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# PROPERTIES OF $\alpha$-OPEN SETS IN IDEAL GENERALIZED TOPOLOGICAL SPACES

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***Abstract***. The aim of this paper is to introduce and characterize the concepts of $\alpha$-open sets and their related notions in ideal generalized topological spaces. Moreover, we obtain the decomposition of $(g, g')$-continuity in ideal generalized topological space.

## 1. Introduction

In [1], Csaszar introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In the same paper he investigated characterizations of generalized continuous functions ( $= (g, g')$-continuous functions). A subfamily $g$ of the power set $P(X)$ of a nonempty set $X$ is called a generalized topology [1] on $X$ if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \underset{i\in I}{\cup} G_i \in g$. We call the pair $(X, g)$ a generalized topological space (briefly GTS) on $X$. The members of $g$ are called $g$-open sets [1] and the complement of a $g$-open set is called a $g$-closed set. The generalized closure of a set $S$ of $X$, denoted by $g\mathrm{Cl}(A)$, is the intersection of all $g$-closed sets containing $A$ and the generalized interior of $A$, denoted by $g\mathrm{Int}(A)$, is the union of $g$-open sets included in $A$. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [4] and Vaidyanathasamy [6]. An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. The aim of this paper is to introduce and characterized the concepts of $g$-$\alpha$-$\mathcal{I}$-open sets and their related notions in ideal generalized topological spaces.

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## 2. Preliminaries

**Definition 2.1.** Given a generalized topological space $(X, g)$ with an ideal $\mathcal{I}$ on $X$ (for short, IGTS) and if $\mathcal{P}(X)$ is the set of all subsets of $X$, the generalized local function of $A$ with respect to $g$ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(g, \mathcal{I}) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $g$-open set containing $x\}$. When there is no ambiguity, we will write $A^*_g$ for $A^*(g, \mathcal{I})$.

**Definition 2.2.** Let $(X, g, \mathcal{I})$ be an IGTS. The set operator $g\mathrm{Cl}^*$ is called a g-$*$-closure and is denoted as $g\mathrm{Cl}^*(A) = A \cup A^*_g$ for $A \subset X$.

We will denote by $g^*(\mathcal{I}, g)$ the generalized topology generated by $g\mathrm{Cl}^*$, that is $g^*(\mathcal{I}, g) = \{U \subset X : g\mathrm{Cl}^*(X \backslash U) = X \backslash U\}$. Clearly, $g^*(\mathcal{I}, g)$ is finer than $g$. The elements of $g^*(\mathcal{I}, g)$ are called $g$-$*$-open sets and the complement of $g$-$*$-open set is a called $g$-$*$-closed set. Also, $g\mathrm{Int}^*(A)$ denotes the interior of $A$ in $g^*(\mathcal{I})$.

**Definition 2.3.** A subset $A$ of an IGTS $(X, g, \mathcal{I})$ is said to be

(i) $g$-$R$-$\mathcal{I}$-open if $A = g\mathrm{Int}(g\mathrm{Cl}^*(A))$.
(ii) $g$-semi-$\mathcal{I}$-open [3] if $A \subset g\mathrm{Cl}^*(g\mathrm{Int}(A))$.
(iii) $g$-pre-$\mathcal{I}$-open [2] if $A \subset g\mathrm{Int}(g\mathrm{Cl}^*(A))$.
(iv) $g$-$b$-$\mathcal{I}$-open if $A \subset g\mathrm{Int}(g\mathrm{Cl}^*(A)) \cup g\mathrm{Cl}^*(g\mathrm{Int}(A))$ .
(v) $g$-$\beta$-$\mathcal{I}$-open if $A \subset g\mathrm{Cl}(g\mathrm{Int}(g\mathrm{Cl}^*(A)))$.
(vi) $g$-$\delta$-$\mathcal{I}$-open if $g\mathrm{Int}(g\mathrm{Cl}^*(A)) \subset g\mathrm{Cl}^*(g\mathrm{Int}(A))$.

The complement of a $g$-pre-$\mathcal{I}$-open (resp. $g$-semi-$\mathcal{I}$-open, $g$-$\beta$-$\mathcal{I}$-open) set is called a $g$-pre-$\mathcal{I}$-closed (resp. $g$-semi-$\mathcal{I}$-closed, $g$-$\beta$-$\mathcal{I}$-closed) set.

**Lemma 2.4.** *Let $(X, g, \mathcal{I})$ be an IGTS. Then*

*(i) A subset $A$ is $g$-pre-$\mathcal{I}$-closed if and only if $gCl(gInt^*(A)) \subset A$* [2]*;*
*(ii) A subset $A$ is $g$-$\beta$-$\mathcal{I}$-closed if and only if $gInt(gCl(gInt^*(A))) \subset A$.*

**Definition 2.5.** The union of all $g$-pre-$\mathcal{I}$-open (resp. $g$-semi-$\mathcal{I}$-open) sets of $(X, g, \mathcal{I})$ containing $A$ is called the $g$-pre-$\mathcal{I}$-interior [2] (resp. $g$-semi-$\mathcal{I}$-interior [3]) of $A$ and is denote by $gp\mathcal{I}\mathrm{Int}(A)$ (resp. $gs\mathcal{I}\mathrm{Int}(A)$).

**Definition 2.6.** The intersection of all $g$-pre-$\mathcal{I}$-closed (resp. $g$-semi-$\mathcal{I}$-closed) sets of $(X, g, \mathcal{I})$ contained in $A$ is called the $g$-pre-$\mathcal{I}$-closure [2] (resp. $g$-semi-$\mathcal{I}$-closure [3]) of $A$ and is denote by $gp\mathcal{I}\mathrm{Cl}(A)$ (resp. $gs\mathcal{I}\mathrm{Cl}(A)$).

**Lemma 2.7.** *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. Then the following hold*

*(i) $gs\mathcal{I}Int(A) = A \cap gCl^*(gInt(A))$.*
*(ii) $gs\mathcal{I}Cl(A) = A \cup gInt^*(gCl(A))$.*
*(iii) $gp\mathcal{I}Int(A) = A \cap gInt(gCl^*(A))$.*
*(iv) $gp\mathcal{I}Cl(A) = A \cup gCl(gInt^*(A))$.*

**Definition 2.8.** A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is said to be

(i) $(g, g')$-pre-$\mathcal{I}$-continuous [2] if the inverse image of every $g'$-open set of $Y$ is $g$-pre-$\mathcal{I}$-open in $X$.
(ii) $(g, g')$-semi-$\mathcal{I}$-continuous [3] if the inverse image of every $g'$-open set of $Y$ is $g$-semi-$\mathcal{I}$-open in $X$.

(iii) $(g, g')$-$b$-$\mathcal{I}$-continuous if the inverse image of every $g'$-open set of $Y$ is $g$-$b$-$\mathcal{I}$-open in $X$.
(iv) $(g, g')$-$\beta$-$\mathcal{I}$-continuous if the inverse image of every $g'$-open set of $Y$ is $g$-$\beta$-$\mathcal{I}$-open in $X$.
(v) $(g, g')$-$\delta$-$\mathcal{I}$-continuous [3] if the inverse image of every $g'$-open set of $Y$ is $g$-$\delta$-$\mathcal{I}$-open in $X$.
(vi) strongly $(g, g')$-$\beta$-$\mathcal{I}$-continuous if the inverse image of every $g'$-open set of $Y$ is strongly $g$-$\beta$-$\mathcal{I}$-open in $X$.

## 3. $(g, g')$-$\alpha$-$\mathcal{I}$-open sets

**Definition 3.1.** A subset $A$ of an IGTS $(X, g, \mathcal{I})$ is said to be $g$-$\alpha$-$\mathcal{I}$-open if and only if $A \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$.
The family of all $g$-$\alpha$-$\mathcal{I}$-open sets of $(X, g, \mathcal{I})$ is denoted by $\alpha\mathcal{I}O(X, g)$. Also, The family of all $g$-$\alpha$-$\mathcal{I}$-open sets of $(X, g, \mathcal{I})$ containing $x$ is denoted by $\alpha\mathcal{I}O(X, g, x)$.

**Remark 3.2.** Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on $X$. If $\mathcal{I} \subset \mathcal{J}$, then $\alpha\mathcal{J}O(X, g) \subset \alpha\mathcal{I}O(X, g')$.

**Proposition 3.3.** (i) Every $g$-open set is $g$-$\alpha$-$\mathcal{I}$-open.
(ii) Every $g$-$\alpha$-$\mathcal{I}$-open set is $g$-semi-$\mathcal{I}$-open.
(iii) Every $g$-$\alpha$-$\mathcal{I}$-open set is $g$-pre-$\mathcal{I}$-open.

*Proof.* Follows from the definitions. □

**Corollary 3.4.** (i) Every $g$-$\alpha$-$\mathcal{I}$-open set is $g$-$b$-$\mathcal{I}$-open.
(ii) Every $g$-$\alpha$-$\mathcal{I}$-open set is $g$-$\beta$-$\mathcal{I}$-open.

