Poincare Journal of Analysis & Applications
Vol. 8, No. 1(I) (2021), Special issue, 265-271

DOI: 10.33786/pjaa.2021.v08i01(i).024

ON r-FUZZY WEAKLY b-OPEN FUNCTIONS

N. RAJESH† AND R. VENNILA

Date of Receiving	:	06. 10. 2020
Date of Revision	:	03. 11. 2020
Date of Acceptance	:	17. 11. 2020

Abstract. In this paper, we introduce and characterize a new class of functions called r-fuzzy weakly b-open (r-fuzzy weakly b-closed) functions between smooth fuzzy topological spaces.

1. Introduction and Prelimanaries

Functions is a well-known notion in topology. The second author studied various types of functions in Topology [4, 5, 21]. The fuzzy concept has invaded almost all branches of Mathematics since its introduction by Zadeh [23]. Fuzzy sets have applications in many fields such as information [17] and control [20]. The theory of fuzzy topological spaces was introduced and developed by Chang [1] and since then various notions in classical topology have been extended to fuzzy topological spaces. Sǒstak [18] and Kubiak [10] introduced the fuzzy topology as an extension of Chang's fuzzy topology. It has been developed in many directions. Sǒstak [19] also published a survey article of the developed areas of fuzzy topological spaces. This paper is organized as follows. In the second section, we provide some background on smooth fuzzy topological spaces and r-fuzzy sets. We introduce and characterize a new class of functions called r-fuzzy weakly b-open (r-fuzzy weakly b-closed) functions between smooth fuzzy topological spaces in the third section. Moreover, conclusion and scope for future work were discussed at the end.

2. Preliminaries

Definition 2.1. A fuzzy point x_t in X is a fuzzy set taking value $t \in I_0$ at x and zero elsewhere, $x_t \in \lambda$ if and only if $t \leq \lambda(x)$. A fuzzy set λ is quasicoincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \bar{q} \mu$.

2010 *Mathematics Subject Classification.* 54A05.
Key words and phrases. Fuzzy topological spaces, r-fuzzy b-open set.
Communicated by. C. Özel and R. Dhavaseelan
† *Corresponding author.*

Definition 2.2. [10, 18] A function $\tau : I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$;
(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.
(3) $\tau(\bigvee_{j\in\Gamma} \mu_j) \geq \bigvee_{j\in\Gamma} \tau(\mu_j)$ for any $\{\mu_j\}_{j\in\Gamma} \in I^X$.

The pair (X, τ) is called a smooth fuzzy topological space.

A fuzzy point in X with support $x \in X$ and the value $\alpha (0 < \alpha \leq 1)$ is denoted by x_α.

Definition 2.3. [13] A fuzzy set $\lambda \in I^X$ is said to be q-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. It is known that $\lambda \leq \mu$ if and only if λ and $1 - \mu$ are not q-coincident, denoted by $\lambda \bar{q} (1 - \mu)$.

Definition 2.4. [3] A fuzzy set λ is said to be r-fuzzy Q-neighbourhood of x_p if $\tau(\lambda) \geq r$ such that $x_p q \lambda$. We will denote the set of all r-fuzzy open Q-neighbourhood of x_p by $\mathcal{Q}(x_p, r)$.

Definition 2.5. [2] Let (X, τ) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator Cl : $I^X \times I_0 \to I^X$ is defined as follows: $\mathrm{Cl}(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, \tau(\bar{1} - \mu) \geq r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $\mathrm{Cl}(\bar{0}, r) = \bar{0}$.
(2) $\lambda \leq \mathrm{Cl}(\lambda, r)$.
(3) $\mathrm{Cl}(\lambda, r) \vee \mathrm{Cl}(\mu, r) = \mathrm{Cl}(\lambda \vee \mu, r)$.
(4) $\mathrm{Cl}(\lambda, r) \leq \mathrm{Cl}(\lambda, s)$ if $r \leq s$.
(5) $\mathrm{Cl}(\mathrm{Cl}(\lambda, r), r) = \mathrm{Cl}(\lambda, r)$.

