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ON Ψ_δ-SETS AND Ψ_δ-FUNCTIONS DEFINED BY IDEALS

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Abstract. In this paper, we introduce two operator associate with Ψ and δ-*int* operators in ideal topological spaces and investigate their properties. We will use the composite of these operators to offer a finer topology on X than τ_δ by utilizing the new notions of $\Psi_\delta O(X)$. Also, we will use a special ideal to define ω_δ-open and ω_δ-closed sets. Finally, we will give some of the fundamental properties of such sets and some existing functions in the literature.

1. Introduction

The study of local function on ideal topological space was introduced by Kuratowski [12]. Many mathematicians such as Jankovic and Hamlet [6, 7]; Samuel [20]; Hayashi [8]; Modak and Islam [11, 17, 15] have enriched this study. Natkaneic [18] have introduced the complement of local function and it is called Ψ-operator. In an ideal topological space $(X, \tau, \mathcal{I})$, the local function $(\cdot)^*$ is defined as $A^*(\mathcal{I}, \tau)$ (or simply A^*) $:= \{x \in X : (\forall U \in \mathcal{U}(x))\,(U \cap A \notin \mathcal{I})\}$ where $\mathcal{U}(x)$ is the collection of all open subsets containing $x \in X$. It is complement function, that is Ψ-operator is defined as $\Psi(A) := X \setminus (X \setminus A)^*$. Hdeib introduced the notion of ω-closeness [9] by which he introduced and investigated the notion of ω-continuity. Fomin introduced the notion of θ-continuity [4] and the notion of δ-continuity has been defined by Noiri [19]. The notions of θ-open and δ-open set have been introduced by Velicko [22]. He also showed that the collection of all θ-open (resp. δ-open) sets in a topological space forms a topology on X denoted by τ_θ (resp. τ_δ)

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In this paper, we will use the composite of Ψ and δ-*int* operators to offer a finer topology on X, which will be denoted by $\Psi_\delta O(X)$, than τ_δ. By utilizing the new notions and using the ideal $\mathcal{I}_c := \{A \subseteq X || A| \leq \aleph_0\}$, we will also define one more topology, which will be denoted by $\omega_\delta O(X)$, finer than $\omega_\theta O(X)$ and τ_δ. In addition, We will discuss some of the fundamental properties of such space and some related maps.

Throughout this paper, (X, τ) (briefly X) means a topological space. For a subset A of a space X, the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively. A subset A of a space X is said to be preopen [13] (resp. regular open [21]) if $A \subseteq int(cl(A))$ (resp. $A = int(cl(A))$).

A point x of X is said to be in δ-closure (resp. θ-closure) [22] of a subset A of X, denoted δ-$cl(A)$ (resp. θ-$cl(A)$) if $int(cl(U)) \cap A \neq \emptyset$ (resp. $cl(U) \cap A \neq \emptyset$) for each open set U of X containing x. A subset A of space X is called to be δ-closed (resp. θ-closed) if δ-$cl(A) = A$ (resp. θ-$cl(A)$). The complement of a δ-closed (resp. θ-closed) is called δ-open (resp. θ-open). The δ-interior of a subset A of X is the union of all regular open sets of X whose contained in A and is denoted by δ-$int(A)$. The θ-interior of a subset A of X is the union of all open sets of X whose closures are contained in A and is denoted by θ-$int(A)$. Recall that a point x of X is called a condensation point of a subset A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A subset A of a space X is ω-closed [9] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. The union of all ω-open sets of X contained in A is called the ω-interior of A and is denoted by ω-$int(A)$. A subset A of X is called ω_θ-open [3] if for each $x \in A$ there exists an open subset $B \subseteq X$ containing x such that $B \setminus \theta$-$int(A)$ is countable. The complement of an ω_θ-open set is called ω_θ-closed.

The family of all ω-open (resp. ω_θ-open, preopen, δ-open, θ-open, regular open) subsets of X is denoted by $\omega O(X)$ (resp. $\omega_\theta O(X)$, $PO(X)$, $\delta O(X) = \tau_\delta$, $\theta O(X) = \tau_\theta$, $RO(X)$). The family of all ω-open (resp. ω_θ-open, preopen, δ-open, θ-open, regular open) sets of X containing a point x of X is denoted by $\omega O(X, x)$ (resp. $\omega_\theta O(X, x)$, $PO(X, x)$, $\delta O(X, x)$, $\theta O(X, x)$, $RO(X, x)$).

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-continuous [10] (resp. ω_θ-continuous [3], δ-continuous [19], θ-continuous [4]) if $f^{-1}[V]$ is ω-open (resp. ω_θ-open, δ-open, θ-open) for every open subset V of Y. A function $f : (X, \tau) \to (Y, \sigma)$ is called weakly ω-continuous [10] (resp. weakly ω_θ-continuous [3]) if for each $x \in X$ and each open subset V in Y containing $f(x)$, there exists an ω-open (resp. ω_θ-open) subset U in X containing x such $f[U] \subseteq cl[V]$.

2. Preliminaries

Definition 2.1. Let X be a set and $\gamma : 2^X \to 2^X$ be any arbitrary set-valued set-function. Then γ is called isotonic [17] if $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$ for all $A, B \in 2^X$. The collection of all isotonic mappings is denoted by $\Gamma(X)$ [1] (or simply Γ).

Definition 2.2. [1] Let X be a set and $\gamma \in \Gamma(X)$.
(1) $\gamma \in \Gamma_0$ if $\gamma(\emptyset) = \emptyset$,
(2) $\gamma \in \Gamma_1$ if $\gamma(X) = X$,
(3) $\gamma \in \Gamma_2$ if $\gamma(\gamma(A)) = \gamma(A)$ for all $A \in 2^X$,

(4) $\gamma \in \Gamma_+$ if $A \subseteq \gamma(A)$ for all $A \in 2^X$,
(5) $\gamma \in \Gamma_-$ if $\gamma(A) \subseteq A$ for all $A \in 2^X$.

We denote by Γ_Δ, if $\Delta \subseteq \{0, 1, 2, +, -\}$, the subset of Γ composed of all $\gamma \in \Gamma$ that satisfy (Γ_n) for $n \in \Delta$. We write, for sake of simplicity, Γ_{12} instead of $\Gamma_{\{1,2\}}$.

Definition 2.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$.
(1) A is said to be a Ψ^*-set [15] if $A \subseteq (\Psi(A))^*$,
(2) A is said to be a Ψ-C-set [16] if $A \subseteq cl(\Psi(A))$,
(3) The collection of all Ψ^*-set (resp. Ψ-C-set) is denoted as $\Psi^*(X, \tau)$ (resp. $\Psi(X, \tau)$),
(4) An ideal topological space $(X, \tau, \mathcal{I})$ is Hayashi-Samuel space [2] if $\tau \cap \mathcal{I} = \{\emptyset\}$.

