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TOPOLOGICAL ALGEBRA IN VIRTUE OF PRE-OPEN SETS

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(Dedicated to Professor Maximilian Ganster on the occasion of his retirement)

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Abstract. In this paper, we elucidate the theory of p - topological group and p - topological ring. Also examine its properties with appropriate examples. Besides, we analyze subrings and ideals of p - topological rings. In addition, we evince that closure of a subring in a p - topological ring is a p - topological ring.

1. Introduction

In Mathematics, we classify structures endowed on a set into two kinds namely, algebraic structure and topological structure. A set endowing more than one structure plays a vital role in the literature. Ring, vector space and algebra are sets bestowed with two algebraic structures. In 1926, Lie developed the concept of continuous groups, a group with continuous binary operation. Based on this, notion of Topological Algebra was developed. By generalizing the continuity, some new structures emerged in [1] and coincidence of those structures on finite set is discussed in [3]. The idea of pre - open sets and pre - continuous functions on a topological space was introduced by Mashhour et. al in [6]. By making use of pre - open sets and pre - continuous functions on algebraic structures, we expound p - topological groups and p - topological rings in this paper.

2. Notation and Preliminaries

The notions pre - open, pre - closed sets follows [11] and pre - interior, pre - closure of a set follows denoted by *pint* and *pcl*. Let X and Y be topological spaces then $f : X \mapsto Y$ is pre - continuous if and only if the inverse image of an open set is pre - open (respectively, inverse image of a closed set is pre - closed). A subset $\mathcal{D}$ of X is pre - connected [9] if $\mathcal{D}$ cannot be written as the union of two disjoint pre - open sets.

On all accounts of this paper, the pair (G, τ) denotes a group G with topology τ and $(\mathcal{R}, \tau)$ betoken a ring $\mathcal{R}$, with addition $+$ and multiplication . in conjunction with

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a topology τ. For any $x \in \mathcal{R}$, $-x$, x^{-1} indicates the additive inverse and multiplicative inverse of x in $\mathcal{R}$. Let $\mathsf{M}, \mathsf{N} \subseteq \mathcal{R}$, then the notions $\mathsf{M} + \mathsf{N}, \mathsf{M}\,.\,\mathsf{N}$ denotes the ring addition and ring multiplication of M, N and $-\mathsf{M}, \mathsf{M}^{-1}$ denotes the additive inverse and multiplicative inverse of M. The set of non - zero elements and units of a ring $\mathcal{R}$ is signified by $\mathcal{R}^*$ and $\mathcal{U}(\mathcal{R})$. For a set $\mathcal{S}$, the power set of $\mathcal{S}$ is denoted by $\mathcal{P}(\mathcal{S})$ and for a topology τ, the collection of pre - open sets is typified by τ_p. The sets of real numbers, rational numbers and irrational numbers are symbolized as $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{S}$.

Lemma 2.1. [5] *Let $(X_i)_{i\in I}$ be a family of topological spaces and $\emptyset \neq D_i \subseteq X_i$ for each $i \in I$. Then, $\prod_{i\in I} D_i = B \subseteq \prod_{i\in I} X_i$ is pre - open in the product space $\prod_{i\in I} X_i$ if and only if D_i is pre - open in X_i for each $i \in I$ and D_i is non - dense for only finitely many $i \in I$.*

Lemma 2.2. [4] *Let Y be a topological space and $M \subseteq Y$. Then*
(i) $int(M) \subseteq pint(M)$.
(ii) $pcl(M) \subseteq cl(M)$.

Definition 2.3. A topological space Y is **submaximal** [5] if every dense subset of Y is open.

Lemma 2.4. [5] *For a topological space X, the following conditions are equivalent:*
(i) *X is submaximal.*
(ii) *Every pre - open set is open.*

3. p - topological group

In this section, we introduce the concept of p - topological group and investigate its basic properties with illustrated examples.

Definition 3.1. A pair (G, τ) is said to be **p - topological group** :

if for each $x, y \in \mathsf{G}$, and an open neighbourhood U of xy, there exist pre - open sets V of x and W of y such that $VW \subseteq U$ and
for an open neighbourhood P of x^{-1} there exists pre - open set Q of x such that $Q^{-1} \subseteq P$.

In other words, multiplication and inversion mappings are pre - continuous.

We observe that,

- Any group with partition topology is a trivial example of p - topological group.
- Every finite left (right) topological group is p - topological group. Since the basis of topology on finite left (right) topological group is cosets of a subgroup (partition topology) and for the partition topology τ on any set X, τ_p is $\mathcal{P}(\mathcal{X})$ and so every subset of X is pre - open.
- Every topological group is p - topological group, but converse need not be true. The following Examples 3.2, 3.3, 3.4 are all p - topological group but not a topological group.

A finite group with indiscrete topology is the only connected finite topological group. But in the case of p - topological group we can provide some more connected topologies. We can define a topology τ on G, for any group G such that (G, τ) is connected p - topological group as follows.

Example 3.2. Let G be any group having cardinality greater than 2 with the topology $\tau = \{\emptyset, \mathsf{G} \setminus \{e\}, \mathsf{G}\}$. Now, τ_p is $\mathcal{P}(\mathcal{G}) \setminus \{e\}$. Let $x \in \mathsf{G}$ and U be an open neighbourhood of x. Then $x = a.b$ for some $a, b \in \mathsf{G}$ and $x = y^{-1}$ for some $y \in \mathsf{G}$.

> **Case 1 :** Suppose $U = \mathsf{G}$, then G itself is the pre - open neighbourhood of a, b and y respectively.
> **Case 2 :** Suppose $U = \mathsf{G}\setminus\{e\}$, then $x \neq e$ and $\{y\}$ is a pre - open neighbourhood of x^{-1} such that $\{y\}^{-1} = \{x\} \subset U$ and so inversion is pre - continuous. We discuss the pre - continuity of multiplication as follows :
> If $a, b \neq e$, then $\{a\}$, $\{b\}$ are pre - open neighbourhoods of a and b such that $\{a\} . \{b\} \subset U$.
> If either a or b is identity, then the other element should be x. $\{e, z\}$ where $z \neq y$ and $\{x\}$ are pre - open neighbourhoods of e and x such that $\{e, z\} . \{x\} \subset U$.

Thus, (G, τ) is a connected p - topological group.