Proposition 2.6. [14] Let (X, τ) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator Int : $I^X \times I_0 \to I^X$ is defined as follows: $\mathrm{Int}(\lambda, r) = \vee\{\mu : \mu \leq \lambda, \tau(\mu) \geq r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $\mathrm{Int}(\bar{1} - \lambda, r) = \bar{1} - \mathrm{Cl}(\lambda, r)$.
(2) $\mathrm{Int}(\bar{1}, r) = \bar{1}$.
(3) $\lambda \geq \mathrm{Int}(\lambda, r)$.
(4) $\mathrm{Int}(\lambda, r) \wedge \mathrm{Int}(\mu, r) = \mathrm{Int}(\lambda \wedge \mu, r)$.
(5) $\mathrm{Int}(\lambda, r) \geq \mathrm{Int}(\lambda, s)$, if $r \leq s$.
(6) $\mathrm{Int}(\mathrm{Int}(\lambda, r), r) = \mathrm{Int}(\lambda, r)$.

Definition 2.7. [12] A fuzzy point x_p is said to be a r-fuzzy θ-cluster point of a fuzzy set λ if and only if for every $\mu \in \mathcal{Q}(x_p, r)$, $\mathrm{Cl}(\mu, r)$ is q-coincident with λ. The set of all r-fuzzy θ-cluster points of λ is called the r-fuzzy θ-closure of λ and will be denoted by $\mathrm{Cl}_\theta(\lambda, r)$. A fuzzy set λ will be called r-fuzzy θ-closed if and only if $\lambda = \mathrm{Cl}_\theta(\lambda, r)$. The complement of a r-fuzzy θ-closed set is called r-fuzzy θ-open and the r-fuzzy θ-interior of λ denoted by $\mathrm{Int}_\theta(\lambda, r)$ is defined as $\mathrm{Int}_\theta(\lambda, r) = \{x_p$: for some $\beta \in \mathcal{Q}(x_p, r)$, $\mathrm{Cl}(\beta, r) \leq \lambda\}$.

Lemma 2.8. [12] *Let λ be a fuzzy set in a smooth fuzzy topological space (X, τ). Then,*

(1) λ is r-fuzzy θ-open if, and only if $\lambda = \mathrm{Int}_\theta(\lambda, r)$;
(2) $1 - \mathrm{Int}_\theta(\lambda, r) = \mathrm{Cl}_\theta(1 - \lambda, r)$ and $\mathrm{Int}_\theta(1 - \lambda, r) = 1 - \mathrm{Cl}_\theta(\lambda, r)$;
(3) $\mathrm{Cl}_\theta(\lambda, r)$ is a r-fuzzy closed set but not necessarily is a r-fuzzy θ-closed set.

Definition 2.9. A fuzzy set λ of a smooth fuzzy topological space (X, τ) is called:

(1) r-fuzzy preopen [14] if $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$;
(2) r-fuzzy regular open [8] if $\lambda = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$;
(3) r-fuzzy α-open [14] if $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r)$;
(4) r-fuzzy preclosed [14] if $\mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \leq \lambda$;
(5) r-fuzzy regular closed [8] if $\lambda = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$;
(6) r-fuzzy α-closed [14] if $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) \leq \lambda$;
(7) r-fuzzy b-open if $\lambda \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \vee \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$;
(8) r-fuzzy b-closed if $\mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \wedge \mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \leq \lambda$.

Definition 2.10. Let (X, τ) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $b\,\mathrm{Cl} : I^X \times I_0 \to I^X$ is defined as $b\,\mathrm{Cl}(\lambda, r) = \wedge\{\mu : \mu \geq \lambda,\ \mu \text{ is } r\text{-fuzzy } b\text{-closed}\}$. An operator $b\,\mathrm{Int} : I^X \times I_0 \to I^X$ is defined as $b\,\mathrm{Int}(\lambda, r) = \vee\{\mu : \mu \leq \lambda,\ \mu \text{ is } r\text{-fuzzy } b\text{-open}\}$.

Definition 2.11. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(1) r-fuzzy b-open if $f(\lambda)$ is a r-fuzzy b-open set in Y for each fuzzy open set λ of X;
(2) r-fuzzy weakly open if $f(\lambda) \leq \mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$ for each fuzzy open set λ of X.