Lemma 2.4. [14] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A \in \tau$, then $A \subseteq \Psi(A)$.*

Lemma 2.5. *Let X be a set and $\gamma \in \Gamma(X)$. If $\gamma \in \Gamma_{1-}$ and $\gamma(A \cap B) = \gamma(A) \cap \gamma(B)$ then the family $\gamma O(X) := \{A \subseteq X : A \subseteq \gamma(A)\}$ is a topology on X.*

Proof. 1) $\gamma \in \Gamma_1 \Rightarrow \gamma(X) = X$ then $X \in \gamma O(X)$.
2) Let $A, B \in \gamma O(X)$.

$$\left.\begin{array}{r} A, B \in \gamma O(X) \\ (A \cap B \subseteq A)(A \cap B \subseteq B) \end{array}\right\} \Rightarrow \left.\begin{array}{c} A \cap B \in \gamma(A) \cap \gamma(B) \\ \gamma(A) \cap \gamma(B) = \gamma(A \cap B) \end{array}\right\} \Rightarrow A \cap B \in \gamma O(X).$$

3) In [1], the author proved that if $\mathcal{A} \subseteq \gamma O(X)$, then $\bigcup \mathcal{A} \in \gamma O(X)$. □

3. Ψ_δ-sets

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We define the operator $\Psi_\delta : 2^X \to 2^X$ as $\Psi_\delta(A) := \Psi(\delta\text{-}int(A))$ for all $A \in 2^X$.

Example 3.2. Let $X := \{a, b\}$, $\tau := \{\emptyset, X, \{a\}\}$ and $\mathcal{I} := \{\emptyset, \{a\}\}$. $\emptyset \in \tau$ but $\Psi_\delta(\emptyset) = \Psi_\delta(\{a\}) = \Psi_\delta(\{b\}) = \{a\}$ and $\Psi_\delta(X) = X$. So there is no $A \in 2^X$ such that $\Psi_\delta(A) = \emptyset$.

Theorem 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the operator $\Psi_\delta : 2^X \to 2^X$ is monotonic.*

Proof. It is clear from the fact that the operators δ-int and Ψ are monotonic. □

Remark 3.4. It is easy to see the following results from the example 3.2.
(a) $\{b\} \nsubseteq \Psi_\delta(\{b\})$ implied that the operator $\Psi_\delta \notin \Gamma_+$,
(b) $\Psi_\delta(\emptyset) \neq \emptyset$ implied that the operator $\Psi_\delta \notin \Gamma_0$.

Theorem 3.5. *An ideal topological space $(X, \tau, \mathcal{I})$ is Hayashi-Samuel space if and only if the operator $\Psi_\delta \in \Gamma_0$.*

Proof. It is obvious from the fact that $\Psi_\delta(\emptyset) = \Psi(\emptyset)$ if and only if $(X, \tau, \mathcal{I})$ is Hayashi-Samuel space [14]. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then δ-$int(A) \subseteq \Psi_\delta(A)$.*

Proof. Let $A \subseteq X$.
$A \subseteq X \Rightarrow \delta\text{-}int(A) \in \tau \overset{\text{Lemma 2.4}}{\Rightarrow} \delta\text{-}int(A) \subseteq \Psi(\delta\text{-}int(A)) = \Psi_\delta(A)$. □

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space and $A \subseteq X$.*
1) If A is closed, then $\Psi_\delta(A) \setminus A = \emptyset$,
2) If A is regular open, then $\Psi_\delta(A) = A$.

Proof. 1) Let $A \in C(X, \tau)$.

$$\left.\begin{array}{r} A \in C(X,\tau) \overset{[14]}{\Rightarrow} \Psi(A) \setminus A = \emptyset \\ \Psi_\delta(A) \subseteq \Psi(A) \end{array}\right\} \Rightarrow \Psi_\delta(A) \setminus A = \emptyset.$$

2) Let $A \in RO(X)$.
$A \in RO(X) \Rightarrow A = \delta\text{-}int(A) = \Psi(\delta\text{-}int(A)) = \Psi_\delta(A)$. □

Definition 3.8. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We define the operator $\Psi_{\delta\text{-}int} : 2^X \to 2^X$ as $\Psi_{\delta\text{-}int}(A) := A \cap \Psi_\delta(A)$ for all $A \in 2^X$.

Theorem 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A, B \subseteq X$ then*
1) $\Psi_{\delta\text{-}int} \in \Gamma_{01-}$.
2) $\Psi_{\delta\text{-}int}(A \cap B) = \Psi_{\delta\text{-}int}(A) \cap \Psi_{\delta\text{-}int}(B)$

Proof. 1) Let $A \subseteq X$.
$A \subseteq X \Rightarrow \Psi_{\delta\text{-}int}(A) \subseteq A \Rightarrow \Psi_{\delta\text{-}int} \in \Gamma_{0-}$.
$\Psi_{\delta\text{-}int}(X) = \Psi(\delta\text{-}int(X)) = \Psi(X) = X \Rightarrow \Psi_{\delta\text{-}int} \in \Gamma_1$.
2) Let $A, B \subseteq X$.

$$\begin{array}{rcl} \Psi_{\delta\text{-}int}(A \cap B) & = & (A \cap B) \cap \Psi_\delta(A \cap B) \\ & = & (A \cap B) \cap \Psi(\delta\text{-}int(A) \cap \delta\text{-}int(B)) \\ & = & (A \cap \Psi(\delta\text{-}int(A))) \cap (B \cap \Psi(\delta\text{-}int(A))) \\ & = & \Psi_{\delta\text{-}int}(A) \cap \Psi_{\delta\text{-}int}(B) \quad \square \end{array}$$

Definition 3.10. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is said to be a Ψ_δ-set if $\Psi_{\delta\text{-}int}(A) = A$. The collection of all Ψ_δ-sets will be denoted by $\Psi_\delta O(X)$ or $\Psi_\delta\,(X, \tau, \mathcal{I})$. Also, the collection of all Ψ_δ-sets containing x of X will be denoted by $\Psi_\delta O(X, x)$.

Theorem 3.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The followings are equivalent:*
(1) A is Ψ_δ-set,
(2) $A \subseteq \Psi_{\delta\text{-}int}(A)$,
(3) $A \subseteq \Psi_\delta(A)$.

Proof. (1) $\Rightarrow$ (2) : This is obvious from the definition of Ψ_δ-set.
(2) $\Rightarrow$ (3) : It is obvious from the definition of $\Psi_{\delta\text{-}int}$.
(3) $\Rightarrow$ (1) : Let $A \subseteq \Psi_\delta(A)$.

$$\left.\begin{array}{r} A \subseteq \Psi_\delta(A) \Rightarrow A \cap \Psi_\delta(A) = A \\ \Psi_{\delta\text{-}int}(A) = A \cap \Psi_\delta(A) \end{array}\right\} \Rightarrow \Psi_{\delta\text{-}int}(A) = A \Rightarrow A \in \Psi_\delta O(X).$$ □

Theorem 3.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\Psi_\delta O(X)$ is a topology on X.*

Proof. Let $(X, \tau, \mathcal{I})$ be an ideal topological space.

$$\left.\begin{array}{r} \text{Theorem 3.9} \\ \text{Lemma 2.5} \end{array}\right\} \Rightarrow \left.\begin{array}{r} \Psi_{\delta\text{-}int}O(X) \text{ is a topology on } X \\ \Psi_{\delta\text{-}int}O(X) = \Psi_\delta O(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow \Psi_\delta O(X)$ is a topology on X. □

Theorem 3.13. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. A subset A of X is Ψ_δ-set if and only if for every $x \in A$, there exists an open subset $B \subseteq X$ containing x such that $B \setminus \delta\text{-}int(A) \in \mathcal{I}$.*