As in the above example, we can provide more connected p - topological groups for an infinite group as follows :

Example 3.3. Let G be an infinite group and $a \in \mathsf{G}$ be an arbitrary non - identity element with the topology $\tau = \{\emptyset, \mathsf{G} \setminus \{e, a\}, \mathsf{G}\}$. Then $\tau_p = \mathcal{P}(\mathcal{G}) \setminus \{\{e\}, \{a\}, \{e, a\}\}$. Let $x = b.c$ for some $b, c \in \mathsf{G}$ and U be an open neighbourhood of x. If $U = \mathsf{G}$ then (G, τ) is a p - topological group. Let us assume that $U = \mathsf{G} \setminus \{e, a\}$. Then $x \notin \{e, a\}$ and the cases follows:

> **Case 1 :** If $b, c \notin \{e, a\}$, then $\{b\}, \{c\}$ are pre - open neighbourhoods of b and c such that $\{b\} . \{c\} = \{x\} \subset U$.
> **Case 2 :** If $b = c = a$, then $\{a, y_1\}$ where $y_1 \notin \left\{e, a^{-1}, y_1^{-1}, ay_1^{-1}, y_1^{-1}a\right\}$ is a pre - open neighbourhood of b, c such that $\{a, y_1\} . \{a, y_1\} \subset U$.
> **Case 3 :** Suppose either b or $c \in \{e, a\}$
> If either b or c is e, then the other element should be x. Without loss of generality, let us fix $b = e$. Then $\{e, y_2\}$, where $y_2 \notin \left\{x^{-1}, ax^{-1}, x^{-1}a\right\}$ and $\{x\}$ are pre - open neighbourhoods of b and c such that $\{e, y_2\} . \{x\} \subset U$.
> If $b = a$, then $c = a^{-1}x$. Now, $\{a, y_3\}$ and $\left\{a^{-1}x, y_4\right\}$, where $y_3 \notin \left\{x^{-1}a, ax^{-1}a, y_4^{-1}, ay_4^{-1}\right\}$ and $y_4 \notin \left\{a^{-1}, e\right\}$ are pre - open neighbourhoods of b and c respectively such that $\{a, y_3\} . \left\{a^{-1}x, y_4\right\} \subset U$.
> If $c = a$, then $b = xa^{-1}$. Now, $\left\{xa^{-1}, y_5\right\}$ and $\{a, y_6\}$, where $y_5 \notin \left\{a^{-1}, e\right\}$ and $y_6 \notin \{ax^{-1}, ax^{-1}a,\ y_5^{-1}, y_5^{-1}a\}$ are pre - open neighbourhoods of b and c respectively such that $\left\{xa^{-1}, y_5\right\} . \{a, y_6\} \subset U$.

Thus, multiplication is pre - continuous on G. Now, the pre continuity of inversion follows:

> **Case 1 :** If $x^{-1} \neq a$, then $\left\{x^{-1}\right\}$ is a pre - open neighbourhood of x^{-1} such that $\left\{x^{-1}\right\}^{-1} = \{x\} \subset U$.
> **Case 2 :** If $x^{-1} = a$, then $\left\{x^{-1}, y\right\}$ where $y \neq x$ is a pre - open neighbourhood of x^{-1} such that $\left\{x^{-1}, y\right\}^{-1} = \left\{x, y^{-1}\right\} \subset U$.

Hence (G, τ) is a connected p - topological group. By the above example, We can see that, in an infinite group G, for any $n \in \mathbb{N}$ we can provide n number of connected topologies $\tau_i, i = 1, 2, 3...n$ such that (G, τ_i) is p - topological group.

Example 3.4. Consider the group $\mathsf{G} = (\mathbb{Z}_3, \oplus)$ with a topology $\tau = \{\emptyset, \{0, 2\}, \mathsf{G}\}$. For given τ, $\tau_p = \{\emptyset, \{0\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \mathsf{G}\}$. Now, consider $2 \in \mathsf{G}$ which can be written as $2 = 1 \oplus 1$. Let $\{0, 2\}$ be an open neighbourhood of 2. Then there does not exist pre - neighbourhoods U, V of 1 such that $UV \subseteq \{0, 2\}$. Thus (G, τ) is not a p - topological group.

Example 3.5. Consider the group of real numbers under usual addition $(\mathbb{R}, +)$ with the lower limit topology. Since the multiplication is continuous on $\mathbb{R}$, it is pre - continuous. But the inversion is not pre - continuous. Indeed, let $[0, a) : a \in \mathbb{R}^+$ be an open neighbourhood of 0. The inverse of 0 is itself. There does not exists any pre - open neighbourhood of 0 whose inverse is contained in $[0, a)$.

Proposition 3.6. Let M be an open set in p - topological group. Then for each $x \in \mathsf{G}$,

(i) $int(x\mathsf{M}) \subseteq xint(\mathsf{M})$.
(ii) $int(\mathsf{M}x) \subseteq int(\mathsf{M})x$.
(iii) $int(\mathsf{M}^{-1}) \subseteq [int(\mathsf{M})]^{-1}$.
(iv) $xcl(\mathsf{M}) \subseteq cl(x\mathsf{M})$.
(v) $cl(\mathsf{M})x \subseteq cl(\mathsf{M}x)$.
(vi) $[cl(\mathsf{M})]^{-1} \subseteq cl(\mathsf{M}^{-1})$.

Proof is trivial by the facts that, M is open $\Leftrightarrow int\mathsf{M} = \mathsf{M}$ and $pcl(\mathsf{M}^{-1}) \subseteq cl(\mathsf{M}^{-1})$ (Lemma 2.4 (ii)). Though the result is trivial an interesting fact is as follows :

In the above proposition, the set M cannot be assumed to be pre - open and the reverse containment need not be hold. Let $\mathsf{G} = (\mathbb{Z}_4, \oplus)$ with $\tau = \{\emptyset, \{0, 1\}, \{2\}, \{0, 1, 2\}, \{3\}, \{0, 1, 3\}, \{2, 3\}, \mathsf{G}\}$. Then $\tau_p = \mathcal{P}(\mathbb{Z}_4)$ and so (G, τ) is a p - topological group. Let $x = 1 \in \mathsf{G}$ and $\{3\} \in \tau$ then $x\mathsf{M} = \mathsf{M}x = \{0\}$ and $\mathsf{M}^{-1} = \{1\}$.