3. r-Fuzzy weakly b-open functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be r-fuzzy weakly b-open if $f(\lambda) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$ for each $\tau(\lambda) \geq r$.

Remark 3.2. It is evident that, every r-fuzzy weakly open function is r-fuzzy weakly b-open and every r-fuzzy b-open function is also r-fuzzy weakly b-open. But the converse need not be true in general.

Example 3.3. Let $X = \{a, b, c\}$ and $\mu = \left(\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.4}\right)$, and $\beta = \left(\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3}\right)$. Define $\tau : L^X \to L$ and $\sigma : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu \\ 0 & \textit{otherwise} \end{cases} \qquad \sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \beta \\ 0 & \textit{otherwise.} \end{cases}$$

Then the identity function $f : (X, \tau) \to (Y, \sigma)$ defined by $g(a) = c$, $g(b) = a$ and $g(c) = b$ is $\frac{1}{2}$-fuzzy weakly b-open but not $\frac{1}{2}$-fuzzy b-open.

Definition 3.4. A smooth fuzzy topological space (X, τ) is r-fuzzy regular if for each $x \in X$, $\rho \in [0, 1]$, $\tau(\lambda) \geq r$ with $x_p \leq \lambda$, there exists $\tau(\mu) \geq r$ such that $x_p \leq \mathrm{Cl}(\mu, r) \leq \lambda$.

Theorem 3.5. *For a surjective function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*

(1) f is r-fuzzy weakly b-open;
(2) $f(\mathrm{Int}_\theta(\lambda, r)) \leq b\,\mathrm{Int}(f(\lambda), r)$ for every $\lambda \in I^X$;
(3) $\mathrm{Int}_\theta(f^{-1}(\beta), r) \leq f^{-1}(b\,\mathrm{Int}(\beta, r))$ for every $\beta \in I^Y$;
(4) $f^{-1}(b\,\mathrm{Cl}(\beta, r)) \leq \mathrm{Cl}_\theta(f^{-1}(\beta), r)$ for every $\beta \in I^Y$;
(5) For each r-fuzzy θ-open set $\lambda \in I^X$, $f(\lambda)$ is r-fuzzy b-open in Y;

(6) For any $\beta \in I^Y$ and any r-fuzzy θ-closed set $\lambda \in I^X$ containing $f^{-1}(\beta)$, where X is a r-fuzzy regular space, there exists a r-fuzzy b-closed $\delta \in I^Y$ containing β such that $f^{-1}(\delta) \leq \lambda$.