Proof. $(\Rightarrow)$: Let A be a Ψ_δ-set and $x \in A$.

$$\begin{array}{rcl} x \in A \in \Psi_\delta O(X) & \Rightarrow & x \in \Psi_\delta(A) \\ & \Rightarrow & x \notin (X \setminus \delta\text{-}int(A))^* \\ & \Rightarrow & (\exists B \in \mathcal{U}(x))(B \cap (X \setminus \delta\text{-}int(A)) \in \mathcal{I}) \\ & \Rightarrow & (\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) \in \mathcal{I}). \end{array}$$

$(\Leftarrow)$: Let A be a subset of X and $x \in A$.

$$\begin{array}{rcl} \left.\begin{array}{r} x \in A \\ \text{Hypothesis} \end{array}\right\} & \Rightarrow & (\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) \in \mathcal{I}) \\ & \Rightarrow & (\exists B \in \mathcal{U}(x))(B \cap (X \setminus \delta\text{-}int(A)) \in \mathcal{I}) \\ & \Rightarrow & x \notin (X \setminus \delta\text{-}int(A))^* \\ & \Rightarrow & x \in \Psi_\delta(A) \\ & \Rightarrow & A \subseteq \Psi_\delta(A) \end{array}$$

Then A is Ψ_δ-set. □

Theorem 3.14. *Let (X,τ) be a topological space. Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X. If $\mathcal{I} \subseteq \mathcal{J}$, then $\Psi_\delta O(X,\tau,\mathcal{J})$ is finer than $\Psi_\delta O(X,\tau,\mathcal{I})$.*

Proof. Let $\mathcal{I} \subseteq \mathcal{J}$ and $x \in A \in \Psi_\delta O(X,\tau,\mathcal{I})$.

$$\left.\begin{array}{r} x \in A \\ A \in \Psi_\delta O(X,\tau,\mathcal{I}) \end{array}\right\} \Rightarrow x \in A \in \Psi_\delta O(X,\tau,\mathcal{I}) \Rightarrow \left.\begin{array}{r} (\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) \in \mathcal{I}) \\ \mathcal{I} \subseteq \mathcal{J} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) \in \mathcal{J}) \Rightarrow A \in \Psi_\delta O(X,\tau,\mathcal{J}).$$ □

Theorem 3.15. *Let (X,τ) be a topological space. If $\mathcal{I} := \{\emptyset\}$, then $\Psi_\delta O(X) = \tau_\delta$.*

Proof. Let $\mathcal{I} = \{\emptyset\}$ and $A \in \Psi_\delta O(X)$.

$$\begin{array}{rcl} A \in \Psi_\delta O(X) & \Leftrightarrow & (\forall x \in A)(\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) = \emptyset) \\ & \Leftrightarrow & (\forall x \in A)(\exists B \in \mathcal{U}(x))(B \subseteq \delta\text{-}int(A)) \\ & \Leftrightarrow & (\forall x \in A)(x \in \delta\text{-}int(A)) \\ & \Leftrightarrow & A = \delta\text{-}int(A) \\ & \Leftrightarrow & A \in \tau_\delta \quad \square \end{array}$$

Remark 3.16. Theorem 3.15 need not be true for an ideal which contains sets other than emptyset as shown by the following example. Let $X := \{a,b,c\}$, $\tau := \{\emptyset, X, \{a,c\}\}$ and $\mathcal{I} := \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$. Then $\tau_\delta = \{\emptyset, X\}$ and $X^* = \{b\}$. Now let $A \subsetneq X$. Then $\Psi_\delta(A) = \Psi(\delta\text{-}int(A)) = \Psi(\emptyset) = X \setminus X^* = \{a,c\}$ and so $\Psi_\delta O(X) = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\} \neq \tau_\delta$.

Corollary 3.17. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then every δ-open set is Ψ_δ-set.

Proof. Let $\mathcal{I}$ be an ideal on X.
$\mathcal{I}$ is an ideal $\Rightarrow \{\emptyset\} \subseteq \mathcal{I} \Rightarrow \tau_\delta = \Psi_\delta O(X,\tau,\{\emptyset\}) \subseteq \Psi_\delta O(X,\tau,\mathcal{I}) = \Psi_\delta O(X)$. □

Theorem 3.18. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then A is Ψ_δ-set if and only if for every $x \in A$, there exists an open set B containing x and a set $V \in \mathcal{I}$ such that $B \setminus V \subseteq \delta\text{-}int(A)$.*

Proof. $(\Rightarrow)$: Let A be Ψ_δ-set and $x \in A$.

$$\left.\begin{array}{r} x \in A \in \Psi_\delta O(X) \Rightarrow (\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) \in \mathcal{I}) \\ V := B \setminus \delta\text{-}int(A) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists B \in \mathcal{U}(x))(\exists V \in \mathcal{I})(B \setminus V \subseteq \delta\text{-}int(A)).$$

$(\Leftarrow)$: Let $A \subseteq X$ and $x \in A$.

$$\begin{array}{rcl} \left.\begin{array}{r} x \in A \\ \text{Hypothesis} \end{array}\right\} & \Rightarrow & (\exists B \in \mathcal{U}(x))(\exists V \in \mathcal{I})(B \setminus V \subseteq \delta\text{-}int(A)) \\ & \Rightarrow & (\exists B \in \mathcal{U}(x))(\exists V \in \mathcal{I})(B \subseteq V \cap \delta\text{-}int(A)) \\ & \Rightarrow & (\exists B \in \mathcal{U}(x))(\exists V \in \mathcal{I})(B \setminus \delta\text{-}int(A) \subseteq V) \\ & \Rightarrow & (\exists B \in \mathcal{U}(x))(B \setminus \delta\text{-}int(A) \in \mathcal{I}) \\ & \Rightarrow & A \in \Psi_\delta O(X) \quad \square \end{array}$$

Theorem 3.19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If for each $x \in X$ there exists an open subset B containing x such that $B \in \mathcal{I}$, then $\Psi_\delta O(X)$ is discrete topology.*

Proof. Let $A \subseteq X$ and $x \in A$.

$$\left.\begin{array}{r} \left.\begin{array}{r} x \in A \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \exists B \in \mathcal{U}(x) \cap \mathcal{I} \\ V := B \end{array}\right\} \Rightarrow (B \in \mathcal{U}(x))(V \in \mathcal{I})(B \setminus V = \emptyset \subseteq \delta\text{-}int(A))$$

Then A is Ψ_δ-set. $\square$

Theorem 3.20. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then $\Psi_\delta(A)$ is a Ψ_δ-set.*

Proof. Let $A \subseteq X$.

$$\begin{array}{rcl} A \subseteq X & \Rightarrow & \delta\text{-}int(A) \subseteq \Psi_\delta(A) \\ & \Rightarrow & \delta\text{-}int(A) \subseteq \delta\text{-}int(\Psi_\delta(A)) \\ & \Rightarrow & \Psi(\delta\text{-}int(A)) \subseteq \Psi(\delta\text{-}int(\Psi_\delta(A))) \\ & \Rightarrow & \Psi_\delta(A) \subseteq \Psi_\delta(\Psi_\delta(A)) \end{array}$$

Then $\Psi_\delta(A) \in \Psi_\delta O(X)$. $\square$

Theorem 3.21. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. $\Psi_{\delta\text{-}int}(A)$ is a Ψ_δ-set.*