(i) $int(x\mathsf{M}) = \emptyset$, $xint(\mathsf{M}) = \{0\} \Rightarrow xint(\mathsf{M}) \nsubseteq int(x\mathsf{M})$.
(ii) $int(\mathsf{M}x) = \emptyset$, $int(\mathsf{M})x = \{0\} \Rightarrow int(\mathsf{M})x \nsubseteq int(\mathsf{M}x)$.
(iii) $int(\mathsf{M}^{-1}) = \emptyset$, $[int(\mathsf{M})]^{-1} = \{1\} \Rightarrow [int(\mathsf{M})]^{-1} \nsubseteq int(\mathsf{M}^{-1})$.
(iv) $cl(x\mathsf{M}) = \{0, 1\}$, $xcl(\mathsf{M}) = \{0\} \Rightarrow cl(x\mathsf{M}) \nsubseteq xcl(\mathsf{M})$.
(v) $cl(\mathsf{M}x) = \{0, 1\}$, $cl(\mathsf{M})x = \{0\} \Rightarrow cl(\mathsf{M}x) \nsubseteq cl(\mathsf{M})x$.
(vi) $cl(\mathsf{M}^{-1}) = \{0, 1\}$, $[cl(\mathsf{M})]^{-1} = \{1\} \Rightarrow cl(\mathsf{M}^{-1}) \nsubseteq [cl(\mathsf{M})]^{-1}$.

4. p - topological ring

In this section, we investigate the conception of p - topological rings and scrutinize its properties with some examples.

Definition 4.1. p - topological ring is a ring $\mathcal{R}$ with a topology τ gratifies the following provisions:

(i) for each $x, y \in \mathcal{R}$ and open neighbourhood S of $x + y$, there exist pre - open sets T of x and W of y such that $\mathsf{T} + \mathsf{W} \subseteq \mathsf{S}$,
(ii) for each $x \in \mathcal{R}$ and open neighbourhood P of $-x$, there exists pre - open set Q of x such that $-\mathsf{Q} \subseteq \mathsf{P}$,

(iii) for each $x, y \in \mathcal{R}$ and open neighbourhood S of $x \,.\, y$, there exist pre - open sets T of x and W of y such that $\mathsf{T} \,.\, \mathsf{W} \subseteq \mathsf{S}$.

In essence, ring addition, multiplication, and additive inversion mappings are pre - continuous.

We notice that,

- Any ring $\mathcal{R}$ with either discrete or indiscrete topology are idle examples of p - topological ring. In general, any topological ring is p - topological ring.
- A ring $\mathcal{R}$ with partition topology is p - topological ring, where τ_p is discrete for a partition topology.

Example 4.2. Consider $\mathcal{R}$ to be an infinite ring under ring addtion $+$ and multiplication $*$ with additive identity e. Let $a \in \mathcal{R}$ and τ be a topology on $\mathcal{R}$, $\tau = \{\emptyset, \mathcal{R} \setminus \{e, a\}, \mathcal{R}\}$. Then $\tau_p = \mathcal{P}(\mathcal{R}) \setminus \{\{e\}, \{a\}, \{e, a\}\}$. Let x be an arbitrary element of $\mathcal{R}$ such that $x = b + c$, $x = y * z$ for some $b, c, y, z \in \mathcal{R}$ and $\mathcal{U}$ be an open neighbourhood of x. Suppose $\mathcal{U} = \mathcal{R}$, then constrain contentment is obvious. On presuming $\mathcal{U} = \mathcal{R} \setminus \{e, a\}$, then $x \notin \{e, a\}$ and the cases follow :

Ring addition:
Case 1: If $b, c \notin \{e, a\}$, then $\{b\}, \{c\}$ are pre - open neighbourhoods of b and c such that $\{b\} + \{c\} = \{x\} \subset \mathcal{U}$.
Case 2: Whether $b = c = a$, then $\{a, y_1\}$ where $y_1 \notin \{e, -a, -y_1, a - y_1, y_1 - a\}$ is a pre - open neighbourhood of b, c such that $\{a, y_1\} + \{a, y_1\} \subset \mathcal{U}$.
Case 3: Suppose either b or $c \in \{e, a\}$
If either b or c is e, then the other element should be x. Without loss of generality, let us fix $b = e$. Then $\{e, y_2\}$, where $y_2 \notin \{-x, a - x, -x + a\}$ and $\{x\}$ are pre - open neighbourhoods of b and c such that $\{e, y_2\} + \{x\} \subset \mathcal{U}$.
If $b = a$, then $c = -a + x$. Now, $\{a, y_3\}$ and $\{-a + x, y_4\}$, where $y_3 \notin \{-x + a,$
$a - x + a, -y_4, a - y_4\}$ and $y_4 \notin \{-a, e\}$ are pre - open neighbourhoods of b and c respectively such that $\{a, y_3\} + \{-a + x, y_4\} \subset \mathcal{U}$.
If $c = a$, then $b = x - a$. Now, $\{x - a, y_5\}$ and $\{a, y_6\}$, where $y_5 \notin \{-a, e\}$ and $y_6 \notin \{a - x, a - x + a, -y_5, -y_5 + a\}$ are pre - open neighbourhoods of b and c respectively such that $\{a, y_3\} + \{-a + x, y_4\} \subset \mathcal{U}$.

Thus, Ring addition is pre - continuous on $\mathcal{R}$.

Ring inversion:
Case 1: If $-x \neq a$, then $\{-x\}$ is a pre - open neighbourhood of $-x$ such that $-\{-x\} = \{x\} \subset \mathcal{U}$.
Case 2: Suppose $-x = a$, then $\{-x, y\}$ where $y \neq x$ is a pre - open neighbourhood of $-x$ such that $-\{-x, y\} = \{x, -y\} \subset \mathcal{U}$.

Hence, ring inversion is pre - continuous on $\mathcal{R}$.