Proof. (1)⇔(2): Let $\lambda \in I^X$ and x_p be a fuzzy point in $\mathrm{Int}_\theta(\lambda, r)$. Then there exists a fuzzy open q-neighbourhood μ of x_p such that $\mu \leq \mathrm{Cl}(\mu, r) \leq \lambda$. Then $f(\mu) \leq f(\mathrm{Cl}(\mu, r)) \leq f(\lambda)$. Since f is r-fuzzy weakly b-open, $f(\mu) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r) \leq b\,\mathrm{Int}(f(\lambda), r)$. Then $f(x_p)$ is a fuzzy point in $b\,\mathrm{Int}(f(\lambda), r)$. Hence $x_p \in f^{-1}(b\,\mathrm{Int}(f(\lambda), r))$. Thus $\mathrm{Int}_\theta(\lambda, r) \leq f^{-1}(b\,\mathrm{Int}(f(\lambda), r))$, and so $f(\mathrm{Int}(\lambda), r) \leq b\,\mathrm{Int}(f(\lambda), r)$. Conversely, let $\tau(\mu) \geq r$. Since $\mu \leq \mathrm{Int}_\theta(\mathrm{Cl}(\mu, r), r)$, we have $f(\mu) \leq f(\mathrm{Int}_\theta(\mathrm{Cl}(\mu, r), r)) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r)$. Hence f is r-fuzzy weakly b-open.
(2)⇔(3): Let $\beta \in I^Y$. Then by (2), $f(\mathrm{Int}_\theta(f^{-1}(\beta), r)) \leq b\,\mathrm{Int}(\beta, r)$. Therefore, $\mathrm{Int}_\theta(f^{-1}(\beta), r) \leq f^{-1}(b\,\mathrm{Int}(\beta, r))$. The converse is clear.
(3)⇔(4): Let $\beta \in I^Y$. By (3), $\overline{1} - \mathrm{Cl}_\theta(f^{-1}(\beta), r) = \mathrm{Int}_\theta(\overline{1} - f^{-1}(\beta), r) = \mathrm{Int}_\theta(f^{-1}(\overline{1} - \beta), r) \leq f^{-1}(b\,\mathrm{Int}(\overline{1} - \beta, r)) = f^{-1}(\overline{1} - b\,\mathrm{Cl}(\beta, r)) = \overline{1} - f^{-1}(b\,\mathrm{Cl}(\beta, r))$. Therefore, $f^{-1}(b\,\mathrm{Cl}(\beta, r)) \leq \mathrm{Cl}_\theta(f^{-1}(\beta), r)$. The converse is clear.
(4)⇒(5): Let λ be a r-fuzzy θ-open set in X. Then $\overline{1} - f(\lambda) \in I^Y$ and by (4), $f^{-1}(b\,\mathrm{Cl}(\overline{1} - f(\lambda)), r) \leq \mathrm{Cl}_\theta(f^{-1}(\overline{1} - f(\lambda)), r)$. Therefore, $\overline{1} - f^{-1}(b\,\mathrm{Int}(f(\lambda), r)) \leq \mathrm{Cl}_\theta(\overline{1} - \lambda, r) = \overline{1} - \lambda$. Then we have $\lambda \leq f^{-1}(b\,\mathrm{Int}(f(\lambda), r))$ which implies $f(\lambda) \leq b\,\mathrm{Int}(f(\lambda), r)$. Hence $f(\lambda)$ is r-fuzzy b-open in Y.
(5)⇒(6): Let $\beta \in I^Y$ and $\lambda \in I^X$ be a r-fuzzy θ-closed set such that $f^{-1}(\beta) \leq \lambda$. Since $\overline{1} - \lambda$ is r-fuzzy θ-open in X, by (5), $f(\overline{1} - \lambda)$ is r-fuzzy b-open in Y. Let $\delta = \overline{1} - f(\overline{1} - \lambda)$. Then δ is r-fuzzy b-closed and $\beta \leq \delta$. Now, $f^{-1}(\delta) = f^{-1}(\overline{1} - (f(\overline{1} - \lambda))) = \overline{1} - f^{-1}(f(\lambda)) \leq \lambda$.
(6)⇒(4): Let $\beta \in I^Y$. Then by Corollary 3.6 of [?] $\lambda = \mathrm{Cl}_\theta(f^{-1}(\beta), r)$ is r-fuzzy θ-closed in X and $f^{-1}(\beta) \leq \lambda$. Then there exists a r-fuzzy b-closed set $\delta \in Y^X$ containing β such that $f^{-1}(\delta) \leq \lambda$. Since δ is r-fuzzy b-closed, $f^{-1}(b\,\mathrm{Cl}(\beta, r) \leq f^{-1}(\delta) \leq \mathrm{Cl}_\theta(f^{-1}(\beta), r)$. □

Theorem 3.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is r-fuzzy weakly b-open if, and only if for each fuzzy point x_p in X and each $\tau(\mu) \geq r$ containing x_p, there exists a r-fuzzy b-open set δ containing $f(x_p)$ such that $\delta \leq f(\mathrm{Cl}(\mu, r))$.*

Proof. Let $x_p \in X$ and $\tau(\mu) \geq r$ such that μ containing x_p. Since f is r-fuzzy weakly b-open, $f(\mu) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r)$. Let $\delta = b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r)$. Hence $\delta \leq f(\mathrm{Cl}(\mu, r))$ with δ containing $f(x_p)$. Conversely, let $\tau(\mu) \geq r$ and let $y_p \in f(\mu)$. Then $\delta \leq f(\mathrm{Cl}(\mu, r))$ for some r-fuzzy b-open set δ in Y containing y_p. Hence we have, $y_p \in \delta \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r)$. Hence $f(\mu) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r)$ and f is r-fuzzy weakly b-open. □

