{N}$ and $y \in E_i$ such that $\varphi(g) \in y * W_i$. Since $N = \ker\varphi \subseteq V_i$ for each $i \in \mathbb{N}$. We have $g \in \varphi^{-1}(y * W_i) = (g_y * N) * \varphi^{-1}(W_i) \subseteq (g_y * N) * V_i \subseteq g_y * V_i^2 \subseteq g_y * U_i$. Therefore, $g \in F_i * U_i$, and hence $G = \bigcup_{i\in\mathbb{N}}F_i * U_i$. So we have that G is sM-bounded. □

Remark 2.8. The class of sM-bounded groups is not productive in general.

Example 2.9. Consider the additive group of reals R endowed with usual topology. We note that $(R, +, \tau)$ with usual topology is ITG but not sM-bounded group. Moreover $(R^2, +, \tau_R^2)$ is also ITG. Let us prove that R^2 is not sM-bounded. Then by Lemma 1.10, for each $n \in N$, pick a semi open nbhd U_n of (0,..., 0), $U_n = \Pi_{i=1}^k V_{n,i} \times \Pi_{i>k}R_i$. For $V_{n,i}$ we have four possibilities: $(-1, 1)$, $[0, 1)$, $(-1, 0]$ and $[-1, 1]$.
Case I: $V_{n,i} = (-1, 1)$ if $j \le i$ and $V_{n,i} = R$ otherwise. For $n = 1$ we have $U_1 = (-1, 1) \times R$. For any $\{F_i : i \in N\}$ finite subsets of R^2, we can define a point in R^2 outside of $\bigcup_{i=1}^{\infty} F_i + U_1$. Indeed, we consider the point $x = (x_i)_{i\in N} \in R^2$, where ith coordinate x_i satisfies $x_i \in R-(p_i(F_i)+p_i(U_1))$, where $p_i : R^2 \to R$ is the projection. It is not difficult to see that $x \notin \bigcup_{i\in N} F_i + U_1$. Hence, the group R^2 is not sM-bounded group. For first case $(R^2, +, \tau_R^2)$ is not sM-bounded group. Therefore, there is no need to check the remaining possibilities.

Theorem 2.10. *Let $(G, *, \tau_1)$ be an extremely disconnected, sM-bounded group and $(H, *, \tau_2)$ be hereditarily semi-Lindelöf, ITG. Then the product $(G \times H, \tau_1 \times \tau_2)$ is hereditarily sM-bounded.*

Proof. Let S be the subset of the $G \times H$. Moreover let $(U_i)_{i\in N}$ and $(V_i)_{i\in N}$ are progressions of semi open nbhds of e_G and e_H respectively. As G is sM-bounded so there are finite sets P_i in G such that $\bigcup_{i\in N} P_i * W_i = G$, where $W_i = U_i \cap U_i^{-1}$. For each $i \in N$ and each $p \in P_i$ define, $Q(i,p) = \{h \in H$: there is g in a W_i such that $(g,h) \in S\}$. Fix $i \in N$ and $p \in P_i$. Since H is hereditarily semi Lindelöf, hence $Q(i,p)$ is semi Lindelöf, there is a finite set $M(i,p) \subset Q(i,p)$ such that $\bigcup_{i\in N} M(i,p) * V_i = H$. For each $y \in M(i,p)$ there is $g(y,i,p) \in p * W_i$ such that $(g(y,i,p),y) \in S$. The set $A(i,p) = \{(g(y,i,p),y) : y \in M(i,p)\}$ is a finite subset of S. Take any $(g,h) \in S$ there are $k \in N$ and $p \in P_k$ such that $g \in p * W_k$. Therefore $h \in Q(k,p)$ and there exists some $b \in Q(k,p)$ such that $h \in b * V_k$. Thus $(g(b,k,p),b) \in A(k,p)$, where $g(b,k,p) \in p * W_k$. Using the symmetry of W_k, we get $(g,h) \in g(b,k,p) * (W_k)^2 \times b * V_k$. This means that the finite sets $A_k = \bigcup_{p\in P_k} A(k,p)$ witness that $G \times H$ is sM-bounded. □

Theorem 2.11. *Let* (G,τ) *be a semi P-space. Then semi-Lindelöf, ITG* $(G,*,\tau)$ *is sR-bounded.*

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.12. *Every open subgroup* $(H,*)$ *of sR-bounded group* $(G,*,\tau)$ *is sR-bounded.*

Proof. The proof is easy. □

3. Strictly s-Menger-bounded groups

The following concept is defined in terms of infinite topological game.