The following examples show that the converses of Proposition 3.3 is not true in general.

**Example 3.5.** Let $X = \{a, b, c, d\}$, $g = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the set $A = \{b, c, d\}$ is $g$-$\alpha$-$\mathcal{I}$-open but not $g$-open.

**Example 3.6.** Let $X = \{a, b, c\}$, $g = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then the set $\{a, c\}$ is $g$-$\alpha$-$\mathcal{I}$-open but not $g$-open. Also, the set $\{b\}$ is $g$-pre-$\mathcal{I}$-open but not $g$-$\alpha$-$\mathcal{I}$-open.

**Example 3.7.** Let $X = \{a, b, c\}$, $g = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{b, c\}$ is $g$-semi-$\mathcal{I}$-open but not $g$-$\alpha$-$\mathcal{I}$-open.

**Proposition 3.8.** Let $A$ be a subset of an IGTS $(X, g, \mathcal{I})$. If $B$ is a $g$-semi-$\mathcal{I}$-open set of $X$ such that $B \subset A \subset g\mathrm{Int}(g\mathrm{Cl}^*(B))$, then A is a $g$-$\alpha$-$\mathcal{I}$-open set of $X$.

*Proof.* Since $B$ is a $g$-semi-$\mathcal{I}$-open set of $X$, we have $B \subset g\mathrm{Cl}^*(g\mathrm{Int}(B))$. Thus, $A \subset g\mathrm{Int}(g\mathrm{Cl}^*(B)) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Cl}^*(g\mathrm{Int}(B)))) = g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(B))) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$, and so $A$ is a $g$-$\alpha$-$\mathcal{I}$-open set of $X$. □

**Proposition 3.9.** Let $(X, g, \mathcal{I})$ be an IGTS. Then a subset of $X$ is $g$-$\alpha$-$\mathcal{I}$-open if and only if it is both $g$-$\delta$-$\mathcal{I}$-open and $g$-pre-$\mathcal{I}$-open.

*Proof.* Let $A$ be a $g$-$\alpha$-$\mathcal{I}$-open set. Since every $g$-$\alpha$-$\mathcal{I}$-open set is $g$-semi-$\mathcal{I}$-open, by Proposition 3.3 $A$ is $g$-$\delta$-$\mathcal{I}$-open. Now we prove that $A \subset g\text{Int}(g\text{Cl}^*(A))$. Since $A$ is a $g$-$\alpha$-$\mathcal{I}$-open, we have $A \subset g\text{Int}(g\text{Cl}^*(g\text{Int}(A))) \subset g\text{Int}(g\text{Cl}^*(A))$. Hence $A$ is $g$-pre-$\mathcal{I}$-open. Conversely, let $A$ be a $g$-$\delta$-$\mathcal{I}$-open and $g$-pre-$\mathcal{I}$-open set. Then we have $g\text{Int}(g\text{Cl}^*(A)) \subset g\text{Cl}^*(g\text{Int}(A))$ and hence $g\text{Int}(g\text{Cl}^*(A)) \subset g\text{Int}(g\text{Cl}^*(g\text{Int}(A)))$. Since $A$ is $g$-pre-$\mathcal{I}$-open, we have $A \subset g\text{Int}(g\text{Cl}^*(A))$. Therefore, we obtain that $A \subset g\text{Int}(g\text{Cl}^*(g\text{Int}(A)))$; hence $A$ is $g$-$\alpha$-$\mathcal{I}$-open. □

**Theorem 3.10.** *A subset $A$ is $g$-$\alpha$-$\mathcal{I}$-open if and only if it is $g$-semi-$\mathcal{I}$-open and $g$-pre-$\mathcal{I}$-open.*

*Proof.* Let $A$ be $g$-semi-$\mathcal{I}$-open and $g$-pre-$\mathcal{I}$-open. Then, $A \subset g\text{Int}(g\text{Cl}^*(A)) \subset g\text{Int}(g\text{Cl}^*(g\text{Cl}^*(g\text{Int}(A)))) = g\text{Int}(g\text{Cl}^*(g\text{Int}(A)))$. This shows that $A$ is $g$-$\alpha$-$\mathcal{I}$-open. The converse is obvious. □

**Corollary 3.11.** The following properties are equivalent for subsets of an IGTS $(X, g, \mathcal{I})$:

(i) Every $g$-pre-$\mathcal{I}$-open set is $g$-semi-$\mathcal{I}$-open.

(ii) A subset $A$ of $X$ is $g$-$\alpha$-$\mathcal{I}$-open if and only if it is $g$-pre-$\mathcal{I}$-open.

*Proof.* (i)$\Rightarrow$ (ii): Let $A$ be a $g$-pre-$\mathcal{I}$-open subset of $X$. Then by (i), $A$ is $g$-semi-$\mathcal{I}$-open, by Theorem 3.10, it is $g$-$\alpha$-$\mathcal{I}$-open.
(ii)$\Rightarrow$ (i): Let $A$ be a $g$-pre-$\mathcal{I}$-open subset of $X$. Then by (ii), $A$ is $g$-$\alpha$-$\mathcal{I}$-open, by Theorem 3.3, it is $g$-semi-$\mathcal{I}$-open. □

**Corollary 3.12.** The following properties are equivalent for subsets of an IGTS $(X, g, \mathcal{I})$:

(i) Every $g$-semi-$\mathcal{I}$-open set is $g$-pre-$\mathcal{I}$-open.

(ii) A subset $A$ of $X$ is $g$-$\alpha$-$\mathcal{I}$-open if and only if it is $g$-semi-$\mathcal{I}$-open.

*Proof.* (i)$\Rightarrow$ (ii): Let $A$ be a $g$-semi-$\mathcal{I}$-open subset of $X$. Then by (i), $A$ is $g$-pre-$\mathcal{I}$-open, by Theorem 3.10, it is $g$-$\alpha$-$\mathcal{I}$-open.
(ii)$\Rightarrow$ (i): Let $A$ be a $g$-semi-$\mathcal{I}$-open subset of $X$. Then by (ii), $A$ is $g$-$\alpha$-$\mathcal{I}$-open, by Theorem 3.3, it is $g$-pre-$\mathcal{I}$-open. □

**Proposition 3.13.** Let $A$ be a subset of an IGTS $(X, g, \mathcal{I})$. If $A$ is $g$-pre-$\mathcal{I}$-closed and $g$-$\alpha$-$\mathcal{I}$-open, then it is $g$-open.

*Proof.* Suppose $A$ is $g$-pre-$\mathcal{I}$-closed and $g$-$\alpha$-$\mathcal{I}$-open. Then by Lemma 2.4 $g\text{Cl}(g\text{Int}^*(A)) \subset A$ and $A \subset g\text{Int}(g\text{Cl}^*(g\text{Int}(A)))$. Now $g\text{Cl}(g\text{Int}(A)) \subset g\text{Cl}(g\text{Int}(A)) \subset g\text{Cl}(g\text{Int}^*(A)) \subset A$ and so $A \subset g\text{Int}(g\text{Cl}^*(g\text{Int}(A)) \subset A \subset g\text{Int}(A)$. Therefore, $A$ is $g$-open. □

**Lemma 3.14.** *If $A$ is any subset of an IGTS $(X, g, \mathcal{I})$, then $gInt(gCl^*(A))$ is $g$-$R$-$\mathcal{I}$-open.*

**Proposition 3.15.** Let $A$ be a subset of an IGTS $(X, g, \mathcal{I})$. If $A$ is $g$-$\alpha$-$\mathcal{I}$-open and $g$-$\beta$-$\mathcal{I}$-closed, then it is $g$-$R$-$\mathcal{I}$-open.

*Proof.* Let $A$ be $g$-$\alpha$-$\mathcal{I}$-open and $g$-$\beta$-$\mathcal{I}$-closed. We have by Lemma 2.4, $A \subset g\text{Int}(g\text{Cl}^*(g\text{Int}(A)))$ and $g\text{Int}(g\text{Cl}^*(g\text{Int}(A))) \subset g\text{Int}(g\text{Cl}(g\text{Int}^*(A))) \subset A$; hence $A = g\text{Int}(g\text{Cl}^*(g\text{Int}(A)))$. Thus, by Lemma 3.14, $A$ is $g$-$R$-$\mathcal{I}$-open. □

**Lemma 3.16.** *Let $A$ be a subset of an IGTS $(X, g, \mathcal{I})$. Then $A$ is $g$-open and $g$-closed if and only if it is $g$-$\alpha$-$\mathcal{I}$-open and $g$-pre-$\mathcal{I}$-closed.*

*Proof.* Let $A$ be $g$-$\alpha$-$\mathcal{I}$-open and $g$-pre-$\mathcal{I}$-closed. We have $A \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$ and $g\mathrm{Cl}^*(g\mathrm{Int}(A)) \subset A$ and hence $A \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) \subset g\mathrm{Int}(g\mathrm{Cl}(g\mathrm{Int}^*(A))) \subset g\mathrm{Cl}(g\mathrm{Int}^*(A)) \subset A$. Thus, $A = g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) = g\mathrm{Cl}(g\mathrm{Int}^*(A))$; hence $A$ is $g$-open and $g$-closed. □

**Remark 3.17.** The intersection of two $g$-$\alpha$-$\mathcal{I}$-open sets need not be $g$-$\alpha$-$\mathcal{I}$-open in general. Let $(X, g, \mathcal{I})$ as in Example 3.6. Then the sets $\{a, c\}$ and $\{b, c\}$ are $g$-$\alpha$-$\mathcal{I}$-open sets of $(X, g, \mathcal{I})$ but their intersection $\{c\}$ is not an $g$-$\alpha$-$\mathcal{I}$-open set of $(X, g, \mathcal{I})$.

