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ON CONTRA we^*-CONTINUOUS FUNCTIONS

MURAD ÖZKOÇ† AND PINAR ŞAŞMAZ

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Abstract. The main goal of this paper is to introduce and study a new type of contra continuity called contra we^*-continuity via we^*-open sets defined by us. We obtain several characterizations and some fundamental properties of contra we^*-continuous functions. Moreover, we investigate the relationships between contra we^*-continuous functions and other related strong and weak forms of contra continuity. Furthermore, we introduce the concepts of we^*-compactness and strongly contra we^*-closed graphs to look into some different properties.

1. Introduction

Some forms of open sets such as β-open sets [1], a-open sets [10], e^*-open sets [13] and w-open sets [5] play important role in general topology. The same way the notion of continuity is now a vital concept in topology and applications. By using these sets many authors introduced and studied various types of generalizations of continuity. For instance, the concept of w-continuity [15] were introduced by Hdeib in 1989. Contra continuity [7] were introduced and investigated by Dontchev in 1996. Next, in 2007, Al-Omari and Noorani were introduced contra w-continuity [2], in 2008, Ekici were introduced contra e^*-continuity [12], in 2009, Ekici were introduced e^*-continuity [13]. Recently, Aljarrah et. al. were introduced and studied contra $w\beta$-continuity [3] in 2014.

In this paper, we define and study the notion of contra we^*-continuity which is weaker than both the notion of contra continuity [7] defined by Dontchev and the notion of contra e^*-continuity [12] defined by Ekici. Also, we obtain several characterizations of contra we^*-continuous functions and investigate some of their fundamental properties. Moreover, we investigate the relationships between contra $w\beta$-continuous functions and separation axioms and we^*-compactness and contra we^*-closedness of graphs.

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† *Corresponding author.*

2. Preliminaries

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open, clopen) sets of X is denoted $C(X)$(resp. $O(X), CO(X)$). The co-countable topology on X, τ_{coc}; the topology whose open sets are the empty set and complements of subsets of X which are at most countable. In this paper, a space X is called locally indiscrete if every open set is closed in X. We recall the following definitions which will be used throughout this paper.

Definition 2.1. A subset A of a space X is called:

(1) regular open [20] if $A = int(cl(A))$. The complement of a regular open set is called regular closed. A point $x \in X$ is said to be the δ-cluster point [22] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. If $A = \delta$-$cl(A)$, then A is called δ-closed [22], and the complement of a δ-closed set is called δ-open. The set $\{x|(\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by δ-$int(A)$.

(2) β-open [1] if $A \subseteq cl(int(cl(A)))$. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by β-$cl(A)$. The union of all β-open sets of X contained in A is called the β-interior of A and is denoted by β-$int(A)$.

(3) a-open [10] if $A \subseteq int(cl(\delta\text{-}int(A))$. The complement of an a-open set is called a-closed [10]. The intersection of all a-closed sets containing A is called the a-closure [10] of A and is denoted by a-$cl(A)$. The union of all a-open sets of X contained in A is called the a-interior [10] of A and is denoted by a-$int(A)$.

(4) e^*-open [13] if $A \subseteq cl(int(\delta\text{-}cl(A))$. The complement of an e^*-open set is called e^*-closed [13]. The intersection of all e^*-closed sets containing A is called the e^*-closure [13] of A and is denoted by e^*-$cl(A)$. The union of all e^*-open sets of X contained in A is called the e^*-interior [13] of A and is denoted by e^*-$int(A)$.

The family of all open (resp. closed, a-open, a-closed, e^*-open, e^*-closed, regular open, regular closed) subsets of X is denoted by $O(X)$ (resp. $C(X)$, $aO(X)$, $aC(X)$, $e^*O(X)$, $e^*C(X)$, $RO(X)$, $RC(X)$). The family of all open (resp. closed, a-open, a-closed, e^*-open, e^*-closed, regular open, regular closed) sets of X containing a point x of X is denoted by $O(X,x)$ (resp. $C(X,x)$, $aO(X,x)$, $aC(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $RO(X,x)$, $RC(X,x)$).

Definition 2.2. Let A be a subset of a space X. The intersection of all open sets in X containing A is called the kernel [18] of A and is denoted by $ker(A)$.

Lemma 2.3. [18] *The following properties hold for subsets A and B of a space X.*
(1) $x \in ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X,x)$.
(2) $A \subseteq ker(A)$.
(3) If A is open in X, then $A = ker(A)$.
(4) If $A \subseteq B$, then $ker(A) \subseteq ker(B)$.