Definition 3.1. Consider that $(G,*,\tau)$ is an ITG and an infinite topological game is played between two players, Tom and Jerry. Tom chooses a semi open nbhd U_0 of e in G. Then, Jerry chooses a finite set $F_0 \subset G$. In the second turn, Tom chooses another semi open nbhd U_1 of e in G and Jerry chooses a finite set $F_1 \subset G$. The game continues in the same way until we have U_0, U_1, ... of semi open nbhds of e and F_0, F_1, ... of finite subsets of G. Jerry wins if $\bigcup_{i\in N} F_i * U_i = G$. Otherwise, Tom wins. Then we say that the ITG $(G,*,\tau)$ is strictly sM-bounded if there exists a winning strategy for the Jerry.

Remark 3.2. Every strictly sM-bounded group is sM-bounded group but converse is not true in general.

The following example shows that there is an sM-bounded group which is not strictly sM-bounded.

Example 3.3. Let R be a real line, and R^N its countable powers. The additive group R^N is ITG. R^N is not strictly sM-bounded because R^N is not strictly M-bounded [8]. So we just have to prove R^N is sM-bounded. Let $\beta = \{(p,q) : p < q; p,q \in R\}$ be a base for real line R, then there exists four types of semi open sets in R of the form : (p,q), $[p,q)$, $(p,q]$ and $[p,q]$.
Case I: Consider that $(U_n)_{n\in N}$ be semi open nbhds of (0,..., 0) $\in R^N$. Then by Lemma 1.10, for each $n \in N$, pick a semi open nbhd V_n of (0,..., 0), $V_n = \Pi_{i=1}^{k} V_{n,i} \times \Pi_{i>k} R_i$ such that $V_n+ \ldots +V_n \subset U_n$ (n times taken V_n). Define an increasing sequence $m_1 < m_2 <\ldots$ in N such that $\{j \in N : V_{n,i} \neq R = R_j\} \subset \{1, 2, \ldots, m_n\}$ for each n. To each m_i associate the set $Q_i \subset R^N$ defined by $Q_i = \Pi_{n\in N} P_{i,j}$, where $P_{i,j} = (-m_{i+1}, m_{i+1})$, for $j \leq m_i$, and $P_{i,j} = R$, for $j > m_i$. Observe that $X \subset \bigcup_{n\in N} Q_n$. "Indeed, if