**Theorem 3.18.** *If $\{A_\alpha\}_{\alpha\in\Omega}$ be a family of $g$-$\alpha$-$\mathcal{I}$-open sets in $(X, g, \mathcal{I})$, then $\bigcup\limits_{\alpha\in\Omega} A_\alpha$ is $g$-$\alpha$-$\mathcal{I}$-open in $(X, g, \mathcal{I})$.*

*Proof.* Since $\{A_\alpha : \alpha \in \Omega\} \subset g\text{-}\alpha\mathcal{I}O(X)$, then $A_\alpha \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A_\alpha)))$ for every $\alpha \in \Omega$. Thus, $\underset{\alpha\in\Omega}{\cup} A_\alpha \subset \underset{\alpha\in\Omega}{\cup} g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A_\alpha))) \subset g\mathrm{Int}(g\mathrm{Cl}^*(\underset{\alpha\in\Omega}{\cup} g\mathrm{Int}(A_\alpha))$ $= g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(\underset{\alpha\in\Omega}{\cup} A_\alpha))$. Therefore, we obtain $\underset{\alpha\in\Omega}{\cup} A_\alpha \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(\underset{\alpha\in\Omega}{\cup} A_\alpha))$. Hence any union of $g$-$\alpha$-$\mathcal{I}$-open sets is $g$-$\alpha$-$\mathcal{I}$-open. □

**Definition 3.19.** In an IGTS $(X, g, \mathcal{I})$, $A \subset X$ is said to be $g$-$\alpha$-$\mathcal{I}$-closed if $X\backslash A$ is $g$-$\alpha$-$\mathcal{I}$-open in $X$.
The family of all $g$-$\alpha$-$\mathcal{I}$-closed sets of $(X, g, \mathcal{I})$ is denoted by $\alpha\mathcal{I}C(X, g)$.

**Theorem 3.20.** *If $A$ is a $g$-$\alpha$-$\mathcal{I}$-closed set in an IGTS $(X, g, \mathcal{I})$ if and only if $gCl(gInt^*(gCl(A))) \subset A$.*

*Proof.* Follows from the definitions. □

**Theorem 3.21.** *If $A$ is a $g$-$\alpha$-$\mathcal{I}$-closed set in an IGTS $(X, g, \mathcal{I})$, then $gCl(gInt(gCl^*(A))) \subset A$.*

*Proof.* Since $A \in \alpha\mathcal{I}C(X, g)$, $X\backslash A \in \alpha\mathcal{I}O(X, g)$. Hence, $X\backslash A \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(X\backslash A)) \subset g\mathrm{Int}(g\mathrm{Cl}(g\mathrm{Int}(X\backslash A))) = X\backslash g\mathrm{Cl}(g\mathrm{Int}(g\mathrm{Cl}(A))) \subset X\backslash(g\mathrm{Cl}(g\mathrm{Int}(g\mathrm{Cl}^*(A)))$. Therefore, we obtain $g\mathrm{Cl}(g\mathrm{Int}(g\mathrm{Cl}^*(A)) \subset A$. □

**Proposition 3.22.** Let $(X, g, \mathcal{I})$ be an IGTS. If a subset of $X$ is $g$-$\beta$-$\mathcal{I}$-closed and $g$-$\delta$-$\mathcal{I}$-open, then it is $g$-$\alpha$-$\mathcal{I}$-closed.

*Proof.* Follows from the definitions. □

**Theorem 3.23.** *Arbitrary intersection of $g$-$\alpha$-$\mathcal{I}$-closed sets is always $g$-$\alpha$-$\mathcal{I}$-closed.*

*Proof.* Follows from Theorems 3.18 and 3.21. □

**Definition 3.24.** Let $(X, g, \mathcal{I})$ be an IGTS, $S$ a subset of $X$ and $x$ be a point of $X$. Then

(i) $x$ is called an $g$-$\alpha$-$\mathcal{I}$-interior point of $S$ if there exists $V \in \alpha\mathcal{I}O(X, g)$ such that $x \in V \subset S$.

ii) the set of all $g$-$\alpha$-$\mathcal{I}$-interior points of $S$ is called $g$-$\alpha$-$\mathcal{I}$-interior of $S$ and is denoted by $g\alpha\mathcal{I}\mathrm{Int}(S)$.

**Theorem 3.25.** *Let $A$ and $B$ be subsets of $(X, g, \mathcal{I})$. Then the following properties hold:*

(i) $g\alpha\mathcal{I}Int(A) = \cup\{T : T \subset A \text{ and } A \in \alpha\mathcal{I}O(X, g)\}$.
(ii) $g\alpha\mathcal{I}Int(A)$ *is the largest* $g$-$\alpha$-$\mathcal{I}$-*open subset of* $X$ *contained in* $A$.
(iii) $A$ *is* $g$-$\alpha$-$\mathcal{I}$-*open if and only if* $A = g\alpha\mathcal{I}Int(A)$.
(iv) $g\alpha\mathcal{I}Int(g\alpha\mathcal{I}Int(A)) = g\alpha\mathcal{I}Int(A)$.
(v) *If* $A \subset B$, *then* $g\alpha\mathcal{I}Int(A) \subset g\alpha\mathcal{I}Int(B)$.
(vi) $g\alpha\mathcal{I}Int(A \cap B) = g\alpha\mathcal{I}Int(A) \cap g\alpha\mathcal{I}Int(B)$.
(vii) $g\alpha\mathcal{I}Int(A \cup B) \subset g\alpha\mathcal{I}Int(A) \cup g\alpha\mathcal{I}Int(B)$.

*Proof.* $(i)$. Let $x \in \cup\{T : T \subset A$ and $A \in \alpha\mathcal{I}O(X, g)\}$. Then, there exists $T \in \alpha\mathcal{I}O(X, g, x)$ such that $x \in T \subset A$ and hence $x \in g\alpha\mathcal{I}\mathrm{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $A \in \alpha\mathcal{I}O(X, g)\} \subset g\alpha\mathcal{I}\mathrm{Int}(A)$. For the reverse inclusion, let $x \in g\alpha\mathcal{I}\mathrm{Int}(A)$. Then there exists $T \in \alpha\mathcal{I}O(X, g, x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $A \in \alpha\mathcal{I}O(X, g)\}$. This shows that $g\alpha\mathcal{I}\mathrm{Int}(A) \subset \cup\{T : T \subset A$ and $A \in \alpha\mathcal{I}O(X, g)\}$. Therefore, we obtain $g\alpha\mathcal{I}\mathrm{Int}(A) = \cup\{T : T \subset A$ and $A \in \alpha\mathcal{I}O(X, g)\}$. The proof of $(ii) - (v)$ are obvious.
$(vi)$. By $(v)$, we have $g\alpha\mathcal{I}\mathrm{Int}(A) \subset g\alpha\mathcal{I}\mathrm{Int}(A \cup B)$ and $g\alpha\mathcal{I}\mathrm{Int}(B) \subset g\alpha\mathcal{I}\mathrm{Int}(A \cup B)$. Then we obtain $g\alpha\mathcal{I}\mathrm{Int}(A) \cup g\alpha\mathcal{I}\mathrm{Int}(B) \subset g\alpha\mathcal{I}\mathrm{Int}(A \cup B)$ Since $g\alpha\mathcal{I}\mathrm{Int}(A) \subset A$ and $g\alpha\mathcal{I}\mathrm{Int}(B) \subset B$, we obtain $g\alpha\mathcal{I}\mathrm{Int}(A \cup B) \subset g\alpha\mathcal{I}\mathrm{Int}(A) \cup g\alpha\mathcal{I}\mathrm{Int}(B)$. It follows that $g\alpha\mathcal{I}\mathrm{Int}(A \cap B) = g\alpha\mathcal{I}\mathrm{Int}(A) \cap g\alpha\mathcal{I}\mathrm{Int}(B)$.
$(vii)$. Since $A \cap B \subset A$ and $A \cap B \subset B$, we have $g\alpha\mathcal{I}\mathrm{Int}(A \cap B) \subset g\alpha\mathcal{I}\mathrm{Int}(A)$ and $g\alpha\mathcal{I}\mathrm{Int}(A \cap B) \subset g\alpha\mathcal{I}\mathrm{Int}(B)$ . Therefore, $g\alpha\mathcal{I}\mathrm{Int}(A) \cup g\alpha\mathcal{I}\mathrm{Int}(B) \subset g\alpha\mathcal{I}\mathrm{Int}(A \cap B)$. □