Theorem 3.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. Then the following statements are equivalent:*

(1) f is r-fuzzy weakly b-open;
(2) $b\,\mathrm{Cl}(f(\lambda), r) \leq f(\mathrm{Cl}(\lambda, r))$ for each $\tau(\lambda) \geq r$;
(3) $b\,\mathrm{Cl}(f(\mathrm{Int}(\beta, r)), r) \leq f(\beta)$ for each $\tau(\overline{1} - \beta) \geq r$.

Proof. (1)⇔(3): Let $\tau(\overline{1} - \beta) \geq r$. Then $f(\overline{1} - \beta) = \overline{1} - f(\beta) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\overline{1} - \beta, r)), r)$ and so $\overline{1} - f(\beta) \leq \overline{1} - b\,\mathrm{Cl}(f(\mathrm{Int}(\beta, r)), r)$. Hence $b\,\mathrm{Cl}(f(\mathrm{Int}(\beta, r)), r) \leq f(\beta)$. The converse is clear.

(3)$\Leftrightarrow$(2): Let $\tau(\lambda) \geq r$. Since $\mathrm{Cl}(\lambda, r)$ is a r-fuzzy closed and $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ by (3), $b\,\mathrm{Cl}(f(\lambda), r) \leq b\,\mathrm{Cl}(f(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r) \leq f(\mathrm{Cl}(\lambda, r))$. The converse is clear. □

The proof of the following theorem is obvious and thus omitted.

Theorem 3.8. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent:*

(1) f is r-fuzzy weakly b-open;
(2) for each $\tau(\overline{1} - \beta) \geq r$, $f(\mathrm{Int}(\beta, r)) \leq b\,\mathrm{Int}(f(\beta), r)$;
(3) for each $\tau(\lambda) \geq r$, $f(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$;
(4) for each r-fuzzy regular open set $\lambda \in I^X$, $f(\lambda) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$;
(5) for every r-fuzzy preopen set $\lambda \in I^X$, $f(\lambda) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$;
(6) for every r-fuzzy α-open set $\lambda \in I^X$, $f(\lambda) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$.

Definition 3.9. A function $f : (X, \tau) \to (Y, \sigma)$ is said to satisfy the r-fuzzy weakly b-open interiority condition of $b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r) \leq f(\lambda)$ for every $\tau(\lambda) \geq r$.

Theorem 3.10. *If $f : (X, \tau) \to (Y, \sigma)$ is r-fuzzy weakly b-open and satisfies fuzzy weakly b-open interiority condition, then f is fuzzy b-open.*

Proof. Let $\tau(\lambda) \geq r$. Since f is r-fuzzy weakly b-open, we have $f(\lambda) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$. However, because f satisfies the r-fuzzy weakly b-open interiority condition, $f(\lambda) = b\,\mathrm{Int}(f(\mathrm{Cl}(\lambda, r)), r)$ and hence $f(\lambda)$ is r-fuzzy b-open. □

Theorem 3.11. *Let (X, τ) be a r-fuzzy regular space. Then $f : (X, \tau) \to (Y, \sigma)$ is r-fuzzy weakly b-open if and only if f is r-fuzzy b-open.*

Proof. Let $\tau(\lambda) \geq r$ and $\lambda \neq \overline{0}$. For each fuzzy point x_p in λ, let $\tau(\mu_{x_p}) \geq r$ such that $x_p \in \mu_{x_p} \leq \mathrm{Cl}(\mu_{x_p}, r) \leq \lambda$. Hence we have $\lambda = \vee\{\mu_{x_p}\colon x_p \in \lambda\}\ \vee\{\mathrm{Cl}(\mu_{x_p}, r)\colon x_p \in \lambda\}$ and, $f(\lambda) = \vee\{f(\mu_{x_p})\colon x_p \in \lambda\} \leq \vee\{b\,\mathrm{Int}(f(\mathrm{Cl}(\mu_{x_p}, r)), r)\colon x_p \in \lambda\} \leq b\,\mathrm{Int}(f(\vee\{\mathrm{Cl}(\mu_{x_p}, r)), r)\colon x_p \in \lambda\}) = b\,\mathrm{Int}(f(\lambda), r)$. Thus, f is r-fuzzy b-open. □

Definition 3.12. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be r-fuzzy contra-pre-b-closed provided that $f(\lambda)$ is r-fuzzy b-open for each r-fuzzy b-closed subset $\lambda \in I^X$.