Lemma 2.4. [6] *Let X be a topological space. Then the intersection of an a-open set and an e^*-open set is e^*-open.*

Definition 2.5. A subset A of a space X is said to be w-open [5] (resp. $w\beta$-open [4]) if for every $x \in A$ there exists an open (resp. β-open) set U containing x such that $U \setminus A$ is countable. The complement of an w-open set (resp. $w\beta$-open set) is said to be w-closed (resp. $w\beta$-closed).

Definition 2.6. A function $f : X \to Y$ is said to be contra continuous [7] (resp. contra w-continuous [2]) if $f^{-1}[V]$ is closed (resp. w-closed) in X for every $V \in O(Y)$.

3. we^*-open sets

Definition 3.1. Let A be a subset of a space X. A is said to be we^*-open (resp. wa-open) if for every $x \in A$, there exists an e^*-open (resp. a-open) set U containing x such that $U \setminus A$ is countable. The complement of an we^*-open (resp. wa-open) set is called an we^*-closed (resp. wa-closed). The family of all we^*-open (resp. wa-open) sets of X will be denoted by $we^*O(X)$ (resp. $waO(X)$). The family of all we^*-open (resp. wa-open) sets of X containing a point x of X will be denoted by $we^*O(X,x)$ (resp. $waO(X,x)$).

Remark 3.2. From Definition 2.5 and Definition 3.1, we have the following diagram.

$$\begin{array}{ccccccc} \text{open} & \rightarrow & \beta\text{-open} & \rightarrow & e^*\text{-open} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ w\text{-open} & \rightarrow & w\beta\text{-open} & \rightarrow & we^*\text{-open} & \leftarrow & wa\text{-open} \end{array}$$

Example 3.3. Consider the real numbers $\mathbb{R}$ with usual topology. The set of rational numbers $\mathbb{Q}$ is we^*-open but it is not wa-open.

Lemma 3.4. *Let X be a topological space. Then the following properties hold.*

(1) *The union of any family of we^*-open sets is we^*-open.*
(2) *The intersection of an wa-open set and an we^*-open set is we^*-open.*

Proof. (1) Let $x \in \bigcup \mathcal{A}$.

$$\left.\begin{array}{r} x \in \bigcup \mathcal{A} \Rightarrow (\exists A \in \mathcal{A})(x \in A) \\ \mathcal{A} \subseteq we^*O(X) \end{array}\right\} \Rightarrow (\exists U \in e^*O(X,x))(|U \setminus (\bigcup \mathcal{A})| \leq |U \setminus A| \leq \aleph_0).$$

(2) Let $A \in waO(X)$, $B \in we^*O(X)$ and $x \in A \cap B$.

$$\left.\begin{array}{r} x \in A \cap B \Rightarrow (x \in A)(x \in B) \\ (A \in waO(X))(B \in we^*O(X)) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists W \in aO(X,x))(|W \setminus A| \leq \aleph_0)(\exists V \in e^*O(X,x))(|V \setminus B| \leq \aleph_0) \\ U := W \cap V \end{array}\right\} \overset{\text{Lemma 2.4}}{\Rightarrow}$$

$$\Rightarrow (\exists U \in e^*O(X,x))(|U \setminus (A \cap B)| = |(U \setminus A) \cup (U \setminus B)| \leq |(W \setminus A) \cup (V \setminus B)| \leq \aleph_0).$$

Definition 3.5. Let A be a subset of a space X. The union of all we^*-open subsets of X contained in A is called the we^*-interior of A and is denoted by $we^*\text{-}int(A)$.

Theorem 3.6. *Let A be a subset of a space X. Then the following properties hold:*

(1) $we^*\text{-}int(A) \subseteq A$,
(2) $we^*\text{-}int(A) \in we^*O(X)$,
(3) $x \in we^*\text{-}int(A)$ *if and only if there exists* $U \in we^*O(X,x)$ *such that* $U \subseteq A$,
(4) $A \subseteq B \Rightarrow we^*\text{-}int(A) \subseteq we^*\text{-}int(B)$,
(5) $we^*\text{-}int(A) \cup we^*\text{-}int(B) \subseteq we^*\text{-}int(A \cup B)$,
(6) $we^*\text{-}int(A \cap B) \subseteq we^*\text{-}int(A) \cap we^*\text{-}int(B)$,
(7) $A \in we^*O(X)$ *if and only if* $A = we^*\text{-}int(A)$,

(8) $we^*\text{-}int(we^*\text{-}int(A)) = we^*\text{-}int(A)$.