**Theorem 3.26.** *If $(X, g, \mathcal{I})$ is an IGTS, then $g\alpha\mathcal{I}Int(A) = A \cap gInt(gCl^*(gInt(A)))$ holds for every subset $A$ of $X$.*

*Proof.* Since $A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) = g\mathrm{Int}(g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))))$ $= g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)) \cap (g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A) \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))))$ $= g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))))))$, $A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$ is an $g$-$\alpha$-$\mathcal{I}$-open set contained in $A$ and so $A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) \subset g\alpha\mathcal{I}\mathrm{Int}(A)$. Since $g$-$\alpha\mathcal{I}\mathrm{Int}(A)$ is $g$-$\alpha$-$\mathcal{I}$-open, $g$-$\alpha\mathcal{I}\mathrm{Int}(A) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(g\alpha\mathcal{I}\mathrm{Int}(A)))) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$ and so $g\alpha\mathcal{I}\mathrm{Int}(A) \subset A \cap g\mathrm{Int}(g\mathrm{Cl}^*(\mathrm{Int}(A)))$. Hence $g\alpha\mathcal{I}\mathrm{Int}(A) = A \cap \mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$. □

**Theorem 3.27.** *If $(X, g, \mathcal{I})$ is an IGTS, then $g\alpha\mathcal{I}Cl(A) = A \cap gCl(gInt^*(gCl(A)))$ holds for every subset $A$ of $X$.*

*Proof.* Follows from Theorem 3.26. □

**Theorem 3.28.** *If $(X, g, \mathcal{I})$ is an IGTS, then $g\alpha\mathcal{I}Int(A) = gp\mathcal{I}Int(A)$ holds for every $g$-$\delta$-$\mathcal{I}$-open subset $A$ of $X$.*

*Proof.* Since every $g$-$\alpha$-$\mathcal{I}$-open set is $g$-pre-$\mathcal{I}$-open, $g\alpha\mathcal{I}\mathrm{Int}(A) \subset gp\mathcal{I}\mathrm{Int}(A)$. By Theorem 3.26, $g\alpha\mathcal{I}\mathrm{Int}(A) = A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$. Since $A$ is $g$-$\delta$-$\mathcal{I}$-open, $g\alpha\mathcal{I}\mathrm{Int}(A) \supset A \cap g\mathrm{Int}(g\mathrm{Int}(g\mathrm{Cl}^*(A))) = A \cap g\mathrm{Int}(g\mathrm{Cl}^*(A)) = gp\mathcal{I}\mathrm{Int}(A)$ by Lemma 2.7 and so $g\alpha\mathcal{I}\mathrm{Int}(A) \supset gp\mathcal{I}\mathrm{Int}(A)$. Therefore, $g\alpha\mathcal{I}\mathrm{Int}(A) = gp\mathcal{I}\mathrm{Int}(A)$. □

**Definition 3.29.** Let $(X, g, \mathcal{I})$ be an IGTS, $S$ a subset of $X$ and $x$ be a point of $X$. Then

(i) $x$ is called an $g$-$\alpha$-$\mathcal{I}$-cluster point of $S$ if $V \cap S \neq \emptyset$ for every $V \in \alpha\mathcal{I}O(X, g, x)$.

(ii) the set of all $g$-$\alpha$-$\mathcal{I}$-cluster points of $S$ is called $g$-$\alpha$-$\mathcal{I}$-closure of $S$ and is denoted by $g\alpha\mathcal{I}\mathrm{Cl}(S)$.

**Theorem 3.30.** *Let $A$ and $B$ be subsets of $(X, g, \mathcal{I})$. Then the following properties hold:*

*(i) $g$-$\alpha\mathcal{I}Cl(A) = \cap\{F : A \subset F$ and $F \in \alpha\mathcal{I}C(X, g)\}$.*
*(ii) $g\alpha\mathcal{I}Cl(A)$ is the smallest $g$-$\alpha$-$\mathcal{I}$-closed subset of $X$ containing $A$.*
*(iii) $A$ is $g$-$\alpha$-$\mathcal{I}$-closed if and only if $A = g\alpha\mathcal{I}Cl(A)$.*
*(iv) $g\alpha\mathcal{I}Cl(g\alpha\mathcal{I}Cl(A) = g\alpha\mathcal{I}Cl(A)$.*
*(v) If $A \subset B$, then $g\alpha\mathcal{I}Cl(A) \subset g\alpha\mathcal{I}Cl(B)$.*
*(vi) $g\alpha\mathcal{I}Cl(A \cup B) = g\alpha\mathcal{I}Cl(A) \cup g\alpha\mathcal{I}Cl(B)$.*
*(vii) $g\alpha\mathcal{I}Cl(A \cap B) \subset g\alpha\mathcal{I}Cl(A) \cap g\alpha\mathcal{I}Cl(B)$.*

*Proof.* $(i)$. We shall only proof (i) as the proofs of the other statements are obvious. Suppose that $x \notin g\alpha\mathcal{I}\mathrm{Cl}(A)$. Then there exists $V \in \alpha\mathcal{I}O(X, g)\}$ such that $V \cap A \neq \emptyset$. Since $X\backslash V$ is $g$-$\alpha$-$\mathcal{I}$-closed set containing $A$ and $x \notin X\backslash V$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in \alpha\mathcal{I}C(X, g)\}$. Then there exists $F \in \alpha\mathcal{I}C(X, g)$ such that $A \subset F$ and $x \notin F$. Since $X\backslash V$ is $g$-$\alpha$-$\mathcal{I}$-closed set containing $x$, we obtain $(X\backslash F) \cap A = \emptyset$. This shows that $x \notin g\alpha\mathcal{I}\mathrm{Cl}(A)$. Therefore, we obtain $g\alpha\mathcal{I}\mathrm{Cl}(A) = \cap\{F : A \subset F$ and $F \in \alpha\mathcal{I}C(X, g)\}$. $\square$

**Theorem 3.31.** *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. A point $x \in g\alpha\mathcal{I}Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \alpha\mathcal{I}O(X, g, x)$.*

*Proof.* Suppose that $x \in g\alpha\mathcal{I}\mathrm{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in \alpha\mathcal{I}O(X, g, x)$. Suppose that there exists $U \in \alpha\mathcal{I}O(X, g, x)$ such that $U \cap A = \emptyset$. Then $A \subset X\backslash U$ and $X\backslash U$ is $g$-$\alpha$-$\mathcal{I}$-closed. since $A \subset X\backslash U$, $g\alpha\mathcal{I}\mathrm{Cl}(A) \subset g\alpha\mathcal{I}\mathrm{Cl}(X\backslash U)$. Since $x \in g\alpha\mathcal{I}\mathrm{Cl}(A)$, we have $x \in g\alpha\mathcal{I}\mathrm{Cl}(X\backslash U)$. Since $X\backslash U$ is $g$-$\alpha$-$\mathcal{I}$-closed, we have $x \in X\backslash U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in \alpha\mathcal{I}O(X, g, x)$. We shall show that $x \in g\alpha\mathcal{I}\mathrm{Cl}(A)$. Suppose that $x \notin g\alpha\mathcal{I}\mathrm{Cl}(A)$. Then there exists $U \in \alpha\mathcal{I}O(X, g, x)$ such that $U \cap A = \emptyset$. This is a contradiction to $U \cap A \neq \emptyset$; hence $x \in g\alpha\mathcal{I}\mathrm{Cl}(A)$. $\square$

**Theorem 3.32.** *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. Then the following propeties hold:*

*(i) $g\alpha\mathcal{I}Int(X\backslash A) = X\backslash g\alpha\mathcal{I}Cl(A)$;*
*(i) $g\alpha\mathcal{I}Cl(X\backslash A) = X\backslash g\alpha\mathcal{I}Int(A)$.*

*Proof.* $(i)$. Let $x \in X\backslash g\alpha\mathcal{I}\mathrm{Cl}(A)$. Then, there exists $V \in \alpha\mathcal{I}O(X, g, x)$ such that $V \cap A \neq \emptyset$; hence we obtain $x \in g\alpha\mathcal{I}\mathrm{Int}(X\backslash A)$. This shows that $X\backslash g\alpha\mathcal{I}\mathrm{Cl}(A) \subset g$-$\alpha\mathcal{I}\mathrm{Int}(X\backslash A)$. Let $x \in g\alpha\mathcal{I}\mathrm{Int}(X\backslash A)$. Since $g\alpha\mathcal{I}\mathrm{Int}(X\backslash A) \cap A = \emptyset$, we obtain $x \notin g\alpha\mathcal{I}\mathrm{Cl}(A)$; hence $x \in X\backslash g\alpha\mathcal{I}\mathrm{Cl}(A)$. Therefore, we obtain $g\alpha\mathcal{I}\mathrm{Int}(X\backslash A) = X\backslash \alpha\mathcal{I}\mathrm{Cl}(A)$.
$(ii)$. Follows from $(i)$. $\square$

**Definition 3.33.** A subset $B_x$ of an IGTS $(X, g, \mathcal{I})$ is said to be an $g$-$\alpha$-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an $g$-$\alpha$-$\mathcal{I}$-open set $U$ such that $x \in U \subset B_x$.