Proof. Let $A \subseteq X$.

$$\begin{array}{rcl} A \subseteq X & \Rightarrow & \delta\text{-}int(A) \subseteq A \cap \Psi_\delta(A) \\ & \Rightarrow & \delta\text{-}int(A) \subseteq \Psi_{\delta\text{-}int}(A) \\ & \Rightarrow & \delta\text{-}int(A) \subseteq \delta\text{-}int(\Psi_{\delta\text{-}int}(A)) \\ & \Rightarrow & \Psi(\delta\text{-}int(A)) \subseteq \Psi(\delta\text{-}int(\Psi_{\delta\text{-}int}(A)) \\ & \Rightarrow & A \cap \Psi(\delta\text{-}int(A)) \subseteq A \cap \Psi(\delta\text{-}int(\Psi_{\delta\text{-}int}(A)) \\ & \Rightarrow & \Psi_{\delta\text{-}int}(A) \subseteq \Psi_\delta(\Psi_{\delta\text{-}int}(A)). \end{array}$$

Then $\Psi_{\delta\text{-}int}(A) \in \Psi_\delta O(X)$. $\square$

Corollary 3.22. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\Psi_{\delta\text{-}int} \in \Gamma_2$.

Proof. Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A\subseteq X$.

$$\Psi_{\delta\text{-}int}(A)\in\Psi_\delta O(X)\Rightarrow\Psi_{\delta\text{-}int}(\Psi_{\delta\text{-}int}(A))=\Psi_{\delta\text{-}int}(A)$$

This means that $\Psi_{\delta\text{-}int}\in\Gamma_2$. □

Theorem 3.23. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A\subseteq X$. $x\in\Psi_{\delta\text{-}int}(A)$ if and only if there exists a Ψ_δ-set B in X containing x such that $B\subseteq A$.*

Proof. $(\Rightarrow)$: Let A be any subset of X and $x\in\Psi_{\delta\text{-}int}(A)$.

$$\left.\begin{array}{r}x\in\Psi_{\delta\text{-}int}(A)\\ B:=\Psi_{\delta\text{-}int}(A)\end{array}\right\}\Rightarrow(B\in\Psi_\delta O(X,x))(B\subseteq A).$$

$(\Leftarrow)$: Let $x\in X$.

$$\begin{array}{rcl}\left.\begin{array}{r}x\in X\\ \text{Hypothesis}\end{array}\right\} & \Rightarrow & (\exists B\in\Psi_\delta O(X))(x\in B\subseteq A)\\ & \Rightarrow & x\in B=\Psi_{\delta\text{-}int}(B)\subseteq\Psi_{\delta\text{-}int}(A)\\ & \Rightarrow & x\in\Psi_{\delta\text{-}int}(A).\end{array}$$ □

Theorem 3.24. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. If $(X,\tau,\mathcal{I})$ is a Hayashi-Samuel space, then $(X,\Psi_\delta O(X),\mathcal{I})$ is a Hayashi-Samuel space.*

Proof. Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space and $A\in\Psi_\delta O(X)\cap\mathcal{I}$.

$$\left.\begin{array}{r}\left.\begin{array}{r}A\in\mathcal{I}\\ \text{Hypothesis}\end{array}\right\}\Rightarrow\begin{array}{c}\Psi(A)=\emptyset\\ \Psi_\delta(A)\subseteq\Psi(A)\end{array}\end{array}\right\}\Rightarrow\left.\begin{array}{c}\Psi_\delta(A)=\emptyset\\ A\in\Psi_\delta O(X)\end{array}\right\}\Rightarrow A=\emptyset.$$ □

Theorem 3.25. *Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuel space. Then every Ψ_δ-set is preopen in X.*

Proof. Let $A\in\Psi_\delta O(X)$.

$$\left.\begin{array}{r}A\in\Psi_\delta O(X)\Rightarrow A\subseteq\Psi_\delta(A)\subseteq\Psi(A)\\ (X,\tau,\mathcal{I})\text{ is a Hayashi-Samuel space }\overset{[16]}{\Rightarrow}\Psi(A)\subseteq int(cl(A))\end{array}\right\}\Rightarrow A\subseteq int(cl(A))$$

Then $A\in PO(X)$. □

Theorem 3.26. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then every Ψ_δ-set is Ψ-C-set.*

Proof. Let A be Ψ_δ-set.

$$A\in\Psi_\delta O(X)\Rightarrow A\subseteq\Psi_{\delta\text{-}int}(A)\subseteq\Psi(A)\subseteq cl(\Psi(A))\Rightarrow A\in\Psi(X,\tau).$$

The following example shows that the converse of the theorem is not true in general. □

Example 3.27. Let $X:=\{a,b,c\}$, $\tau:=\{\emptyset,X,\{a,c\}\}$ and $\mathcal{I}:=\{\emptyset,\{c\}\}$. Let $A:=\{a\}$. Then $\Psi(A)=\{a,c\}$ and $(\Psi(A))^*=X$ so $A\in\Psi^*(X,\tau)\subseteq\Psi(X,\tau)$. Therefore

$$\left.\begin{array}{r}\delta\text{-}int(A)=\emptyset\\ (X,\tau,\mathcal{I})\text{ is a Hayashi-Samuel space}\end{array}\right\}\Rightarrow\Psi_\delta(A)=\Psi(\delta\text{-int}(A))=\emptyset$$

Then $A\notin\Psi_\delta O(X)$. Also, we see that $\Psi^*(X,\tau)\nsubseteq\Psi_\delta O(X)$.

Remark 3.28. A Ψ_δ-set need not be a Ψ^*-set as shown by the following example.

Example 3.29. Let $X := \{a, b, c, d\}$, $\tau := \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} := \{\emptyset, \{c\}\}$. Let $A := \{a\}$. Then $\Psi_\delta(A) = \Psi(\delta\text{-}int(A)) = \Psi(\emptyset) = X \setminus \{b, c, d\} = A$ so $A \in \Psi_\delta O(X)$. Therefore $(\Psi(A))^* = (X \setminus \{b, c, d\})^* = \emptyset$ so $A \notin \Psi^*(X, \tau)$.

Corollary 3.30. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the notions of Ψ^*-set and Ψ_δ-set are independent.

Definition 3.31. A topological space (X, τ) is said to be a Δ-space if $\tau = \delta O(X)$.

Remark 3.32. Note that every regular space is Δ-space. The next example shows that the converse does not hold in general.

Example 3.33. Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Then (X, τ) is Δ-space but not regular.

Theorem 3.34. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If (X, τ) is Δ-space, then every open set is Ψ_δ-set.*

Proof. It is not difficult to see that $\tau = \tau_\delta \subseteq \Psi_\delta O(X)$ when the space is Δ-space. □

4. Ψ_δ-function

Definition 4.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called Ψ_δ-function if $f^{-1}[V]$ is Ψ_δ-set for each open set V of (Y, σ).

Theorem 4.2. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is Ψ_δ-function if and only if $f : (X, \Psi_\delta O(X)) \to (Y, \sigma)$ is continuous.*

Proof. Obvious. □

Remark 4.3. The composition of two Ψ_δ-functions need not be Ψ_δ-functions as shown by the following example.