Ring multiplication:
Case 1: If $y, z \notin \{e, a\}$, then $\{y\}, \{z\}$ are pre - open neighbourhoods of y and z such that $\{y\} * \{z\} = \{x\} \subset \mathcal{U}$.
Case 2: Whether $y = z = a$, then $\{a, y_6\}$ where y_6 satisfying $a * y_6, y_6 * a, y_6^2 \notin \{e, a\}$ is a pre - open neighbourhood of y, z such that $\{a, y_6\} * \{a, y_6\} \subset \mathcal{U}$.
Case 3: Suppose either y or $z \in \{e, a\}$

If $y = a$, then $\{a, y_7\}$ and $\{z, y_8\}$, where y_7, y_8 suffices $a * y_8, y_7 * z, y_7 * y_8 \in \mathcal{U}$ are pre - open neighbourhoods of y and z respectively such that $\{a, y_7\} * \{z, y_8\} \subset \mathcal{U}$.
If $z = a$, then $\{y, y_9\}$ and $\{a, y_0\}$, where y_9, y_0 satiate $y * y_0, y_9 * a, y_9 * y_0 \notin \{e, a\}$ are pre - open neighbourhoods of y and z respectively such that $\{y, y_9\} * \{a, y_0\} \subset \mathcal{U}$.

Therefore, Ring multiplication is pre - continuous on $\mathcal{R}$.

Thence, $(\mathcal{R}, \tau)$ is a connected (not pre - connected) p - topological ring. In the above example, there is no necessity of commutativity and unity in $\mathcal{R}$.

Example 4.3. Let $\mathcal{R}$ be the ring of real numbers under usual addition and usual multiplication together with $\tau = \{\emptyset, \mathcal{D}, \mathbb{R}\}$ where $\mathcal{D}$ is a dense subset of $\mathbb{R}$ with empty interior under usual topology on $\mathbb{R}$. At most, the possible choices of $\mathcal{D}$ are either $\mathbb{Q}$ or $\mathbb{S}$. Under both cases, $(\mathcal{R}, \tau)$ is not a topological ring. Indeed, let $\mathcal{D}$ be $\mathbb{Q}$, and $\mathcal{D}$ be an open neighbourhood of 0. $0 = s + (-s)$ where $s \in \mathbb{S}$. For each s, the only open neighbourhood of s is $\mathbb{R}$ and $\mathbb{R} + \mathbb{R} \nsubseteq \mathcal{D}$. Also, let $\mathcal{D}$ be $\mathbb{S}$, and $\mathcal{D}$ be an open neighbourhood of s. $s = s + 0$. $\mathbb{R}$ is the only open set containing 0 and $\mathcal{D} + \mathbb{R} \nsubseteq \mathcal{D}$. Hence, ring addition is not continuous. Similarly, by replacing 1 instead of 0, we have that multiplication is not continuous. Let us check whether $(\mathcal{R}, \tau)$ is a p - topological ring or not ? For given τ, $\tau_p = \tau \cup \{\mathcal{U} : \mathcal{U} \subset \mathbb{R},\ \mathcal{U}$ has at least one element $d \ni d \in \mathcal{D}\}$. The pre - continuity of ring addition, inversion and multiplication holds trivially for $\mathbb{R}$ as open neighbourhood of $x + y, x \,.\, y, -x$. So let us assume that $\mathcal{D}$ is the appraising neighbourhood.

Case 1: Assume $\mathcal{D} = \mathbb{S}$
Ring addition:
Let $x, y \in \mathbb{S}$. In this case, $\{x\}, \{y\}$ are pre - open neighbourhoods of x and y such that $\{x\} + \{y\} = \{x + y\} \subset \mathcal{D}$.
Suppose $x \in \mathcal{D}, y \in \mathbb{Q}$, then $\{x\}, \{y, z\}$ where $z \in \mathcal{D}$ such that $z + x \in \mathcal{D}$ are pre - open neighbourhoods of x and y satisfying $\{x\} + \{y, z\} \subset \mathcal{D}$.

Hence Ring addtion is pre - continuous.

Ring Inversion:
If $x \in \mathcal{D}$ then $-x \in \mathcal{D}$, and so $-\mathcal{D} \subseteq \mathcal{D}$. Thus, ring inversion is pre - continuous.
Ring Multiplication:
Suppose both $x, y \in \mathbb{S}$. In this case, $\{x\}, \{y\}$ are pre - open neighbourhoods of x and y such that $\{x\} \,.\, \{y\} = \{x.y\} \subset \mathcal{D}$.
Let $x \in \mathcal{D}, y \in \mathbb{Q}$, then $\{x\}, \{y, z\}$ where $z \in \mathcal{D}$ such that $z.x \in \mathcal{D}$ are pre - open neighbourhoods of x and y satisfying $\{x\} \,.\, \{y, z\} \subset \mathcal{D}$.
Case 2: Assume $\mathcal{D} = \mathbb{Q}$
Ring addition and multiplication:
Suppose both $x, y \in \mathbb{S}$, then it is impossible to find pre - open neighbourhoods $\mathcal{U}, \mathcal{V}$ of x, y such that $\mathcal{U} + \mathcal{V} \subseteq \mathcal{D}$. Hence ring addition is not pre - continuous. Analogously, ring multiplication is not pre - continuous.

In consequence of the above, $(\mathcal{R}, \tau)$ is a pre - connected p - topological ring when $\mathcal{D} = \mathbb{S}$ and not a p - topological ring while $\mathcal{D} = \mathbb{Q}$.

In the upcoming part of the section, For circumvent of replicate proofs, we prove results on ring multiplication alone which also hold for ring addition.

Proposition 4.4. Let M be an open set in a p - topological ring $\mathcal{R}$. Then $x\ .\ \mathsf{M}$ (respectively $\mathsf{M}\ .\ x$) is pre - open for all $x \in \mathcal{U}(\mathcal{R})$.

Proof. Let $b \in x\ .\ \mathsf{M}$, then $b = x\ .\ a$ for some $a \in \mathsf{M}$. Now, $a = x^{-1}\ .\ b$ and by the pre - continuity of ring multiplication there endures pre - open neighbourhoods S and T of x^{-1} and b such that $\mathsf{S}\ .\ \mathsf{T} \subseteq \mathsf{M}$ which implies $b \in \mathsf{T} \subseteq x\ .\ \mathsf{M}$. Hence $x\ .\ \mathsf{M}$ is pre - open. Similarly we can prove that $\mathsf{M}\ .\ x$ is pre - open. □

Corollary 4.5. Let M be an open set in a p - topological ring $\mathcal{R}$. Then $x\ +\ \mathsf{M}$ (respectively $\mathsf{M}\ +\ x$) is pre - open.