**Theorem 3.34.** *A subset of an IGTS $(X, g, \mathcal{I})$ is $g$-$\alpha$-$\mathcal{I}$-open if and only if it is an $g$-$\alpha$-$\mathcal{I}$-neighbourhood of each of its points.*

*Proof.* Let $G$ be an $g$-$\alpha$-$\mathcal{I}$-open set of $X$. Then by definition, it is clear that $G$ is an $g$-$\alpha$-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and $G$ is $g$-$\alpha$-$\mathcal{I}$-open. Conversely, suppose $G$ is an $g$-$\alpha$-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in \alpha\mathcal{I}O(X, g)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each $S_x$ is $g$-$\alpha$-$\mathcal{I}$-open, $G$ is $g$-$\alpha$-$\mathcal{I}$-open in $(X, g, \mathcal{I})$. □

## 4. $(g, g')$-$\alpha$-$\mathcal{I}$-continuous functions

**Definition 4.1.** A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is said to be $(g, g')$-$\alpha$-$\mathcal{I}$-continuous if the inverse image of every $g'$-open set of $Y$ is $g$-$\alpha$-$\mathcal{I}$-open in $X$.

**Proposition 4.2.** (i) Every $(g, g')$-continuous function is $(g, g')$-$\alpha$-$\mathcal{I}$-continuous but not conversely.
(ii) Every $(g, g')$-$\alpha$-$\mathcal{I}$-continuous function is $(g, g')$-semi-$\mathcal{I}$-continuous but not conversely.
(iii) Every $(g, g')$-$\alpha$-$\mathcal{I}$-continuous function is $(g, g')$-pre-$\mathcal{I}$-continuous but not conversely.

*Proof.* Follows from Proposition 3.3 and Examples 3.5, 3.6 and 3.7. □

**Corollary 4.3.** (i) Every $(g, g')$-$\alpha$-$\mathcal{I}$-continuous function is $(g, g')$-$b$-$\mathcal{I}$-continuous but not conversely.
(ii) Every $(g, g')$-$\alpha$-$\mathcal{I}$-continuous function is $(g, g')$-$\beta$-$\mathcal{I}$-continuous but not conversely.

**Theorem 4.4.** *A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is $(g, g')$-$\alpha$-$\mathcal{I}$-continuous if and only if it is $(g, g')$-semi-$\mathcal{I}$-continuous and $(g, g')$-pre-$\mathcal{I}$-continuous.*

*Proof.* This is an immediate consequence of Lemma 3.10. □

**Theorem 4.5.** *For a function $f : (X, g, \mathcal{I}) \to (Y, g')$, the following statements are equivalent:*
*(i) $f$ is $(g, g')$-$\alpha$-$\mathcal{I}$-continuous;*
*(ii) For each point $x$ in $X$ and each $g'$-open set $F$ in $Y$ such that $f(x) \in F$, there is a $g$-$\alpha$-$\mathcal{I}$-open set $A$ in $X$ such that $x \in A$, $f(A) \subset F$;*
*(iii) The inverse image of each $g'$-closed set in $Y$ is $g$-$\alpha$-$\mathcal{I}$-closed in $X$;*
*(iv) For each subset $A$ of $X$, $f(g\alpha\mathcal{I}Cl(A)) \subset g'Cl(f(A))$;*
*(v) For each subset $B$ of $Y$, $g\alpha\mathcal{I}Cl(f^{-1}(B)) \subset f^{-1}(g'Cl(B))$;*
*(vi) For each subset $C$ of $Y$, $f^{-1}(g'Int(C)) \subset g\alpha\mathcal{I}Int(f^{-1}(C))$.*
*(vii) $gCl(gInt^*(gCl(f^{-1}(B)))) \subset f^{-1}(g'Cl(B))$ for each subset $B$ of $Y$.*
*(viii) $f(gCl(gInt^*(gCl(A)))) \subset g'Cl(f(A))$ for each subset $A$ of $X$.*

*Proof.* $(i) \Rightarrow (ii)$: Let $x \in X$ and $F \in g'$ containing $f(x)$. By (i), $f^{-1}(F)$ is $g$-$\alpha$-$\mathcal{I}$-open in $X$. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.
$(ii) \Rightarrow (i)$: Let $F \in g'$ and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is a $g$-$\alpha$-$\mathcal{I}$-open set $U_x$ in $X$ such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is $g$-$\alpha$-$\mathcal{I}$-open in $X$.
$(i) \Leftrightarrow (iii)$: This follows due to the fact that for any subset $B$ of $Y$, $f^{-1}(Y\backslash B) = X\backslash f^{-1}(B)$.
$(iii) \Rightarrow (iv)$: Let $A$ be a subset of $X$. Since $A \subset f^{-1}(f(A))$ we have $A \subset f^{-1}(g'\mathrm{Cl}(f(A)))$. Now, $g'\mathrm{Cl}(f(A))$ is $g'$-closed in $Y$ and hence $g\alpha\mathcal{I}\mathrm{Cl}(A)) \subset f^{-1}(g'\mathrm{Cl}(f(A)))$, for $g\alpha\mathcal{I}\mathrm{Cl}(A)$

is the smallest $g$-$\alpha$-$\mathcal{I}$-closed set containing $A$. Then $f(g\alpha\mathcal{I}\mathrm{Cl}(A)) \subset g'\mathrm{Cl}(f(A))$.
$(iv) \Rightarrow (iii)$: Let $F$ be any $g'$-closed subset of $Y$. Then $f(g\alpha\mathcal{I}\mathrm{Cl}(f^{-1}(F))) \subset g'\mathrm{Cl}(f(f^{-1}(F))) = g'\mathrm{Cl}(F) = F$. Therefore, $g\alpha\mathcal{I}\mathrm{Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is $g$-$\alpha$-$\mathcal{I}$-closed in $X$.
$(iv) \Rightarrow (v)$: Let $B$ be any subset of $Y$. Now, $f(g\alpha\mathcal{I}\mathrm{Cl}(f^{-1}(B))) \subset g'\mathrm{Cl}(f(f^{-1}(B))) \subset g'\mathrm{Cl}(B)$. Consequently, $g\alpha\mathcal{I}\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(g'\mathrm{Cl}(B))$.
$(v) \Rightarrow (iv)$: Let $B = f(A)$ where $A$ is a subset of $X$. Then, $g\alpha\mathcal{I}\mathrm{Cl}(A) \subset g\alpha\mathcal{I}\mathrm{Cl}(f^{-1}(B))$ $\subset f^{-1}(g'\mathrm{Cl}(B)) = f^{-1}(g'\mathrm{Cl}(f(A)))$. This shows that $f(g\alpha\mathcal{I}\mathrm{Cl}(A)) \subset g'\mathrm{Cl}(f(A))$.
$(i) \Rightarrow (vi)$: Let $B \in g'$. Clearly, $f^{-1}(g'\mathrm{Int}(B))$ is $g$-$\alpha$-$\mathcal{I}$-open and we have $f^{-1}(g'\mathrm{Int}(B))$ $\subset g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(g'\mathrm{Int}(B))) \subset g\alpha\mathcal{I}\mathrm{Int}(f^{-1}B)$.
$(vi) \Rightarrow (i)$: Let $B \in g'$. Then $g'\mathrm{Int}(B) = B$ and $f^{-1}(B)\backslash f^{-1}(g'\mathrm{Int}(B)) \subset g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(B))$. Hence we have $f^{-1}(B) = g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is $g$-$\alpha$-$\mathcal{I}$-open in $X$.
$(iii) \Rightarrow (vii)$: Let $B$ be any subset os $Y$. Since $g'\mathrm{Cl}(B)$ is $g'$-closed in $Y$, by (iii), $f^{-1}(g'\mathrm{Cl}(B))$ is $g$-$\alpha$-$\mathcal{I}$-closed and $X\backslash f^{-1}(g'\mathrm{Cl}(B))$ is $g$-$\alpha$-$\mathcal{I}$-open. Then $X\backslash f^{-1}(g'\mathrm{Cl}(B))$ $\subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(f^{-1}(g'\mathrm{Cl}(B))))) = X\backslash g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(f^{-1}(g'\mathrm{Cl}(B)))))$. Hence we obtain $g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(f^{-1}(B)))) \subset f^{-1}(g'\mathrm{Cl}(B))$.
$(vii) \Rightarrow (viii)$: Let $A$ be any ubset of $X$. By(iv), we have $g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(A)) \subset g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(f^{-1}(f(A))))) \subset f^{-1}(g'\mathrm{Cl}(f(A)))$ and hence $f(g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(A)))) \subset g'\mathrm{Cl}(f(A))$.
$(viii) \Rightarrow (i)$: Let $V \in g'$. Then by (v), $f(g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(f^{-1}(Y\backslash V))))) \subset g'\mathrm{Cl}(f(f^{-1}(Y\backslash V))) \subset g'\mathrm{Cl}(Y\backslash V) = Y\backslash V$. Therefore, we have $g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(f^{-1}(Y\backslash V)))) \subset f^{-1}(Y\backslash V) \subset X\backslash f^{-1}(V)$. Consequently, we obtain $f^{-1}(V) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(f^{-1}(V))))$. This shows that $f^{-1}(V)$ is $\alpha$-$\mathcal{I}$-open. Thus, $f$ is $\alpha$-$\mathcal{I}$-continuous. □