Example 4.4. Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} := \{\emptyset, \{c\}\}$. Let $Y := \{1, 2, 3\}$, $\sigma := \{\emptyset, Y, \{1, 2\}\}$ and $\mathcal{J} := \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 3$, $f(b) = 1$, $f(c) = 2$ and $g : (Y, \sigma, \mathcal{J}) \to (X, \tau, \mathcal{I})$ by $g(1) = a$, $g(2) = g(3) = c$. Now $f^{-1}[\{1, 2\}] = \{b, c\} = \Psi(\delta\text{-}int(\{b, c\})) = \Psi_\delta(\{b, c\})$ so $\{b, c\} \in \Psi_\delta O(X)$. Then f is a Ψ_δ-function. On the other hand, $g^{-1}[\{a, b\}] = g^{-1}[\{a\}] = \{1\} \subseteq \{1, 2\} = \Psi(\delta\text{-}int(\{1\}) = \Psi_\delta(\{1\})$ so $\{1\} \in \Psi_\delta O(X)$ and $g^{-1}[\{b\}] = \emptyset$. Then g is a Ψ_δ-function. However, $(f \circ g)^{-1}(\{1, 2\}) = \{2, 3\} \nsubseteq \{1, 2\} = \Psi_\delta(\{2, 3\})$ so $\{2, 3\}$ is not Ψ_δ-set .

Theorem 4.5. *$g : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a Ψ_δ-function and $f : (Y, \sigma) \to (Z, \mu)$ continuous function. Then $f \circ g$ is a Ψ_δ-function.*

Proof. Let g be Ψ_δ-function and let f be continuous.

$$\left.\begin{array}{r} g : (X, \tau, \mathcal{I}) \to (Y, \sigma) \text{ is } \Psi_\delta\text{-function} \Rightarrow g : (X, \Psi_\delta O(X)) \to (Y, \sigma) \text{ is con.} \\ f : (Y, \sigma) \to (Z, \mu) \text{ is con.} \end{array}\right\} \Rightarrow$$

$\Rightarrow f \circ g : (X, (\Psi_\delta O(X))) \to (Z, \mu)$ is continuous

$\Rightarrow f \circ g$ is Ψ_δ-function. □

Theorem 4.6. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is Ψ_δ-function if and only if for each $x \in X$ and every open subset V in Y containing $f(x)$, there exists a Ψ_δ-set B in X containing x such that $f(B) \subseteq V$.*

Proof. $(\Rightarrow)$: Let V be an open subset of Y containing $f(x)$.

$$\left.\begin{array}{r} V \in \mathcal{U}(f(x)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[V] \in \Psi_\delta O(X,x) \\ B := f^{-1}[V] \end{array}\right\} \Rightarrow (\exists B \in \mathcal{U}(x))(f[B] \subseteq V).$$

$(\Leftarrow)$: Let A be an open subset of Y and $x \in f^{-1}[A]$.

$$\begin{array}{rcl} \left.\begin{array}{r} x \in f^{-1}[A] \Rightarrow A \in \mathcal{U}(f(x)) \\ \text{Hypothesis} \end{array}\right\} & \Rightarrow & (\exists B \in \Psi_\delta O(X,x))(f[B] \subseteq A) \\ & \Rightarrow & (\exists B \in \Psi_\delta O(X,x))(B \subseteq f^{-1}[A]) \\ & \Rightarrow & x \in B = \Psi_{\delta\text{-}int}(B) \subseteq \Psi_{\delta\text{-}int}(f^{-1}[A]) \\ & \Rightarrow & x \in \Psi_{\delta\text{-}int}(f^{-1}[A]) \end{array}$$

Then we have $f^{-1}[A] \subseteq \Psi_{\delta\text{-}int}(f^{-1}[A])$. Therefore $f^{-1}[A] = \Psi_{\delta\text{-}int}(f^{-1}[A])$ and so f is Ψ_δ-function. $\square$

Definition 4.7. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called weakly Ψ_δ-function at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists a Ψ_δ-set B in X containing x such that $f[B] \subseteq cl(V)$. If f is weakly Ψ_δ-function at every $x \in X$, it is said to be weakly Ψ_δ-functions.

It is not difficult to see that every Ψ_δ-function is a weakly Ψ_δ-function.

Theorem 4.8. *The following properties are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$.*
(1) $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a weakly Ψ_δ-function,
(2) $x \in \Psi_{\delta\text{-}int}(f^{-1}[cl(V)])$ for each open neighborhood V of $f(x)$.

Proof. $(1) \Rightarrow (2)$: Let V be any open neighborhood of $f(x)$.

$$\begin{array}{rcl} \left.\begin{array}{r} V \in \mathcal{U}(f(x)) \\ \text{Hypothesis} \end{array}\right\} & \Rightarrow & (\exists B \in \Psi_\delta O(X,x))(f[B] \subseteq cl(V)) \\ & \Rightarrow & (B \in \Psi_\delta O(X,x))(B \subseteq f^{-1}[cl(V)]) \\ & \Rightarrow & x \in B = \Psi_{\delta\text{-}int}(B) \subseteq \Psi_{\delta\text{-}int}(f^{-1}[cl(V)]). \end{array}$$

$(2) \Rightarrow (1)$: Let $x \in \Psi_\delta(f^{-1}[cl(V)])$ for each open neighborhood of $f(x)$.

$$\left.\begin{array}{r} B := \Psi_{\delta\text{-}int}(f^{-1}[cl(V)]) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (B \in \Psi_\delta O(X,x))(f[B] \subseteq cl(V)). \quad \square$$

Theorem 4.9. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a weakly Ψ_δ-function if and only if for every open set A of Y, $f^{-1}[A] \subseteq \Psi_{\delta\text{-}int}(f^{-1}[cl(A)])$.*

Proof. $(\Rightarrow)$: Let $A \in \sigma$ and $x \in f^{-1}[A]$.

$$\left.\begin{array}{r} x \in f^{-1}[A] \Rightarrow f(x) \in A \\ A \in \tau \end{array}\right\} \Rightarrow \left.\begin{array}{r} A \in \mathcal{U}(f(x)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow x \in \Psi_{\delta\text{-}int}(f^{-1}[cl(A)]).$$

$(\Leftarrow)$: Let $A \in \mathcal{U}(f(x))$.

$$\left.\begin{array}{r} A \in \mathcal{U}(f(x)) \Rightarrow x \in f^{-1}[A] \\ \text{Hypothesis} \end{array}\right\} \Rightarrow x \in \Psi_{\delta\text{-}int}(f^{-1}[cl(A)])$$

Then f is a weakly Ψ_δ-function. $\square$

5. A finer topology than $\omega_\theta O(X)$

In this section, we will obtain a new type of set by taking a special ideal on X and investigate its some fundamental properties.

Definition 5.1. Let (X,τ) be a topological space and $A \subseteq X$. A is said to be an ω_δ-open set, if A is Ψ_δ-set on the ideal $\mathcal{I}_c$ where $\mathcal{I}_c$ is the collection of all countable subset in X. The complement of an ω_δ-open set is called ω_δ-closed. The collection of all ω_δ-open (resp. ω_δ-closed) subsets of X will be denoted by $\omega_\delta O(X)$ (resp. $\omega_\delta C(X)$). Also, the collection of all ω_δ-open (ω_δ-closed) subsets of X containing x will be denoted by $\omega_\delta O(X,x)$ (resp. $\omega_\delta C(X,x)$).