In Proposition 4.4 and Corollary 4.5, the openness of M cannot be extended to pre - openess. Indeed, consider the ring $\mathcal{R} = (\mathbb{Z}_4, \oplus, \odot)$ with $\tau = \{\emptyset, \{1, 2, 3\}, \mathbb{Z}_4\}$ is a p - topological ring by similar argument to Example 4.2. Now, let $\{2\} \in \tau_p$ and $2 \in \mathcal{R}$. Then $2 \oplus \{2\} = 2 \odot \{2\} = \{0\} \notin \tau_p$.

Proposition 4.6. Let $\mathcal{R}$ be a p - topological ring and $\mathsf{M} \subseteq \mathcal{R}$. Then $-\mathsf{M}$ is pre - open if and only if M is pre - open.

Proof. Let $\mathsf{M} \in \tau_p$, then there exists an open set O in $\mathcal{R}$ such that $\mathsf{M} \subseteq O \subseteq cl(\mathsf{M})$. Now, $-\mathsf{M} \subseteq -O \subseteq -(cl(\mathsf{M}))$. Since inversion is pre - continuous, we have $-O$ which is pre - open and $-(cl(\mathsf{M}))$ is the pre - closure of $-\mathsf{M}$. By using Lemma 2.4 (ii), $-\mathsf{M} \subseteq -O \subseteq int(cl(-O)) \subseteq -(cl(\mathsf{M})) \subseteq cl(-\mathsf{M})$. Hence there exists an open set $int(cl(-O))$ such that $-\mathsf{M} \subseteq int(cl(-O)) \subseteq cl(-\mathsf{M})$ and so $-\mathsf{M}$ is pre - open. Proof of the converse is similar. □

By using Lemma 2.1 and Proposition 4.6, Conditions (i) and (ii) of Definition 4.1 can be combined as: for each $x, y \in G$ and each open neighbourhood U of $x - y$ there exist pre - open sets V of x and W of y such that $V - W \subseteq U$.

Proposition 4.7. For a closed set F in a p - topological ring $\mathcal{R}$, $y\ .\ \mathsf{F}$ (respectively $\mathsf{F}\ .\ y$) is pre - closed for all $y \in \mathcal{U}(\mathcal{R})$.

Proof. Let $b \in pcl(y\ .\ \mathsf{F})$ and $\mathcal{W}$ be an open neighbourhood of c where $c = y^{-1}\ .\ b$. Then by the pre - continuity of ring multiplication, there exist pre - open neighbourhoods $\mathcal{C}$ and $\mathcal{D}$ of y^{-1} and b in $\mathcal{R}$ such that $\mathcal{C}\,.\,\mathcal{D} \subset \mathcal{W}$. Since $b \in pcl(y\,.\,\mathsf{F})$ we have $\mathcal{D} \cap y\,.\,\mathsf{F} \neq \emptyset$. Let $d \in \mathcal{D} \cap\ y\,.\,\mathsf{F}$, then $y^{-1}\,.\,d \in \mathsf{F} \cap \mathcal{C}\,.\,\mathcal{D} \subseteq \mathsf{F} \cap \mathcal{W}$ which implies $\mathsf{F} \cap \mathcal{W}$ is non - empty and so c is a limit point of F and since F is closed we have $c \in \mathsf{F}$. Now $b = y\,.\,c$ and so $b \in y.\mathsf{F}$. By the above argument, $pcl(y\,.\,\mathsf{F}) \subseteq y\,.\,\mathsf{F}$ and since $y\,.\,\mathsf{F} \subseteq pcl(y\,.\,\mathsf{F})$ is trivial we have $y\,.\,\mathsf{F} = pcl(y\,.\,\mathsf{F})$. Hence $y\,.\,\mathsf{F}$ is pre - closed. Proof of $\mathsf{F}\,.\,y$ is similar. □

Corollary 4.8. For a closed set F in a p - topological ring $\mathcal{R}$, $y\ +\ \mathsf{F}$ (respectively $\mathsf{F}\ +\ y$) is pre - closed.

The choice of $x \in \mathcal{U}(\mathcal{R})$ in Proposition 4.4 is justified by the following example.

Example 4.9. Let the ring $\mathcal{R} = (\mathbb{Z}_6, \oplus, \odot)$ with $\tau = \{\emptyset, \{1, 2, 4, 5\}, \mathbb{Z}_6\}$. The pair $(\mathcal{R}, \tau)$ is a p - topological ring and the verification follows from Example 4.2. Let $x = 3$ and $\mathsf{M} = \{1, 2, 4, 5\}$ then $3 \odot \{1, 2, 4, 5\} = \{0, 3\}$ which is not a pre - open set.

Since the results of the Proposition 4.4 and 4.7 relay on units of the ring $\mathcal{R}$, a natural question arises, Does a non - unit element suffice any conditions or not ? The following theorem has some answer to this.

Theorem 4.10. *Let $\mathcal{R}$ be a p - topological ring with unity and* M *be a subset of $\mathcal{R}$. Then for all $y \in \mathcal{R}$ and $x \in \mathcal{U}(\mathcal{R})$, the following are valid:*

(i) $int(y\,.\,\mathsf{M}) \subseteq y\,.\,pint(\mathsf{M})$
(ii) $x\,.\,int(\mathsf{M}) \subseteq pint(x\,.\,\mathsf{M})$
(iii) $x\,.\,int(\mathsf{M}) \subseteq pint(x\,.\,int(\mathsf{M}))$
(iv) $y\,.\,pcl(\mathsf{M}) \subseteq cl(y\,.\,\mathsf{M})$
(v) $pcl(x\,.\,\mathsf{M}) \subseteq x\,.\,cl(\mathsf{M})$
(vi) $pcl(x\,.\,cl(\mathsf{M})) \subseteq x\,.\,cl(\mathsf{M})$

Proof.