Theorem 3.13. *If $f : (X, \tau) \to (Y, \sigma)$ is r-fuzzy weakly b-open and Y has the property that union of r-fuzzy b-closed sets is r-fuzzy b-closed and if for each r-fuzzy b-closed subset β of X and each fiber $f^{-1}(y_p) \leq \overline{1} - \beta$ there exists $\tau(\mu) \geq r$ for which $\beta \leq \mu$ and $f^{-1}(y_p)\ \bar{q}\ \mathrm{Cl}(\mu)$, then f is r-fuzzy contra-pre-b-closed.*

Proof. Assume that $\beta \in I^X$ is r-fuzzy b-closed set and let $y_p \in \overline{1} - f(\beta)$. Thus, $f^{-1}(y_p) \leq \overline{1} - \beta$ and hence there exists $\tau(\mu) \geq r$ for which $\beta \leq \mu$ and $f^{-1}(y_p)\ \bar{q}\ \mathrm{Cl}(\mu, r)$. Therefore, $y_p \in \overline{1} - f(\mathrm{Cl}(\mu, r)) \leq \overline{1} - f(\beta)$. Since f is r-fuzzy weakly b-open, $f(\mu) \leq b\,\mathrm{Int}(f(\mathrm{Cl}(\mu, r)), r)$. Hence $y_p \in b\,\mathrm{Cl}(\overline{1} - f(\mathrm{Cl}(\mu, r)), r) \leq \overline{1} - f(\beta)$. Let $\lambda_{y_p} = b\,\mathrm{Cl}(\overline{1} - f(\mathrm{Cl}(\mu, r)), r)$. Then λ_{y_p} is a r-fuzzy b-closed set of Y containing y_p. Hence $\overline{1} - f(\beta) = \vee\{\lambda_{y_p}\colon y_p \in \overline{1} - f(\beta)\}$ is r-fuzzy b-closed and $f(\beta)$ is r-fuzzy b-open. □

Definition 3.14. Two non-zero fuzzy sets λ and μ in a smooth fuzzy topological space (X, τ) are said to be r-fuzzy b-separated if $\lambda\ \bar{q}\ b\,\mathrm{Cl}(\mu, r)$ and $\mu\ \bar{q}\ b\,\mathrm{Cl}(\lambda, r)$ or equivalently, if there exist two r-fuzzy b-open sets ρ and η such that $\lambda \leq \rho$ and $\mu \leq \eta$, $\lambda\ \bar{q}\ \eta$ and $\mu\ \bar{q}\ \rho$.

Definition 3.15. A smooth fuzzy topological space (X,τ) which cannot be expressed as the union of two r-fuzzy b-separated sets is said to be a r-fuzzy b-connected space.

Theorem 3.16. *If $f:(X,\tau)\to(Y,\sigma)$ is a r-fuzzy weakly b-open surjective function of a smooth fuzzy topological space (X,τ) to a r-fuzzy b-connected space (Y,σ), then (X,τ) is r-fuzzy connected.*