Theorem 5.2. *The following properties are equivalent for a subset A of X.*
(1) A is ω_δ-open,
(2) For every $x \in A$, there exists an open subset $B \subseteq X$ containing x such that $B \setminus \delta$-int(A) is countable,
(3) For every $x \in A$, there exists an open subset $B \subseteq X$ containing x and a countable subset V such that $B \setminus V \subseteq \delta$-int$(A)$.

Proof. It is obvious from Theorem 3.13 and Theorem 3.18. □

Remark 5.3. The following diagram holds for a subset A of a space X.

$$\begin{array}{ccccc} \omega_\theta\text{-open} & \rightarrow & \omega_\delta\text{-open} & \longrightarrow & \omega\text{-open} \\ \uparrow & & \uparrow & & \uparrow \\ \theta\text{-open} & \rightarrow & \delta\text{-open} & \longrightarrow & \text{open} \end{array}$$

The following examples show that these implications are not reversible.

Example 5.4. (1) Let $\mathbb{R}$ be the real line with topology $\tau := \{A \subseteq \mathbb{R} | 0 \in A\} \cup \{\emptyset\}$. Then the set of natural numbers $\mathbb{N}$ is ω_δ-open but it is not δ-open.
(2) Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{I}\}$ where $\mathbb{I}$ is the set of irrational numbers. Let $A := \mathbb{I} \cup \{0\}$. Then A is ω-open but it is not ω_δ-open.
(3) Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{I} \setminus \{\pi\}, \mathbb{R} \setminus \{\pi\}, \mathbb{Q}\}$ where $\mathbb{I}$ is the set of irrational numbers and $\mathbb{Q}$ is the set of rational numbers. Let $A := \mathbb{I} \setminus \{\pi\}$. Then A is ω_δ-open but it is not ω_θ-open.

The following examples show that the notions of open set and ω_δ-open set are independent.

Example 5.5. (1) Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, X, \{a\}\}$. Let $A := \{b\}$. Then A is ω_δ-open but it is not open.
(2) Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{R} \setminus (0,1)\}$. Then the set $\mathbb{R} \setminus (0,1)$ is open but it is not ω_δ-open.

Theorem 5.6. *Let (X,τ) be a Δ-space. Then every open set is ω_δ-open.*

Proof. It is obvious from the fact that $\tau = \tau_\delta (= \delta O(X))$ when the space is Δ-space. □

Theorem 5.7. *Let A be an ω_δ-closed set of X. Then δ-cl$(A) \subseteq K \cup V$ for a closed subset K and a countable subset V.*

Proof. Let $A \in \omega_\delta C(X)$.

$$\begin{array}{lll} A \in \omega_\delta C(X) & \Rightarrow & X \setminus A \in \omega_\delta O(X) \\ & \Rightarrow & (\forall x \in X \setminus A)(\exists B \in \mathcal{U}(x))(\exists V \in \mathcal{I}_c)(B \setminus V \subseteq \delta\text{-}int(X \setminus A) \\ & \Rightarrow & \delta\text{-}cl(A) \subseteq (X \setminus B) \cup V \\ & \Rightarrow & (K := X \setminus B \in C(X,\tau))(V \in \mathcal{I}_c)(\delta\text{-}cl(A) \subseteq K \cup V). \quad \square \end{array}$$

Definition 5.8. The ω_δ-interior of a subset A of a space X is defined as $\Psi_{\delta\text{-}int}(A)$ and denoted by ω_δ-$int(A)$.

Corollary 5.9. The ω_δ-interior of a subset A of a space X is the union of all ω_δ-sets of X contained in A.

Proof. It is obvious from Theorem 3.23. □

Definition 5.10. The intersection of all ω_δ-closed sets of X containing a subset A is called the ω_δ-closure of A and is denoted by ω_δ-$cl(A)$.

Lemma 5.11. *Let A be a subset of a space X. Then the following properties hold.*
(1) ω_δ-$cl(A)$ is ω_δ-closed in X,
(2) ω-$cl(A) \subseteq \omega_\delta$-$cl(A) \subseteq \delta$-$cl(A)$,
(3) ω_δ-$cl(X \setminus A) = X \setminus \omega_\delta$-$int(A)$,
(4) $x \in \omega_\delta$-$cl(A)$ if and only if $A \cap B \neq \emptyset$ for each ω_δ-open set B containing x,
(5) A is ω_δ-closed in X if and only if $A = \omega_\delta$-$cl(A)$.
(6) ω_δ-$cl(\omega_\delta$-$cl(A)) = \omega_\delta$-$cl(A)$.

Theorem 5.12. *Let A be a subset of a space X. Then A is ω_δ-closed if and only if $(\delta\text{-}cl(A))^*(\tau, \mathcal{I}_c) \subseteq A$.*

Proof. Let $A \subseteq X$.

$$\begin{array}{lll} (\delta\text{-}cl(A)(\mathcal{I}_c,\tau))^* \subseteq A & \Leftrightarrow & (X \setminus \delta\text{-}int(X \setminus A))^* \subseteq A \\ & \Leftrightarrow & X \setminus A \subseteq X \setminus (X \setminus \delta\text{-}int(X \setminus A))^* \\ & \Leftrightarrow & X \setminus A \subseteq \Psi(\delta\text{-}int(X \setminus A)) \\ & \Leftrightarrow & X \setminus A \in \Psi_\delta O(X,\tau,\mathcal{I}_c) = \omega_\delta O(X) \\ & \Leftrightarrow & A \in \omega_\delta C(X). \end{array}$$
□

Definition 5.13. Let (X, τ) be a topological space and $A \subseteq X$. Then A is said to be a (ω_δ, δ)-set if ω_δ-$int(A) = \delta$-$int(A)$ and A is said to be a (ω_δ, ω)-set if ω_δ-$int(A) = \omega$-$int(A)$.

Remark 5.14. Every ω_δ-open set is an (ω_δ, ω)-set and every δ-open set is an (ω_δ, δ)-set but not conversely.

Example 5.15. Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{I}\}$ where $\mathbb{I}$ is the set of irrational numbers. Then the set of natural numbers $\mathbb{N}$ is both (ω_δ, ω)-set and (ω_δ, δ)-set but it is neither ω_δ-open nor δ-open.

Corollary 5.16. Let (X, τ) be a topological space and $A \subseteq X$. Then:
(1) A is ω_δ-open if and only if A is ω-open and an (ω_δ, ω)-set.
(2) A is δ-open if and only if A is ω_δ-open and an (ω_δ, δ)-set.

Proof. Obvious. □

Recall that a space X is called locally countable if each $x \in X$ has a countable neighborhood. A space X is called anti-locally countable if nonempty open subsets of X are uncountable.

Theorem 5.17. *Let (X, τ) be a locally countable space and $A \subseteq X$. Then the family of $\omega_\delta O(X)$ is the discrete topology.*

Proof. It is obvious from Theorem 3.19. $\square$

Corollary 5.18. Let (X, τ) be a locally countable space and $A \subseteq X$. Then A is ω_δ-open if and only if A is ω-open.