**Corollary 4.6.** Let $f : (X, g, \mathcal{I}) \to (Y, g', \mathcal{I})$ be an $g$-$\alpha$-$\mathcal{I}$-continuous function, then

(i) $f(g\mathrm{Cl}^*(U)) \subset g'\mathrm{Cl}(f(U))$ for every $g$-pre-$\mathcal{I}$-open set $U$ of $X$,
(ii) $g\mathrm{Cl}^*(f^{-1}(V)) \subset f^{-1}(g'\mathrm{Cl}(V))$ for every $g'$-pre-$\mathcal{I}$-open set $V$ of $Y$.

*Proof.* (1). Let $U$ be any $g$-pre-$\mathcal{I}$-open set of $X$, then $U \subset g\mathrm{Int}(g\mathrm{Cl}^*(U))$. Therefore, by Theorem 4.5, we have $f(g\mathrm{Cl}^*(U)) \subset f(g\mathrm{Cl}(U)) \subset f(g\mathrm{Cl}(g\mathrm{Int}(g\mathrm{Cl}^*(U)))) \subset f(g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(U)))) \subset g'\mathrm{Cl}(f(U))$.
(2). Let $V$ be any $g'$-pre-$\mathcal{I}$-open set of $Y$. By Theorem 4.5, $g\mathrm{Cl}^*(f^{-1}(V)) \subset g\mathrm{Cl}(f^{-1}(V))$ $\subset g\mathrm{Cl}(f^{-}(g'\mathrm{Int}(g'\mathrm{Cl}^*(V)))) \subset g\mathrm{Cl}(g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(f^{-1}(g'\mathrm{Int}(g'\mathrm{Cl}^((V)))))))$ $\subset g\mathrm{Cl}(g\mathrm{Int}^*(g\mathrm{Cl}(f^{-1}(g'\mathrm{Int}(g'\mathrm{Cl}^((V))))))) \subset f^{-1}(g'\mathrm{Cl}(g'\mathrm{Int}(g'\mathrm{Cl}^(V)))) \subset f^{-1}(g'\mathrm{Cl}(V))$. □

**Theorem 4.7.** *Let $f : (X, g, \mathcal{I}) \to (Y, g')$ be a $(g, g')$-$\alpha$-$\mathcal{I}$-continuous function. Then for each subset $V$ of $Y$, $f^{-1}(g'Int(V)) \subset gCl^*(f^{-1}(V))$.*

*Proof.* Let $V$ be any subset of $Y$. Then $g'\mathrm{Int}(V)$ is $g'$-open in $Y$ and so $f^{-1}(g'\mathrm{Int}(V))$ is $g$-$\alpha$-$\mathcal{I}$-open in $X$. Hence $f^{-1}(g'\mathrm{Int}(V)) \subset g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(f^{-1}(g'\mathrm{Int}(V))))) \subset g\mathrm{Cl}^*(f^{-1}(V))$. □

**Theorem 4.8.** *Let $f : (X, g, \mathcal{I}) \to (Y, g')$ be a bijective. Then $f$ is $(g, g')$-$\alpha$-$\mathcal{I}$-continuous if and only if $g'Int(f(U)) \subset f(g\alpha\mathcal{I}Int(U))$ for each subset $U$ of $X$.*

*Proof.* Let $U$ be any subset of $X$. Then by Theorem 4.5, $f^{-1}(g'\mathrm{Int}(f(U))) \subset g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(f(U)))$. Since $f$ is bijection, $g'\mathrm{Int}(f(U)) = f(f^{-1}(g'\mathrm{Int}(f(U))) \subset f(g\alpha\mathcal{I}\mathrm{Int}(U))$. Conversely, let $V$ be any subset of $Y$. Then $g'\mathrm{Int}(f(f^{-1}(V))) \subset f(g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(V)))$. Since $f$ is bijection, $g'\mathrm{Int}(V) = g'\mathrm{Int}(f(f^{-1}(V))) \subset f(g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(V)))$; hence $f^{-1}(g'\mathrm{Int}(V) \subset g\alpha\mathcal{I}\mathrm{Int}(f^{-1}(V))$. Therefore, by Theorem 4.5, $f$ is $(g,g')$-$\alpha$-$\mathcal{I}$-continuous. $\square$

**Definition 4.9.** A subset $K$ of an IGTS $(X,g,\mathcal{I})$ is said to be $g$-$\alpha$-$\mathcal{I}$-compact relative to $X$, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of $K$ by $g$-$\alpha$-$\mathcal{I}$-open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $K \backslash \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X,g,\mathcal{I})$ is said to be $g$-$\alpha$-$\mathcal{I}$-compact if $X$ is $g$-$\alpha$-$\mathcal{I}$-compact subsets of $X$.

**Definition 4.10.** A subset $K$ of an IGTS $(X,g,\mathcal{I})$ is said to be countably $g$-$\alpha$-$\mathcal{I}$-compact relative to $X$, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of $K$ by countable $g$-$\alpha$-$\mathcal{I}$-open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $K \backslash \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, g\mathcal{I})$ is said to be countably $g$-$\alpha$-$\mathcal{I}$-compact if $X$ is countable $g$-$\alpha$-$\mathcal{I}$-compact subset of $X$.

**Definition 4.11.** A subset $K$ of an IGTS $(X,g,\mathcal{I})$ is said to be $g$-$\alpha$-$\mathcal{I}$-Lindelof relative to $X$, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of $K$ by $g$-$\alpha$-$\mathcal{I}$-open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $K \backslash \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X,g,\mathcal{I})$ is said to be $g$-$\alpha$-$\mathcal{I}$-Lindelof if $X$ is $g$-$\alpha$-$\mathcal{I}$-Lindelof subset of $X$.

**Lemma 4.12.** [5] *For any function $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$, $f(\mathcal{I})$ is an ideal on $Y$.*

**Theorem 4.13.** *If $f : (X,g,\mathcal{I}) \to (Y,g')$ is a $(g,g')$-$\alpha$-$\mathcal{I}$-continuous surjection and $(X,g,\mathcal{I})$ is $g$-$\alpha$-$\mathcal{I}$-compact,then $(Y,g',f(\mathcal{I}))$ is $g'$-$f(\mathcal{I})$-compact.*

*Proof.* Let $\{V_\lambda : \lambda \in \Lambda\}$ be a $g'$-open cover of $Y$. Then $\{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is a $g$-$\alpha$-$\mathcal{I}$-open cover of $X$ and hence, there exist a finite subset $\Lambda_0$ of $\lambda$ such that $X \backslash \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\} \in \mathcal{I}$. Since $f$ is surjective, $Y \backslash \bigcup\{V_\lambda : \lambda \in \Lambda_0\} = f(X \backslash \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\}) \in \mathcal{I}$. Therefore, $(Y,g',f(\mathcal{I}))$ is $g'$-$f(\mathcal{I})$-compact.

$\square$

The proofs of the next two Theorems follows from that of Theorem 4.13 and therefore we omit them.

**Theorem 4.14.** *If $f : (X,g,\mathcal{I}) \to (Y,g')$ is a $(g,g')$-$\alpha$-$\mathcal{I}$-continuous surjection and $(X,g,\mathcal{I})$ is $g$-$\alpha$-$\mathcal{I}$-Lindelof, then $(Y,g',f(\mathcal{I}))$ is $g'$-$f(\mathcal{I})$-Lindelof.*

**Theorem 4.15.** *If $f : (X,g,\mathcal{I}) \to (Y,g')$ is a $(g,g')$-$\alpha$-$\mathcal{I}$-continuous surjection and $(X,g,\mathcal{I})$ is countably $g$-$\alpha$-$\mathcal{I}$-compact, then $(Y,g',f(\mathcal{I}))$ is countably $g'$-$f(\mathcal{I})$-compact.*

**Definition 4.16.** An GTS $(X,g)$ is said to be $g$-connected if it cannot be expressed as the union of two nonempty disjoint sets $g$-open sets.