(i) Let $b \in int(y\,.\,\mathsf{M})$ then $b = y\,.\,a$ for some $a \in \mathsf{M}$. Since ring multiplication is pre - continuous, there endure pre - open neighbourhoods $\mathcal{S}$ and $\mathcal{T}$ of y and a such that $\mathcal{S}\,.\,\mathcal{T} \subseteq int(\mathsf{M})$. Now, $y\,.\,\mathcal{T} \subseteq \mathcal{S}\,.\,\mathcal{T} \subseteq int(y\,.\,\mathsf{M}) \subseteq y\mathsf{M}$ implies that $y\,.\,\mathcal{T} \subseteq y\,pint(\mathsf{M})$. Since $b = y\,.\,a \in y\,.\,\mathcal{T}$, we have $b \in y\,.\,pint(\mathsf{M})$. Thus, $int(y\,.\,\mathsf{M}) \subseteq y\,.\,pint(\mathsf{M})$.

(ii) Let $b \in x\,.\,int(\mathsf{M})$ then $b = x\,.\,a$ for some $a \in Int(\mathsf{M})$. Since ring multiplication is pre - continuous, there prevail pre - open sets $\mathcal{S}$ and $\mathcal{T}$ of x^{-1} and b such that $\mathcal{S}\,.\,\mathcal{T} \subseteq Int(\mathsf{M})$. Then, $x^{-1}\,.\,\mathcal{T} \subseteq \mathcal{S}\,.\,\mathcal{T} \subseteq int(\mathsf{M}) \subseteq \mathsf{M}$ which implies that $\mathcal{T} \subseteq x\,.\,\mathsf{M}$. Since $\mathcal{T}$ is an pre - open neighbourhood of b we have $b \in pint(x\,.\,\mathsf{M})$. Thus, $x\,.\,int(\mathsf{M}) \subseteq pint(x\,.\,\mathsf{M})$.

(iii) Let $b \in x\,.\,int(\mathsf{M})$ then $b = x\,.\,a$ for some $a \in int(\mathsf{M})$. Since ring multiplication is pre - continuous, there endure pre - open sets $\mathcal{S}$ and $\mathcal{T}$ of x^{-1} and b such that $S\,.\,\mathcal{T} \subseteq int(\mathsf{M})$. Now, $x^{-1}\,.\,\mathcal{T} \subseteq \mathcal{S}\,.\,\mathcal{T} \subseteq int(\mathsf{M}) \subseteq pint(\mathsf{M})$ implies that $\mathcal{V} \subseteq x\,.\,int(\mathsf{M})$. Since $\mathcal{V}$ is an pre - open neighbourhood in $\mathcal{R}$ such that $b \in \mathcal{V}$ and $b \in pint(x\,.\,int(\mathsf{M}))$. Thus, $x\,.\,int(\mathsf{M}) \subseteq pint(x\,.\,int(\mathsf{M}))$.

(iv) Consider $b \in y\,.\,pcl(\mathsf{M})$, then $b = y\,.\,a$ for some $a \in pcl(\mathsf{M})$. Let $\mathcal{W}$ be an open neighbourhood of b. By pre - continuity of ring multiplication, there prevail pre - open sets $\mathcal{S}$ and $\mathcal{T}$ of y and a such that $\mathcal{S}\,.\,\mathcal{T} \subseteq \mathcal{W}$. Since $a \in pcl(\mathsf{M})$ we have $\mathsf{M} \cap \mathcal{T}$ is non - empty and so there prevail an element $c \in \mathsf{M} \cap \mathcal{T}$ which implies $y\,.\,c \in y\,.\,\mathsf{M} \cap \mathcal{S}\,.\,\mathcal{T} \subseteq (y\,.\,\mathsf{M}) \cap \mathcal{W}$. We have, $(y\,.\,\mathsf{M}) \cap \mathcal{W}$ is non - empty and so every open neighbourhood of b intersects $y\,.\,\mathsf{M}$. Thus, $b \in cl(y\,.\,\mathsf{M})$.

(v) Let $b \in pcl(x\,.\,\mathsf{M})$ and $c = x^{-1}\,.\,b$ in $\mathcal{R}$. Let $\mathcal{W}$ be an open neighbourhood of c. By pre - continuity of ring multiplication, there endure pre - open sets $\mathcal{S}$ and $\mathcal{T}$ of x^{-1} and b such that $\mathcal{S}\,.\,\mathcal{T} \subseteq \mathcal{W}$. Since $b \in pcl(x\,.\,\mathsf{M})$, there endure some d in $\mathcal{R}$ such that $d \in (x\,.\,\mathsf{M}) \cap \mathcal{T}$ which implies that $d \in \mathsf{M} \cap \mathcal{S}\,.\,\mathcal{T} \subseteq \mathsf{M} \cap \mathcal{W}$. We have $\mathsf{M} \cap \mathcal{W}$ is non - empty which implies arbitrary open neighbourhood of c intersects M. Thus, $c \in cl(\mathsf{M}) \Rightarrow x\,.\,c = b \in x\,.\,cl(\mathsf{M})$.

(vi) Let $b \in pcl(x\,.\,cl(\mathsf{M}))$ and $\mathcal{W}$ be an open neighourhood of c where $c = x^{-1}\,.\,b$. Now, by pre - continuity of ring multiplication, there exist pre - open sets $\mathcal{S}$ and $\mathcal{T}$ of x^{-1} and b such that $\mathcal{S}\,.\,\mathcal{T} \subseteq \mathcal{W}$ and so $\mathcal{T} \subset x\,.\,\mathcal{W}$. Since $b \in pcl(x\,.\,cl(M))$, we have $\mathcal{T} \cap (x\,.\,cl(\mathsf{M})) \neq \emptyset$ which implies $x\,.\,\mathcal{W} \cap x\,.\,(cl(\mathsf{M})) \neq \emptyset$. Thus, $\mathcal{W} \cap cl(\mathsf{M}) \neq \emptyset$ and so some points of $\mathcal{W}$ are limit points of M. Since $\mathcal{W}$ is an open neighbourhood of those points, we have $\mathcal{W} \cap \mathsf{M} \neq \emptyset$. Hence $c \in cl(\mathsf{M})$ and so $x \,.\, c = b \in x \,.\, cl(\mathsf{M})$.