Proof. It is obvious from the fact that every ω_δ-open set is ω-open. $\square$

Theorem 5.19. *Let (X, τ) be a topological space and $A \subseteq X$. Then the following properties hold.*
(1) If X is an anti-locally countable space, then $(X, \omega_\delta O(X))$ is anti-locally countable,
(2) If X is an anti-locally countable regular space and A is δ-open, then δ-$cl(A) = \omega_\delta$-$cl(A)$.

Proof. (1) Let (X, τ) be an anti-locally countable space and let A be an ω_δ-open.

$$\left.\begin{array}{r} x \in A \in \omega_\delta O(X) \Rightarrow (\exists B \in \mathcal{U}(x))(\exists V \in \mathcal{I}_c)(B \setminus V \subseteq \delta\text{-}int(A)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow \aleph_0 < |B \setminus V| \leq |\delta\text{-}int(A)| \leq |A|$
Then $(X, \omega_\delta O(X))$ is anti-locally countable.
(2) It is not difficult to see that ω_δ-$cl(A) \subseteq \delta$-$cl(A)$.
Let $x \in \delta$-$cl(A)$ and $G \in \omega_\delta O(X, x)$. We will show that $G \cap A \neq \emptyset$.

$$G \in \omega_\delta O(X, x) \Rightarrow (\exists V \in \mathcal{U}(x))(\exists U \subseteq X)(|U| \leq \aleph_0)(V \setminus U \subseteq \delta\text{-}int(G))$$

$$\left.\begin{array}{r} \Rightarrow (V \in \mathcal{U}(x))(|U| \leq \aleph_0)((V \cap A) \setminus U \subseteq \delta\text{-}int(G) \cap A) \\ (A \in \delta O(X))(x \in \delta\text{-}cl(A)) \\ (X, \tau) \text{ is regular} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (V \cap A \neq \emptyset)(|U| \leq \aleph_0)(V, A \in \omega_\delta O(X))((V \cap A) \setminus U \subseteq \delta\text{-}int(G) \cap A) \\ (X, \tau) \text{ is anti-locally countable} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (V \cap A \in \omega_\delta O(X, x))(|V \cap A| > \aleph_0)(|U| \leq \aleph_0)((V \cap A) \setminus U \subseteq \delta\text{-}int(G) \cap A)$$

$$\Rightarrow (|(V \cap A) \setminus U| > \aleph_0)(|(V \cap A) \setminus U| \leq |\delta\text{-}int(G) \cap A| \leq |G \cap A|)$$

$$\Rightarrow G \cap A \neq \emptyset.$$

$\square$

Corollary 5.20. Let (X, τ) be an anti-locally countable space and $A \subseteq X$. Then the following properties hold.
(1) If A is δ-closed, then δ-$int(A) = \omega_\delta$-$int(A)$,
(2) The family of all (ω_δ, δ)-sets contains all of the δ-closed subset of X.

Theorem 5.21. *If X is a Lindelöf space, then $A \setminus \delta$-$int(A)$ is countable for every closed subset $A \in \omega_\delta O(X)$.*

Proof. Let $A \in \omega_\delta O(X) \cap C(X, \tau)$.

$$\left.\begin{array}{r} A \in \omega_\delta O(X) \Rightarrow (\forall x \in A)(\exists B_x \in \mathcal{U}(x))(|B_x \setminus \delta\text{-}int(A)| \leq \aleph_0) \\ \mathcal{A} := \{B_x | (\forall x \in A)(\exists B_x \in \mathcal{U}(x))(|B_x \setminus \delta\text{-}int(A)| \leq \aleph_0)\} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\mathcal{A} \subseteq \tau)(A \subseteq \bigcup \mathcal{A}) \\ A \in C(X, \tau) \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\mathcal{B} := \mathcal{A} \cup \{X \setminus A\} \subseteq \tau)(X = \bigcup \mathcal{B}) \\ (X, \tau) \text{ is Lindelöf} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| \leq \aleph_0)(X = \bigcup \mathcal{B}^*) \\ \mathcal{B}^{**} := \mathcal{B}^* \setminus \{X \setminus A\}\end{array}\right\} \Rightarrow (\mathcal{B}^{**} \subseteq \tau)(|\mathcal{B}^{**}| \leq \aleph_0)(A \subseteq \bigcup \mathcal{B}^{**})$$

$$\Rightarrow |A \setminus \delta\text{-}int(A)| \leq |(\cup \mathcal{B}^{**}) \setminus \delta\text{-}int(A)| \leq \aleph_0. \qquad \square$$

Definition 5.22. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function and let $\mathcal{I}_c$ be the collection of all countable subset of X. f is said to be an ω_δ-continuous (resp. weakly ω_δ-continuous) if $f : (X, \tau, \mathcal{I}_c) \to (Y, \sigma)$ is Ψ_δ-function (resp. weakly Ψ_δ-function).

Remark 5.23. The following diagram holds for a function $f : X \to Y$.

$$\begin{array}{ccccc} \text{weakly } \omega_\theta\text{-continuous} & \to & \text{weakly } \omega_\delta\text{-continuous} & \longrightarrow & \text{weakly } \omega\text{-continuous} \\ \uparrow & & \uparrow & & \uparrow \\ \omega_\theta\text{-continuous} & \to & \omega_\delta\text{-continuous} & \longrightarrow & \omega\text{-continuous} \\ \uparrow & & \uparrow & & \uparrow \\ \theta\text{-continuous} & \to & \delta\text{-continuous} & \longrightarrow & \text{continuous} \end{array}$$

The following examples show that these implications are not reversible.

Example 5.24. Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, (2,3)\}$. Let $Y := \{a, b, c\}$ and $\sigma := \{\emptyset, Y, \{a, c\}, \{a\}, \{c\}\}$. Define a function $f : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows $f(x) := \begin{cases} a, & \text{if } x \in (0,1) \\ b, & \text{if } x \notin (0,1) \end{cases}$. Then f is weakly ω_δ-continuous but it is not ω_δ-continuous.

Example 5.25. Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{I} \setminus \{\pi\}, \mathbb{R} \setminus \{\pi\}, \mathbb{Q}\}$ where $\mathbb{I}$ is the set of irrational numbers and $\mathbb{Q}$ is the set of rational numbers. Let $Y := \{a, b, c\}$ and $\sigma := \{\emptyset, Y, \{b, c\}, \{a\}\}$. Define a function $f : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows $f(x) := \begin{cases} a, & \text{if } x \in \mathbb{I} \setminus \{\pi\} \\ b, & \text{if } x \notin \mathbb{I} \setminus \{\pi\} \end{cases}$. Then f is ω_δ-continuous but it is not weakly ω_θ-continuous.

Example 5.26. Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{I}\}$ where $\mathbb{I}$ is the set of irrational numbers. Let $Y := \{a, b, c, d\}$ and $\sigma := \{\emptyset, Y, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows $f(x) := \begin{cases} a, & \text{if } x \in \mathbb{I} \cup \{1\} \\ b, & \text{if } x \notin \mathbb{I} \cup \{1\} \end{cases}$. Then f is ω-continuous but it is not weakly ω_δ-continuous.

Example 5.27. Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, X, \{a\}\}$. Define a function $f : (X, \tau) \to (X, \tau)$ as follows $f(a) = b$, $f(b) = c$, $f(c) = a$. Then f is ω_δ-continuous but it is not continuous.

For the other implications, the counterexamples are as shown in [3].