Proof. If possible, let (X,τ) be not r-fuzzy connected. Then there exist fuzzy separated sets λ and μ in X such that $\overline{1}=\lambda\vee\mu$. Since λ and μ are r-fuzzy separated, there exist $\tau(\rho)\geq r$ and $\tau(\eta)\geq r$ such that $\lambda\leq\rho$, $\mu\leq\eta$, $\lambda\,\bar{q}\,\eta$ and $\mu\,\bar{q}\,\rho$. Hence we have $f(\lambda)\leq f(\rho)$, $f(\mu)\leq f(\eta)$, $f(\lambda)\,\bar{q}\,f(\eta)$ and $f(\mu)\,\bar{q}\,f(\rho)$. Since f is r-fuzzy weakly b-open, $f(\rho)\leq b\operatorname{Int}(f(\operatorname{Cl}(\rho,r)),r)$ and $f(\eta)\leq b\operatorname{Int}(f(\operatorname{Cl}(\eta,r)),r)$ and since ρ and η are r-fuzzy open and also r-fuzzy closed, we have $f(\operatorname{Cl}(\rho,r))=f(\rho)$, $f(\operatorname{Cl}(\eta,r))=f(\eta)$. Hence $f(\rho)$ and $f(\eta)$ are r-fuzzy b-open in Y. Therefore, $f(\lambda)$ and $f(\eta)$ are r-fuzzy b-separated sets in Y and $\overline{1}=f(\overline{1})=f(\lambda\vee\mu)=f(\lambda)\vee f(\mu)$. Hence this contrary to the fact that Y is r-fuzzy b-connected. Thus, X is r-fuzzy connected. □

Definition 3.17. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be fuzzy weakly b-closed if $b\operatorname{Cl}(f(\operatorname{Int}(\lambda,r)),r)\leq f(\lambda)$ for each $\tau(\overline{1}-\lambda)\geq r$.

The proof of the following two Theorems are obvious and hence omitted.

Theorem 3.18. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following conditions are equivalent:*

(1) f is r-fuzzy weakly b-closed;
(2) $b\operatorname{Cl}(f(\lambda),r)\leq f(\operatorname{Cl}(\lambda,r))$ for every $\tau(\lambda)\geq r$;
(3) $b\operatorname{Cl}(f(\lambda),r)\leq f(\operatorname{Cl}(\lambda),r)$ for every r-fuzzy regular open set $\lambda\in I^X$;
(4) For each $\mu\in I^Y$ and every $\tau(\eta)\geq r$ with $f^{-1}(\mu)\leq\eta$, there exists a r-fuzzy b-open set $\delta\in I^Y$ with $\mu\leq\delta$ and $f^{-1}(\mu)\leq\operatorname{Cl}(\eta,r)$;
(5) For each fuzzy point y_p in Y and each $\tau(\eta)\geq r$ with $f^{-1}(y_p)\leq\eta$, there exists a r-fuzzy b-open set $\delta\in I^Y$ with $y_p\leq\delta$ and $f^{-1}(\delta)\leq\operatorname{Cl}(\eta,r)$;
(6) $b\operatorname{Cl}(f(\operatorname{Int}(\operatorname{Cl}(\lambda,r),r)),r)\leq f(\operatorname{Cl}(\lambda,r))$ for each $\lambda\in I^X$;
(7) $b\operatorname{Cl}(f(\operatorname{Int}(\operatorname{Cl}_\theta(\lambda,r),r)),r)\leq f(\operatorname{Cl}_\theta(\lambda,r))$ for each $\lambda\in I^X$;
(8) $b\operatorname{Cl}(f(\lambda),r)\leq f(\operatorname{Cl}(\lambda,r))$ for each r-fuzzy b-open set $\lambda\in I^X$.

Theorem 3.19. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following conditions are equivalent:*

(1) f is fuzzy weakly b-closed;
(2) $b\operatorname{Cl}(f(\operatorname{Int}(\lambda,r)),r)\leq f(\lambda)$ for each r-fuzzy b-closed set $\lambda\in I^X$;
(3) $b\operatorname{Cl}(f(\operatorname{Int}(\lambda,r)),r)\leq f(\lambda)$ for each r-fuzzy α-closed set $\lambda\in I^X$.

Theorem 3.20. *If $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly b-closed injective, then for each $\lambda\in I^Y$ and each $\tau(\mu)\geq r$ with $f^{-1}(\lambda)\leq\mu$ there exists a r-fuzzy b-closed set $\rho\in I^Y$ with $\lambda\leq\rho$ and $f^{-1}(\rho)\leq\operatorname{Cl}(\mu)$.*

Proof. Follows from the definitions. □

Conclusion: In this paper, r-fuzzy weakly b-open function is proposed in smooth fuzzy topological space. The concepts of r-fuzzy regular, r-fuzzy contra pre-b-closed, r-fuzzy b- separated and r-fuzzy b-connected sets are studied. An example is provided to express the relationship between r-fuzzy weakly open function and r-fuzzy weakly b-open function.