$\square$

Corollary 4.11. Let $\mathcal{R}$ be a p - topological ring and S be a subset of $\mathcal{R}$. Then for all $y \in \mathcal{R}$ the following are valid:

(i) $int(y + \mathsf{S}) \subseteq y + pint(\mathsf{S})$
(ii) $y + \mathsf{int}(\mathsf{S}) \subseteq pint(y + \mathsf{S})$
(iii) $y + \mathsf{int}(\mathsf{S}) \subseteq pint(y + \mathsf{int}(\mathsf{S}))$
(iv) $pcl(y + \mathsf{S}) \subseteq y + \mathsf{cl}(\mathsf{S})$
(v) $y + \mathsf{pcl}(\mathsf{S}) \subseteq cl(y + \mathsf{S})$
(vi) $pcl(y + \mathsf{cl}(\mathsf{S})) \subseteq y + \mathsf{cl}(\mathsf{S})$
(vii) $-(pcl(S)) \subset cl(-S)$.

Theorem 4.12. *Let* S *and* T *be any subsets of a* p *- topological ring* $\mathcal{R}$. *Then* $pcl(\mathsf{S}) \,.\, pcl(\mathsf{T}) \subseteq cl(\mathsf{S} \,.\, \mathsf{T})$.

Proof. Let $x \in pcl(\mathsf{S}) \,.\, pcl(\mathsf{T})$ and $\mathcal{W}$ be any open neighbourhood of x in $\mathcal{R}$ where $x = a \,.\, b$ for some $a \in pcl(\mathsf{S})$ and $b \in pcl(\mathsf{T})$. Since ring multiplication is pre - continuous, there endure pre - open sets $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{R}$ containing a and b such that $\mathcal{C} \,.\, \mathcal{D} \subseteq \mathcal{W}$. Since $a \in pcl(\mathsf{S})$ and $b \in pcl(\mathsf{T})$ there exist $c \in \mathsf{S} \cap \mathcal{C}$ and $d \in \mathsf{T} \cap \mathcal{D}$. Now $c \,.\, d \in (\mathsf{S} \,.\, \mathsf{T}) \cap (\mathcal{C} \,.\, \mathcal{D}) \subseteq \mathsf{S} \,.\, \mathsf{T} \cap \mathcal{W}$ which implies that $\mathsf{S} \,.\, \mathsf{T} \cap \mathcal{W} \neq \emptyset$. Hence x is a limit point of $\mathsf{S} \,.\, \mathsf{T}$ and so $x \in cl(\mathsf{S} \,.\, \mathsf{T})$. $\square$

Corollary 4.13. Let S and T be any subsets of a p - topological ring $\mathcal{R}$. Then $pcl(\mathsf{S}) + pcl(\mathsf{T}) \subseteq cl(\mathsf{S} + \mathsf{T})$.

One may remind that, A bijective mapping $f : X \mapsto Y$ is p **- homeomorphism** [2] if f is pre - continuous and $f(A)$ is pre - open for every open set A of X.

Theorem 4.14. *Let* $\mathcal{R}$ *be a* p *- topological ring and* $a \in \mathcal{R}$. *Then for all* $x \in \mathcal{U}(\mathcal{R})$, *the mappings* $\lambda_a(x) = a \,.\, x$ *and* $\rho_a(x) = x \,.\, a$ *are* p *- homeomorphism.*

Proof. Let $x \in \mathcal{R}$ and $\mathcal{U}_1$ be an open set containing $a \,.\, x$. By pre - continuity of ring multiplication, for each open neighbourhood $\mathcal{U}_1$ of $a \,.\, x$ there exist pre - open neighbourhoods V_1 and $\mathcal{W}_1$ of a and x such that $\mathcal{V}_1 \,.\, \mathcal{W}_1 \subseteq \mathcal{U}_1$ which implies $a \,.\, \mathcal{W}_1 \subseteq \mathcal{U}_1$ and so $\lambda_a(\mathcal{W}_1) \subseteq \mathcal{U}_1$. Thus, $\lambda_a(x)$ is pre - continuous. Let $x \in \mathcal{R}$ and $\mathcal{U}_2$ be an open neighbourhood of x. The element x can be written as $x = a^{-1} \,.\, a \,.\, x$. Since each left translation is pre - continuous there exist pre - open sets $\mathcal{V}_2$ and $\mathcal{W}_2$ of a^{-1} and $a \,.\, x$ such that $\mathcal{V}_2 \,.\, \mathcal{W}_2 \subseteq \mathcal{U}_2$. Hence each left translation is p - homeomorphism. The proof of $\rho_a(x)$ is similar. $\square$

Corollary 4.15. Let $\mathcal{R}$ be a p - topological ring and $a, x \in \mathcal{R}$. Then

(i) The mappings $\lambda_a(x) = a + x$ and $\rho_a(x) = x + a$ are p - homeomorphism.
(ii) Inversion mapping $I(a) = -a$ is p - homeomorphism.

Theorem 4.16. *Let* $\mathcal{R}$ *and* $\mathcal{S}$ *be* p *- topological division rings with* $\mathcal{S}$ *is submaximal and* $\mathfrak{f}$ *be a* p *- irresolute ring homomorphism at unity and identity. Then* $\mathfrak{f}$ *is* p *- irresolute ring homomorphism.*

Proof. Let $x \in \mathcal{R}$ and $\mathcal{A}$ be a pre - open set in $\mathcal{S}$ containing $\mathfrak{f}(x) = y$. Since $\mathcal{S}$ is submaximal, by Lemma 2.4, $\mathcal{A}$ is open. By Theorem 4.14, left multiplication of an open set is pre - open, we have $y^{-1} . \mathcal{A}$ is pre - open in $\mathcal{S}$ containing $1_{\mathcal{S}}$. Since $\mathfrak{f}$ is p - irresolute at unity $1_{\mathcal{R}}$, we can find a pre - open neighbourhood $\mathcal{B}$ in $\mathcal{R}$ containing $1_{\mathcal{R}}$ such that $\mathfrak{f}(\mathcal{B}) \subset y^{-1} . \mathcal{A}$. Given that $\mathfrak{f}$ is homomorphism and so $\mathfrak{f}(x . \mathcal{B}) = \mathfrak{f}(x) . \mathfrak{f}(\mathcal{B}) \subseteq \mathcal{A}$. By similar of $\mathfrak{f}$ at identity $e_{\mathcal{R}}$, we have $\mathfrak{f}$ is p - irresolute ring homomorphism on $\mathcal{R}$. □

5. Subrings and Ideals

In this section, we study subrings and ideals of p - topological rings and offer some results.