Definition 5.28. A function $f : X \to Y$ is said to be (ω_δ, δ)-continuous (resp. (ω_δ, ω)-continuous) if for every open subset A of Y, $f^{-1}(A)$ is an (ω_δ, δ)-set (resp. (ω_δ, ω)-set) of X.

Remark 5.29. Every δ-continuous function is (ω_δ, δ)-continuous and every ω_δ-continuous function is (ω_δ, ω)-continuous but not conversely.

Example 5.30. Let $\mathbb{R}$ be the real line with topology $\tau := \{\emptyset, \mathbb{R}, \mathbb{I}\}$ where $\mathbb{I}$ is the set of irrational numbers and $\sigma := \{\emptyset, \mathbb{R}, \mathbb{N}\}$. Define a function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ as follows $f(x) := x$ for every $x \in \mathbb{R}$.
(1) f is (ω_δ, ω)-continuous but it is not ω_δ- continuous.
(2) f is (ω_δ, δ)-continuous but it is not δ-continuous.

Definition 5.31. A function $f : X \to Y$ is co-weakly ω_δ-continuous if for every open subset A in Y, $f^{-1}[Fr(A)]$ is ω_δ-closed in X where $Fr(A) = cl(A) \setminus int(A)$.

Theorem 5.32. *Let $f : X \to Y$ be a function. The following are equivalent.*
(1) f is ω_δ-continuous.
(2) f is ω-continuous and (ω_δ, ω)-continuous
(3) f is weakly ω_δ-continuous and co-weakly ω_δ-continuous.

Proof. (1) $\Leftrightarrow$ (2) It is an immediate consequence of corollary 4.17
(1)$\Rightarrow$(3): Obvious.
(3)$\Rightarrow$ (1) : Let $x \in X$ and $V \in \mathcal{U}(f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in \mathcal{U}(f(x))) \\ f \text{ is weakly } w_\delta\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists W \in \omega_\delta O(X,x))(f[W] \subseteq cl(V))(f(x) \notin Fr(V)) \\ f \text{ is coweakly } \omega_\delta\text{-continuous}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{r}(W \setminus f^{-1}[Fr(V)] \in \omega_\delta O(X,x))(f[W] \subseteq cl(V)) \\ U := W \setminus f^{-1}[Fr(V)]\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in \omega_\delta O(X,x))(f[U] \subseteq cl(V)).$$ □

Definition 5.33. If a space X can not be written as the union of two nonempty disjoint ω_δ-open sets, then x is said to be ω_δ-connected.

Theorem 5.34. *If $f : X \to Y$ is a weakly ω_δ-continuous surjection and X is ω_δ-connected, then Y is connected.*

Proof. Suppose that Y is not connected.
(Y, σ) is not connected $\Rightarrow (\exists U, V \in \sigma \setminus \{\emptyset\})(U \cap V = \emptyset)(Y = U \cup V)$

$$\Rightarrow \left.\begin{array}{r}\exists U, V \in \sigma \cap C(Y, \sigma) \\ \text{Hypothesis}\end{array}\right\} \overset{\text{Theorem 4.9}}{\Rightarrow}$$

$$\Rightarrow \left.\begin{array}{r}(\emptyset \neq f^{-1}[U] = \omega_\delta\text{-}int(f^{-1}[U]))(\emptyset \neq f^{-1}[V] = \omega_\delta\text{-}int(f^{-1}[V])) \\ (U' := f^{-1}[U])(V' := f^{-1}[V])(f \text{ is surjection})\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U', V' \in \omega_\delta O(X) \setminus \{\emptyset\})(U' \cap V' = \emptyset)(X = U' \cup V')$$

This means that X is not ω_δ-connected which is a contradiction. □

6. Conclusion

Undoubtedly, operators are one of the indispensible objects of general topology. In this study, we have been introduced the Ψ_δ-operator. However, this operator has not satisfy the property in Definition 2.2 [5]. In order to generate a topology finer than τ_δ [22] and $\omega_\theta O(X)$ [3], we needed an operator derived from Ψ_δ and satisfied the property in Definition 2.2 [5]. For this aim, we introduced the $\Psi_{\delta\text{-}int}$-operator and used this notion to define the notion of Ψ_δ-set. Moreover, some of its properties and characterizations have been obtained. The Ψ_δ topology and Ψ_δ-function have been introduced via Ψ_δ-sets. It can be even more useful to enrich the continuous functions class.

References

[1] A. Csaszar, Generalized open sets, Acta Math. Hungar, **75**(1-2) (1997), 65-87.

[2] J. Dontchev, Idealization of Ganster-Reilly decomposition theorems, arXIV:math. Gn/9901017v1 [math.GN], 1999.

[3] E. Ekici, S. Jafari and R. Latif, ω_θ-open sets and ω_θ-continuous mappings, King Fahd Univ. of Petr. and Mine. Tech. Report Series **385** (2008), 1–13.

[4] S. Fomin, Extentions of topological spaces, Ann. of Math. **44** (1943), 471-480.

[5] D.E. Habil and A.K. Elzenati, Connectedness in isotonic spaces, Turk. J. Math. TUBITAK, **30** (2006), 247-262.

[6] T.R. Hamlett and D. Jankovic, Ideals in topological spaces and the set operator Ψ, Bollettino U.M.I., **7(4-B)** (1990), 863-874.

[7] T.R. Hamlet and Jankovic, New topologies from old via ideals, Mathematical Association of America, **97**(4) (1990), 295-310.

[8] E. Hayashi, Topologies defined by local properties, Math. Ann., **156** (1964), 205-215.

[9] H.Z. Hdeib, ω-closed mapping, Rev. Colombiana Mat., **16** no.1-2, (1982), 65-78.

[10] H.Z. Hdeib, ω-continuous functions, Dirasat, **XVI** (1989), 136-142.

[11] Md.M. Islam and S. Modak, Operator associated with the * and Ψ operators, J. Taibah Univ. Sci., **12**(4) (2018), 444-449.

[12] Kuratowski, Topology, vol.I,. New York: Academic Press, 1966.

[13] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On pre-continuous and weak precontinous mappings, Proc. Math. Phys. Soc. Egypt, **53** (1982), 47-53.

[14] S. Modak and Md. M. Islam, On * and Ψ operators in topological spaces with ideals, Trans.A. Razmadze Math. Inst., **172** (2018), 491-497.

[15] S. Modak, Some new topologies on ideal topological spaces, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 82(3) (2012), 233-243.

[16] S. Modak, C. Bandyopadhyay, A note on Ψ-operator, Bull. Malyas. Math. Sci. Soc., **30**(1) (2007), 43-48.

[17] S. Modak and Md.M. İslam, New operators in ideal topological spaces and their closure spaces, Journal of Science and Engineering, 3(2) (2019), 112-128.

[18] T. Natkaniec, On I-continuity and I-semicontinuity point, Math. Slovaca, **36**(3) (1986), 297-312.

[19] T. Noiri, On δ-continuous function, J. Korean Math. Soc. **16** (1980), 161-166.

[20] P. Samuel, A topology formed from a given topology and ideal, J. Lond. Math. Soc., **10** (1975), 409-416.

[21] M.H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. **41** (1937), 375-481.

[22] N.V. Veličko, H-closed topological spaces, Trans Amer Math Soc, **78**(2) (1968), 103-108.

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