**Definition 4.17.** An IGTS $(X,g,\mathcal{I})$ is said to be $g$-$\alpha$-$\mathcal{I}$-connected if it cannot be expressed as the union of two nonempty disjoint $g$-$\alpha$-$\mathcal{I}$-open sets.

**Theorem 4.18.** *Let $f : (X,g,\mathcal{I}) \to (Y,g')$ be a $(g,g')$-$\alpha$-$\mathcal{I}$-continuous surjective function and $(X,g,\mathcal{I})$ is a $g$-$\alpha$-$\mathcal{I}$-connected space, then $(Y,g')$ is a $g'$-connected space.*

*Proof.* Suppose $Y$ is not $g'$-connected, Then $Y = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and $A, B \in g'$. Since $f$ is $(g, g')$-$\alpha$-$\mathcal{I}$-continuous $f^{-1}(A), f^{-1}(B) \in S\mathcal{I}O(X, g)$ such that $f^{-1}(A) \neq \emptyset$, $f^{-1}(B) \neq \emptyset$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = X$, which implies that $X$ is not $g$-$\alpha$-$\mathcal{I}$-connected. □

## 5. Decomposition of $(g, g')$-continuity

**Definition 5.1.** For an IGTS $(X, g, \mathcal{I})$, we define

(i) $g$-$\mathcal{I}$-$D(\alpha, p) = \{A \subset X : g\alpha\mathcal{I}\mathrm{Int}(A) = gp\mathcal{I}\mathrm{Int}(A)\}$.
(ii) $g$-$\mathcal{I}$-$D(\alpha, s) = \{A \subset X : g\alpha\mathcal{I}\mathrm{Int}(A) = gs\mathcal{I}\mathrm{Int}(A)\}$.

**Definition 5.2.** A subset $A$ of $(X, g, \mathcal{I})$ is said to be $g$-$\mathcal{I}$-$D(\alpha, p)$-set if $A \in g$-$\mathcal{I}$-$D(\alpha, p)$ (resp. $g$-$\mathcal{I}$-$D(\alpha, s)$-set if $A \in g$-$\mathcal{I}$-$D(\alpha, s)$)

**Proposition 5.3.** [3] A subset $A$ of an IGTS $(X, g, \mathcal{I})$ is $g$-semi-$\mathcal{I}$-open if and only if $g\mathrm{Cl}^*(A) = g\mathrm{Cl}^*(g\mathrm{Int}(A))$.

**Theorem 5.4.** *For an IGTS $(X, g, \mathcal{I})$, the relation $S\mathcal{I}O(X, g) \subset g$-$\mathcal{I}$-$D(\alpha, p)$ holds.*

*Proof.* Let $A \in S\mathcal{I}O(X, g)$. Then by Proposition 5.3, $g\mathrm{Cl}^*(A) = g\mathrm{Cl}^*(g\mathrm{Int}(A))$. Then $g\mathrm{Int}(g\mathrm{Cl}^*(A)) = g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$. We have $A \cap g\mathrm{Int}(g\mathrm{Cl}^*(A)) = A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A)))$. Hence $gp\mathcal{I}\mathrm{Int}(A) = g\alpha\mathcal{I}\mathrm{Int}(A)$ by Theorem 3.26 and Lemma 2.7. Therefore, $A \in g$-$\mathcal{I}$-$D(\alpha, p)$. □

The following example shows that the reverse inclusion of Theorem 5.4 is not true in general.

**Example 5.5.** Let $X = \{a, b, c\}$ $g = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then the set $\{a\} \in g$-$\mathcal{I}$-$D(\alpha, p)$ but $\{a\} \notin S\mathcal{I}O(X, g)$.

**Theorem 5.6.** *For an IGTS $(X, g, \mathcal{I})$, $\alpha\mathcal{I}O(X, g) = P\mathcal{I}O(X, g) \cap g$-$\mathcal{I}$-$D(\alpha, p)$ holds.*

*Proof.* Let $A \in P\mathcal{I}O(X, g) \cap g$-$\mathcal{I}$-$D(\alpha, p)$. Then $A = gp\mathcal{I}\mathrm{Int}(A)$ and $g\alpha\mathcal{I}\mathrm{Int}(A) = gp\mathcal{I}\mathrm{Int}(A)$. Hence $A = g\alpha\mathcal{I}\mathrm{Int}(A)$. Therefore, $A \in \alpha\mathcal{I}O(X, g)$. Conversely, let $A \in \alpha\mathcal{I}O(X, g)$. Then $A = g\alpha\mathcal{I}\mathrm{Int}(A) \subset gp\mathcal{I}\mathrm{Int}(A)$. But $gp\mathcal{I}\mathrm{Int}(A) \subset A$. Therefore, $A \in gp\mathcal{I}\mathrm{Int}(A)$ and hence we have $A = g\alpha\mathcal{I}\mathrm{Int}(A) = gp\mathcal{I}\mathrm{Int}(A)$. Hence $A \in g$-$\mathcal{I}$-$D(\alpha, p)$. □

**Theorem 5.7.** *For an IGTS $(X, g, \mathcal{I})$, $\alpha\mathcal{I}O(X, g) = S\mathcal{I}O(X, g) \cap g$-$\mathcal{I}$-$D(\alpha, s)$ holds.*

*Proof.* The proof is similar to the Theorem 5.6. □

**Definition 5.8.** For an IGTS $(X, g, \mathcal{I})$, we define

(i) $g$-$\mathcal{I}$-$D(g, \alpha) = \{A \subset X : g\mathrm{Int}(A) = g\alpha\mathcal{I}\mathrm{Int}(A)\}$.
(ii) $g$-$\mathcal{I}$-$D(g, s) = \{A \subset X : g\mathrm{Int}(A) = gs\mathcal{I}\mathrm{Int}(A)\}$.
(iii) $g$-$\mathcal{I}$-$D(g, p) = \{A \subset X : g\mathrm{Int}(A) = gs\mathcal{I}\mathrm{Int}(A)\}$.

**Theorem 5.9.** *For a subset $A$ of $(X, g, \mathcal{I})$, the following statements are equivalent:*

*(i)* $A \in g$-$\mathcal{I}$-$D(g, p)$.
*(ii)* $gInt(A) = A \cap gCl^*(gInt(A))$.

*Proof.* $(i) \Rightarrow (ii)$: Let $A \in g$-$\mathcal{I}$-$D(g, p)$.Then we have $g\mathrm{Int}(A) = gs\mathcal{I}\mathrm{Int}(A) = A \cap g\mathrm{Cl}^*(g\mathrm{Int}(A))$ by Lemma 2.7. $(ii) \Rightarrow (i)$: Let $g\mathrm{Int}(A) = A \cap g\mathrm{Cl}^*(g\mathrm{Int}(A))$. Then by Lemma 2.7, $g\mathrm{Int}(A) = gs\mathcal{I}\mathrm{Int}(A)$. Therefore, $A \in g$-$\mathcal{I}$-$D(g, p)$. □

**Proposition 5.10.** For each $x \in (X, g, \mathcal{I})$, $\{x\} \in g\text{-}\mathcal{I}\text{-}D(g, s)$.

*Proof.* Let $x \in X$ and $\{x\} \in g$. Then $g\mathrm{Int}(\{x\}) = \{x\} = \{x\} \cap g\mathrm{Cl}^*(\{x\}) = \{x\} \cap g\mathrm{Cl}^*(g\mathrm{Int}(\{x\}))$. Therefore, by Theorem 5.9, $\{x\} \in g\text{-}\mathcal{I}\text{-}D(g, s)$. Now, let $\{x\} \notin g$. Then $g\mathrm{Int}(\{x\}) = \emptyset = \{x\} \cap \emptyset = \{x\} \cap g\mathrm{Cl}^*(g\mathrm{Int}(\{x\}))$. Therefore, by Theorem 5.9, $\{x\} \in g\text{-}\mathcal{I}\text{-}D(g, s)$. □

**Theorem 5.11.** *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. If $A \in g$, then $A \in g\text{-}\mathcal{I}\text{-}D(g, s)$.*

*Proof.* Let $A \in g$. Then $g\mathrm{Int}(A) = A = A \cap g\mathrm{Cl}^*(A) = A \cap g\mathrm{Cl}^*(g\mathrm{Int}(A)) = gs\mathcal{I}\mathrm{Int}(A)$. Hence $A \in g\text{-}\mathcal{I}\text{-}D(g, s)$ by Lemma 2.7. □