Theorem 5.1. *Let $\mathcal{H}$ be an open (closed) subring of a p - topological ring $\mathcal{R}$. Then $\mathcal{H}$ is a p - topological ring.*

Proof. Since $\mathcal{H}$ is open (closed), every open (closed) set in $\mathcal{H}$ is open (closed) in $\mathcal{R}$. By the fact, ring addition, inversion and multiplication is pre - continuous in $\mathcal{R}$ is continuous and hence the result. □

Corollary 5.2. Open (closed) ideal (Left ideal, right ideal) in a p - topological ring is p - topological ring.

The following Example ensures that an ideal (subring) of a p - topological ring need not be p - topological ring.

Example 5.3. Consider $\mathcal{R} = \mathbb{Z}_6$ with $\tau = \{\emptyset, \{1, 2, 3, 5\}, \mathcal{R}\}$ which is a p - topological ring and the proof follows from Example 4.2. Let $\mathcal{Y} = \{0, 3\}$ be an ideal in $\mathcal{R}$. Then $\tau_{\mathcal{Y}} = \{\emptyset, \{3\}, \mathcal{Y}\} = \tau_{\mathcal{Y}_p}$. Take $\{3\}$ as an open neighbourhood of 3. It is impossible to find pre - open neighbourhoods $\mathcal{U}, \mathcal{V}$ of $0, 3$ such that $\mathcal{U} \oplus \mathcal{V} \subseteq \{3\}$.

Theorem 5.4. *Closure of a subring in a p - topological ring is a p - topological ring.*

Proof. Let $\mathcal{R}$ be a p - topological ring and $\mathcal{S}$ be a subring of $\mathcal{R}$ with closure $\overline{\mathcal{S}}$. Let $a, b \in \overline{\mathcal{S}}$ and $\mathcal{U}$ be an open neighbourhood of $a + b$. By the pre - continuity of ring addition there exist pre - open neighbourhoods $\mathcal{V}, \mathcal{W}$ of a and b such that $\mathcal{V} + \mathcal{W} \subseteq \mathcal{U}$. Since $a, b \in \overline{\mathcal{S}}$, every pre - open neighbourhood of a, b intersects $\mathcal{S}$. Thus there exists $x \in \mathcal{V} \cap \mathcal{S}$ and $y \in \mathcal{V} \cap \mathcal{S}$. Since $x, y \in \mathcal{S}$ we have $x + y$ and since $x \in \mathcal{V}, y \in \mathcal{W}$, then $x + y \in \mathcal{U}$. Thus $x + y \in \mathcal{S} \cap \mathcal{U}$ and since $\mathcal{U}$ is an arbitrary open neighbourhood of $a + b$, we have $a + b \in \overline{\mathcal{S}}$. Similarly we prove that $a . b \in \overline{\mathcal{S}}$. Now let $c \in \overline{\mathcal{S}}$ and $\mathcal{M}$ be an open neighbourhood of $-c$. Then by the pre - continuity of inverse, there exists pre - open neighbourhood of $\mathcal{N}$ of c such that $-\mathcal{N} \subseteq \mathcal{M}$. Since $\mathcal{N}$ is pre - open and $c \in \mathcal{N}$, there is a point $z \in \mathcal{S} \cap \mathcal{N}$. Then we have $-z \in \mathcal{S} \cap \mathcal{M}$ and as before this implies $-c \in \overline{\mathcal{S}}$. Thus, $\overline{\mathcal{S}}$ is a subring of $\mathcal{R}$ and by theorem 5.1, $\overline{\mathcal{S}}$ is a p - topological ring. □

Corollary 5.5. Closure of an ideal (Left ideal, right ideal) in a p - topological ring is a p - topological ring.

Theorem 5.6. *Pre - continuous image of a pre - connected set is connected.*

Proof. Let $\mathfrak{f} : \mathcal{X} \mapsto \mathcal{Y}$ be a pre - continuous map. Let $\mathcal{X}$ be connected and $\mathfrak{f}(\mathcal{X}) = \mathcal{Z}$. Let $\mathfrak{g}$ be the restriction of $\mathfrak{f}$ to $\mathcal{Z}$ which is an onto pre - continuous map. We prove that $\mathcal{Z}$ is connected by contradiction. Suppose $\mathcal{Z}$ is not connected then $\mathcal{Z} = \mathcal{M} \cup \mathcal{N}$

where $\mathcal{M}$ and $\mathcal{N}$ are disjoint open sets in $\mathcal{Z}$. Then $\mathfrak{g}^{-1}(\mathcal{M})$ and $\mathfrak{g}^{-1}(\mathcal{N})$ are disjoint non - empty sets whose union is $\mathcal{X}$. Since inverse image of an open set under a pre - continuous map is pre - open which contradicts to $\mathcal{X}$ is pre - connected. $\square$

Theorem 5.7. *Let $\mathcal{R}$ be a p - topological ring and $\mathcal{D}$ a pre - connected component of zero. Then $a + \mathcal{D}$ is a connected component of a for each $a \in \mathcal{R}$ and $\mathcal{D}$ is a closed ideal.*

Proof. Since each translation is p - homeomorphism and by Theorem 5.6, $a + \mathcal{D}$ is a connected component of a for each $a \in \mathcal{R}$. If $a \in \mathcal{D}$, then $\mathcal{D} \cap (a + \mathcal{D}) \neq \emptyset$, so $\mathcal{D} \cup (a + \mathcal{D})$ is a connected set containing zero and thus $a + \mathcal{D} \subseteq \mathcal{D}$. Also $-\mathcal{D}$ is a connected subset of $\mathcal{R}$ containing zero, so $-\mathcal{D} \subseteq \mathcal{D}$. Hence $\mathcal{D}$ is an additive subgroup. For each $b \in \mathcal{R}$, $b \, . \, \mathcal{D}$ and $\mathcal{D} \, . \, b$ are connected sets containing zero so $b \, . \, \mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D} \, . \, b \subseteq \mathcal{D}$. Thus $\mathcal{D}$ is a closed ideal. $\square$

6. Conclusion

In the study of mathematical structures like group and topology, we begin by defining those structures on finite sets. Topological groups mostly deals with an infinite set alone by an assumption in separation. To overcome this, some generalizations of topological groups are defined but they didn't attain similar properties to topological group. By defining p - topological group, we reach a space which has properties close relevant to topological group. Also, we extend the concept of p - topological groups to p - topological rings and discuss some of its properties.

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