Proof. Straightforward.

Definition 3.7. Let A be a subset of a space X. The intersection of all we^*-closed subsets of X containing A is called the we^*-closure of A and is denoted by $we^*\text{-}cl(A)$.

Theorem 3.8. *Let A be a subset of a space X. Then the following properties hold:*

(1) $A \subseteq we^*\text{-}cl(A)$,
(2) $we^*\text{-}cl(A) \in we^*C(X)$,
(3) $x \in we^*\text{-}cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in we^*O(X, x)$,
(4) $A \subseteq B \Rightarrow we^*\text{-}cl(A) \subseteq we^*\text{-}cl(B)$,
(5) $we^*\text{-}cl(A) \cup we^*\text{-}cl(B) \subseteq we^*\text{-}cl(A \cup B)$,
(6) $we^*\text{-}cl(A \cap B) \subseteq we^*\text{-}cl(A) \cap we^*\text{-}cl(B)$,
(7) $A \in we^*C(X)$ *if and only if* $A = we^*\text{-}cl(A)$,
(8) $we^*\text{-}cl(we^*\text{-}cl(A)) = we^*\text{-}cl(A)$,
(9) $we^*\text{-}cl(X \setminus A) = X \setminus we^*\text{-}int(A)$.

Proof. Straightforward.

4. Contra we^*-continuous Functions

Definition 4.1. A function $f : X \to Y$ is said to be contra we^*-continuous (resp. contra wa-continuous) if $f^{-1}[V]$ is we^*-closed (resp. wa-closed) in X for every open set V of Y.

Remark 4.2. From Definition 2.6 and Definition 4.1 we have the following diagram. The converses of these implications need not be true in general as shown by the following example.

$$\begin{array}{ccccc} \text{contra continuous} & \to & \text{contra } w\text{-continuous} & \to & \text{contra } w\beta\text{-continuous} \\ & & & & \downarrow \\ & & \text{contra } wa\text{-continuous} & \to & \text{contra } we^*\text{-continuous} \end{array}$$

Example 4.3. Consider the real numbers $\mathbb{R}$ with usual topology and let $Y := \{a, b\}$ with the topology $\tau := \{\emptyset, Y, \{b\}\}$.

(a) Define the function $f : \mathbb{R} \to Y$ by

$$f(x) = \begin{cases} a & , \quad x \in \mathbb{Q} \\ b & , \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then the function f is contra we^*-continuous but not contra wa-continuous.

(b) Define the function $g : \mathbb{R} \to Y$ by

$$g(x) = \begin{cases} a & , \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ b & , \quad x \in \mathbb{Q} \end{cases}$$

Then g is contra we^*-continuous but g is not contra w-continuous.

Theorem 4.4. *Let $f : X \to Y$ be a function. Then the followings are equivalent:*

(1) *f is contra we^*-continuous;*
(2) *$f^{-1}[F]$ is we^*-open in X for every $F \in C(Y)$;*
(3) *For every $x \in X$ and every $F \in C(Y, f(x))$, there exists $U \in we^*O(X, x)$ such that $f[U] \subseteq F$;*

(4) $f[we^*\text{-}cl(A)] \subseteq ker(f[A])$ *for every subset A of X;*
(5) $we^*\text{-}cl(f^{-1}[B]) \subseteq f^{-1}[ker(B)]$ *for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2) : Let $F \in C(Y)$.

$$\left.\begin{array}{r} F \in C(Y) \Rightarrow \setminus F \in O(Y) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \setminus f^{-1}[F] = f^{-1}[\setminus F] \in we^*C(X)$$

$\Rightarrow f^{-1}[F] \in we^*O(X)$.

(2) $\Rightarrow$ (3) : Let $x \in X$ and $F \in C(Y, f(x))$.

$$\left.\begin{array}{r} F \in C(Y, f(x)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[F] \in we^*O(X, x) \\ U := f^{-1}[F] \end{array}\right\} \Rightarrow (U \in we^*O(X, x))(f[U] \subseteq F).$$

(3) $\Rightarrow$ (4) : Let $A \subseteq X$ and $x \notin f^{-1}[ker(f[A])]$.

$$\left.\begin{array}{r} x \notin f^{-1}[ker(f[A])] \Rightarrow f(x) \notin ker(f[A]) \Rightarrow (\exists F \in C(Y, f(x)))(F \cap f[A] = \emptyset) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in we^*O(X, x))(f[U] \cap f[A] \subseteq F \cap f[A] = \emptyset)$
$\Rightarrow (\exists U \in we^*O(X, x))(f[U \cap A] \subseteq f[U] \cap f[A] = \emptyset)$
$\Rightarrow (\exists U \in we^*O(X, x))(U \cap A = \emptyset)$
$\Rightarrow x \notin we^*\text{-}cl(A)$.