$x = (x_n)_{n\in N} \in X$. Then $\lim_{k\to\infty} \frac{x_{n_k}}{n_{k+1}} = 0$ implies that there is $s \in N$ such that $| x_{n_k} |< n_{k+1}$ for all $k \geq s$. Put $t = \max\{x_{n_1},..., x_{n_s}\}$ and choose i^* and l in N such that $m_{i^*} > t$ and $m_{i^*} \leq n_l < m_{i^*+1}$. Let $k \in N$ be arbitrary. If $n_k > m_{i^*}$, then $P_{i^*,\, n_k} = R$, hence $x_{n_k} \in P_{i^*,\, n_k}$; if $k < s$ and $n_k < m_{i^*}$, then $| x_{n_k} |\leq t < m_{i^*}$ so that $x_{n_k} \in P_{i^*,\, n_k}$; if $k \geq s$ and $n_k \leq m_{i^*}$, then $k < l$, hence $| x_{n_k} |< n_{k+1} < m_{i^*+1}$, which implies $x_{n_k} \in P_{i^*,\, n_k}$, and thus $| x_j |\in P_{i^*,\, j}$ for each $j \in N$. So, $x \in Q_{i^*}$.
For each $n \in N$ choose a finite set $F_n \subset R^N$ such that $Q_n \subset F_n + V_n$. Let $E_n = F_n+...+F_n$ (n times F_n). We prove that $(E_n)_{n\in N}$ witnesses for $(U_n)_{n\in N}$ that R^N is sM-bounded. Let $g \in R^N$; then $g = x_1+...+x_r$, $x_i \in X$ for each $i \leq r$. We have that for each $i \leq r$, $x_i \in Q_{k_i}$ for some $k_i \in \{1, 2,..., r\}$. If we put $k = \max\{k_1,..., k_r, r\}$, then $x_i \in Q_k$ for each $i \leq r$. It follows $g = x_1+...+x_r \in r(F_k + V_k) \subset k(F_k + V_k) \subset E_k + U_k$.
Case II: For $P = [-m_{i+1}, m_{i+1})$ and $P = [-m_{i+1}, m_{i+1}]$ we follow the same argument.
Case III: For $P = (-m_{i+1}, m_{i+1}]$, define a decreasing sequence $m_1 > m_2 >...$ in N, then follow the same way for this decreasing sequence. In each case, it is proved that $(R^N,*,\tau)$ is sM-bounded group.

Theorem 3.4. *Every open subgroup $(H, *)$ of a strictly sM-bounded group $(G, *, \tau)$ is strictly sM-bounded.*

Proof. Consider that G be a strictly sM-bounded group with e_G. Suppose that in turn i, Tom chooses a semi open nbhd U_i of e in H. So by Lemma 1.6, there is a semi open nbhd V_i of e in G such that $V_i \cap H = U_i$ and by definition of ITG, there exists $W_i \in SO(G, e_G)$ such that $W_i^{-1} * W_i \subseteq V_i$. As G is strictly sM-bounded so Jerry can choose the finite F_i corresponding to the winning strategy in G for W_i in such a way that $\bigcup_{i\in N} F_i * W_i = G$. If $x \in F_i$ and $x * W_n$ intersects H, we choose $a_x \in H \cap (x * W_i)$, otherwise we put $a_x = e_G$. It is easy to see that $A_i = \{a_x : x \in F_i\}$ is finite. Let us verify that $\cup_{i\in N} A_i * U_i = H$. It is clear that $\cup_{i\in\mathbb{N}} A_i * U_i \subseteq H$. If $y \in H$, there is $x \in F_i$ such that $y \in x * W_i$ for some $i \in \mathbb{N}$. So, $a_x \in H \cap (x * W_i)$ i.e., $x \in a_x * W_i^{-1}$. Hence, $y \in x * W_i \subseteq a_x * W_i^{-1} * W_i \subseteq a_x * V_i$. Since y and a_x are elements of H, it follows that $a_x^{-1} * y \in V_i \cap H \subseteq U_i$. Thus, $y \in a_i * U_i \subseteq A_i * U_i$. □

Theorem 3.5. *Consider that $(G, *_1, \tau)$ and $(H, *_2, \tau)$ be ITGs. Let $\alpha : (G, \tau_1) \to (H, \tau_{2)}$ be an irresolute epimorphism and G be a strictly sM-bounded. Then H is strictly sM-bounded.*

Proof. Suppose that at step i Tom chooses a nbhd U_i of e_H in H. For each $i \in \mathbb{N}$, the set $V_i = \alpha^{-1}(U_i)$ is a semi open nbhd of e_G in G. As G is strictly sM-bounded, Jerry can choose the finite sets Q_i corresponding to the winning strategy in G for V_i. Then for each $i \in \mathbb{N}$, sets $P_n = \alpha(Q_n)$ are finite and $\bigcup_{i\in N} P_i *_2 U_i = \alpha(\bigcup_{i\in\mathbb{N}} Q_i *_1 V_i) = \alpha(G) = H$. □

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