**Theorem 5.12.** *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. If $A \in g$, then $A \in g\text{-}\mathcal{I}\text{-}D(g, \alpha)$.*

*Proof.* Let $A \in g$. Then $g\mathrm{Int}(A) = A = g\mathrm{Int}(A \cap g\mathrm{Cl}^*(A)) = A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) = A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) = g\alpha\mathcal{I}\mathrm{Int}(A)$. Hence $A \in g\text{-}\mathcal{I}\text{-}D(g, \alpha)$. □

**Theorem 5.13.** *For an IGTS $(X, g, \mathcal{I})$, the following relations hold:*

*(i)* $g\text{-}\mathcal{I}\text{-}D(g, s) \subset g\text{-}\mathcal{I}\text{-}D(g, \alpha)$
*(ii)* $g\text{-}\mathcal{I}\text{-}D(g, p) \subset g\text{-}\mathcal{I}\text{-}D(g, \alpha)$

*holds.*

*Proof.* $(i)$. Let $A \in g\text{-}\mathcal{I}\text{-}D(g, s)$. Then $g\mathrm{Int}(A) = gs\mathcal{I}(A) \supset g\alpha\mathcal{I}(A)$. But $g\mathrm{Int}(A) \subset g\alpha\mathcal{I}(A)$ and hence we have $g\mathrm{Int}(A) = g\alpha\mathcal{I}(A)$. Therefore, $A \in g\text{-}\mathcal{I}\text{-}D(g, \alpha)$. The proof of $(ii)$ is similar to $(i)$. □

**Remark 5.14.** The converses of the Theorem 5.12 and 5.13 are not true in general. Let $(X, g, \mathcal{I})$ as in Example 3.6. Then $\{b\} \in g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ but $\{b\} \notin g$ and $\{b\} \notin g\text{-}\mathcal{I}\text{-}D(g, p)$. Let $(X, g, \mathcal{I})$ as in Example 3.7. Then $\{b, c\} \in g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ but $\{b\} \notin g\text{-}\mathcal{I}\text{-}D(g, s)$.

**Example 5.15.** $g$-semi-$\mathcal{I}$-open sets and $g\text{-}\mathcal{I}\text{-}D(g, s)$ sets are independent of each other. Let $(X, g, \mathcal{I})$ as in Example 3.7. Then $\{a, c\} \in S\mathcal{I}O(X, g)$ but $\{a, c\} \notin g\text{-}\mathcal{I}\text{-}D(g, s)$. Also $\{c\} \notin g\text{-}\mathcal{I}\text{-}D(g, s)$ but $\{a, c\} \in S\mathcal{I}O(X, g)$

**Example 5.16.** $g$-pre-$\mathcal{I}$-open sets and $g\text{-}\mathcal{I}\text{-}D(g, p)$ sets are independent of each other. Let $(X, g, \mathcal{I})$ as in Example 3.6. Then $\{b\} \in P\mathcal{I}O(X, g)$ but $\{b\} \notin g\text{-}\mathcal{I}\text{-}D(g, p)$. Also $\{a\} \notin g\text{-}\mathcal{I}\text{-}D(g, p)$ but $\{a\} \in P\mathcal{I}O(X, g)$

**Example 5.17.** $g$-$\alpha$-$\mathcal{I}$-open sets and $g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ sets are independent of each other. Let $X = \{a, b, c\}$, $g = \{\emptyset, \{c\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\{a, c\} \in \alpha\mathcal{I}O(X, g)$ but $\{a, c\} \notin g\text{-}\mathcal{I}\text{-}D(g, \alpha)$. Also $\{a\} \in g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ but $\{a\} \notin \alpha\mathcal{I}O(X, g)$.

**Theorem 5.18.** *For a subset $A$ of $(X, g, \mathcal{I})$, the following statements are equivalent:*

*(i)* $A \in g$.
*(ii) $A$ is $g$-$\alpha$-$\mathcal{I}$-open and a $g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ set.*
*(iii) $A$ is $g$-pre-$\mathcal{I}$-open and a $g\text{-}\mathcal{I}\text{-}D(g, p)$ set.*

*Proof.* The implication $(i) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (iii)$: Let $A$ be $g$-$\alpha$-$\mathcal{I}$-open and a $g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ set. Then $A \in P\mathcal{I}O(X, g) \cap S\mathcal{I}O(X, g)$ and $g\alpha\mathcal{I}\mathrm{Int}(A) = g\mathrm{Int}(A)$. By Lemma 3.26, we have $A \cap g\mathrm{Int}(g\mathrm{Cl}^*(g\mathrm{Int}(A))) = g\mathrm{Int}(A)$ and subsequently $A \cap g\mathrm{Int}(g\mathrm{Cl}^*(A)) = g\mathrm{Int}(A)$. Now by Lemma 2.7, we have $gp\mathcal{I}\mathrm{Int}(A) = g\mathrm{Int}(A)$. Therefore, $A$ is a $g\text{-}\mathcal{I}\text{-}D(g, \alpha)$ set.

*(iii)* $\Rightarrow$ *(i)*: Let $A$ be $g$-pre-$\mathcal{I}$-open and a $g$-$\mathcal{I}$-$D(g,p)$ set. The by Definition $A = gp\mathcal{I}\mathrm{Int}(A)$ and $g\mathrm{Int}(A) = gp\mathcal{I}\mathrm{Int}(A)$; hence $A \in g$. □

**Theorem 5.19.** *For a subset $A$ of $(X, g, \mathcal{I})$, the following statements are equivalent:*

*(i)* $A \in g$.
*(ii)* *$A$ is $g$-$\alpha$-$\mathcal{I}$-open and a $g$-$\mathcal{I}$-$D(g,s)$ set.*
*(iii)* *$A$ is $g$-semi-$\mathcal{I}$-open and a $g$-$\mathcal{I}$-$D(g,s)$ set.*

*Proof.* Similar to the proof of Theorem 5.18. □

**Definition 5.20.** A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is said to be:

(i) $(g,g')$-$D(\alpha,p)$-$\mathcal{I}$-continuous if $f^{-1}(V) \in g$-$\mathcal{I}$-$D(\alpha,p)$ for every $V \in g'$.
(ii) $(g,g')$-$D(\alpha,s)$-$\mathcal{I}$-continuous if $f^{-1}(V) \in g$-$\mathcal{I}$-$D(\alpha,s)$ for every $V \in g'$.
(iii) $(g,g')$-$D(g,\alpha)$-$\mathcal{I}$-continuous if $f^{-1}(V) \in g$-$\mathcal{I}$-$D(g,\alpha)$ for every $V \in g'$.
(iv) $(g,g')$-$D(g,s)$-$\mathcal{I}$-continuous if $f^{-1}(V) \in g$-$\mathcal{I}$-$D(g,s)$ for every $V \in g'$.
(v) $(g,g')$-$D(g,p)$-$\mathcal{I}$-continuous if $f^{-1}(V) \in g$-$\mathcal{I}$-$D(g,p)$ for every $V \in g'$.

**Theorem 5.21.** *A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is $(g,g')$-continuous if and only if*

*(i) it is $(g,g')$-$\alpha$-$\mathcal{I}$-continuous and $(g,g')$-$D(g,\alpha)$-$\mathcal{I}$-continuous.*
*(ii) it is $(g,g')$-pre-$\mathcal{I}$-continuous and $(g,g')$-$D(g,p)$-$\mathcal{I}$-continuous.*
*(iii) it is $(g,g')$-$\alpha$-$\mathcal{I}$-continuous and $(g,g')$-$D(g,s)$-$\mathcal{I}$-continuous.*
*(iv) it is $(g,g')$-semi-$\mathcal{I}$-continuous and $(g,g')$-$D(g,s)$-$\mathcal{I}$-continuous.*

*Proof.* Follows from Theorems 5.18 and 5.19. □

**Theorem 5.22.** *A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is $(g,g')$-$\alpha$-$\mathcal{I}$-continuous if and only if*

*(i) it is $(g,g')$-semi-$\mathcal{I}$-continuous and $(g,g')$-pre-$\mathcal{I}$-continuous.*
*(ii) it is $(g,g')$-pre-$\mathcal{I}$-continuous and $(g,g')$-$D(\alpha,p)$-$\mathcal{I}$-continuous.*
*(iii) it is $(g,g')$-semi-$\mathcal{I}$-continuous and $(g,g')$-$D(\alpha,s)$-$\mathcal{I}$-continuous.*

*Proof.* Follows from Theorems 3.10, 5.6 and 5.7. □

## 6. Conclusion

The work in this paper is a step forward to strengthen the theoretical foundation of generalized open set in ideal generalized topological space, fuzzy generalized topological space and soft generalized topological space. The theoretical developments studied in this paper may be focused on the applications in image processing, data mining, medical diagnosis etc. The sets studied in this work may be very helpful in applied field and many other new types of sets can be formed from these sets which may be very helpful in applied field of research work.

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