Recommendations for further research directions based on this research work, it may be extended to generalized closed sets via Neutrosophic topological spaces and soft sets. Also for all types of continuous and irresolute functions.

References

[1] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., 24(1968), 182-190.

[2] K. C. Chattopadhyay and S.K. Samanta, Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness, Fuzzy Sets Systems, 54(1993), 207-212.

[3] M. Dimirci, Neighborhood strucure of smooth topological spaces, Fuzzy Sets Systems 92 (1997), 123-128.

[4] C. Duraisamy and R. Vennila, Somewhat λ-Continuous Functions, Applied Mathematical Sciences, 6 (39), 1945-1943, 2012.

[5] C. Duraisamy and R. Vennila, On λ-Continuous Functions, European Journal of Scientific Research, 59 (2), 258-263, 2011.

[6] P. Dwinger, Characterizations of the complete homomorphic images of a completely distributive complete lattice, Indag Math (Proc) 85 (1982), 403-414.

[7] G. Gierz, A compendium of continuous lattices. Springer, Berlin (1980).

[8] S. J. Lee and E. P. Lee, Fuzzy r-regular open sets and fuzzy almost r-continuous maps, Bull. Korean Math. Soc., 39(2002), 441-453.

[9] Y. M. Liu and M. K. Luo, Fuzzy topology. World Scientific, Singapore (1997).

[10] T. Kubiak, On fuzzy topologies, Ph. D. Thesis, A. Mickiewicz, Poznan, 1985.

[11] Y. C. Kim, Initial L-fuzzy closure spaces. Fuzzy Sets Systems, 133 (2003), 277-297.

[12] Y. C. Kim and J. W. Park, R-fuzzy δ-closure and R-fuzzy θ-closure sets, Int. J. Fuzzy Logic Intell. Sys., 10(2000), 557-563.

[13] P. P. Ming and L. Y. Ming, Fuzzy Topology. I. Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence, J. Math. Anal. Appl., 76 (1980), 571-599.

[14] A. A. Ramadan, Y. C. Kim and S. E. Abbas, Weaker forms of continuity in Sŏstak's fuzzy topology, Indian J Pure Appl Math, 34 (3) (2003), 311-333.

[15] F. G. Shi, Theory of Lb-nested sets and La-nested and their applications. Fuzzy System Math 4 (1995), 65-72 (in chinese).

[16] F. G. Shi, J. Zhang and C. Y. Zheng, On L-fuzzy topological spaces, Fuzzy Sets Systems 149 (2005), 473-484.

[17] P. Smets, The degree of belief in a fuzzy event, Inf. Sci., 25 (1981), 1-19.

[18] A. P. Sŏstak, On a fuzzy topological structure, Suppl. Rend. Circ. Matem. Palerms ser II, 11, (1985) 89-103.

[19] A. P. Sŏstak, Basic structures of fuzzy topology, J. Math. Sci., 78(6) (1996) 662-701.

[20] M. Sugeno, An introductory survey of fuzzy control, Inf Sci., 36(1985), 59-83.

[21] R. Vennila and C. Duraisamy, Two New Type of Irresolute Functions in Topological Spaces, International Journal of advanced scientific and technical research, 3 (2), 101-111, 2012.

[22] G.J. Wang, Theory of L-fuzzy topological space, Shaanxi Normal University Press, Xiàn, 1988 (in Chinese).

[23] L. A. Zadeh, Fuzzy sets, Information and Control, 8(1965), 338-353.

(N. Rajesh) Department of Mathematics, Rajah Serfoji Government College, Thanjavur-613005, Tamilnadu, India.
E-mail address: nrajesh_topology@yahoo.co.in

(R. Vennila) Department of Mathematics, Kongu Engineering College, Perundurai, Erode 638 060, Tamilnadu, India.
E-mail address, Corresponding author: author2@abc.com