(4) $\Rightarrow$ (5) : Let $B \subseteq Y$.

$$\left.\begin{array}{r} B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow f[we^*\text{-}cl(f^{-1}[B])] \subseteq ker(f\left[f^{-1}[B]\right]) \subseteq ker(B)$$

$\Rightarrow we^*\text{-}cl(f^{-1}[B]) \subseteq f^{-1}[ker(B)]$.

(5) $\Rightarrow$ (1) : Let $A \in O(Y)$.

$$\left.\begin{array}{r} A \in O(Y) \Rightarrow ker(A) = A \\ \text{Hypothesis} \end{array}\right\} \Rightarrow we^*\text{-}cl(f^{-1}[A]) \subseteq f^{-1}[ker(A)] = f^{-1}[A]$$

$\Rightarrow f^{-1}[A] \in we^*C(X)$.

Theorem 4.5. *Let $f : X \to Y$ be a contra we^*-continuous function. Then the followings hold:*

(1) *f is we^*-continuous if Y is regular.*
(2) *f is we^*-continuous if $we^*\text{-}int(f^{-1}[cl(V)]) \subseteq f^{-1}[V]$ for every $V \in O(Y)$.*

Proof. (1) Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ Y \text{ is regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists W \in O(Y, f(x)))(cl(W) \subseteq V) \\ f \text{ is contra } we^*\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in we^*O(X, x))(f[U] \subseteq cl(W) \subseteq V)$.

(2) Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \Rightarrow cl(V) \in C(Y) \\ f \text{ is contra } we^*\text{-continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[cl(V)] \in we^*O(X) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[V] \subseteq f^{-1}[cl(V)] = we^*\text{-}int(f^{-1}[cl(V)]) \subseteq f^{-1}[V]$
$\Rightarrow we^*\text{-}int(f^{-1}[V]) = f^{-1}[V]$
$\Rightarrow f^{-1}[V] \in we^*O(X)$.

Definition 4.6. A space X is said to be an we^*-space (resp. locally we^*-indiscrete space) if every we^*-open set is open (resp. closed) in X.

Theorem 4.7. *Let $f : X \to Y$ be a contra we^*-continuous function. Then the following properties hold:*

(1) *If X is an we^*-space, then f is contra continuous and contra w-continuous.*
(2) *If X is locally we^*-indiscrete, then f is continuous.*
(3) *If X is an we^*-space and f is a closed surjection, then Y is locally indiscrete.*

Proof. (1) Clear.
(2) Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \\ f \text{ is contra } we^*\text{-continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{c} f^{-1}[V] \in we^*C(X) \\ X \text{ is locally } we^*\text{-indiscrete} \end{array}\right\} \Rightarrow f^{-1}[V] \in O(X).$$

(3) Let $V \in O(Y)$.

$$\left.\begin{array}{r} V \in O(Y) \\ f \text{ is contra } we^*\text{-continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{c} f^{-1}[V] \in we^*C(X) \\ X \text{ is an } we^*\text{-space} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow f^{-1}[V] \in C(X) \\ f \text{ is a closed surjection} \end{array}\right\} \Rightarrow \ f[f^{-1}[V]] = V \in C(X).$$

Definition 4.8. A function $f : X \to Y$ is said to be slightly we^*-continuous if $f^{-1}[V]$ is we^*-open in X for every clopen sets V of Y.

Remark 4.9. It is clear that every contra we^*-continuous is slightly we^*-continuous but the converse is not true in general as shown by the following example.

Example 4.10. Let $(\mathbb{R}, \tau_{coc})$ and $Y := \{1, 2\}$ with the topology $\tau := \{\emptyset, Y, \{1\}\}$. Let $f : (\mathbb{R}, \tau_{coc}) \to (Y, \tau)$ be the function defined by

$$f(x) = \left\{\begin{array}{ccc} 1 & , & x \in \mathbb{R} \setminus \mathbb{Q} \\ 2 & , & x \in \mathbb{Q} \end{array}\right. .$$

Then f is slightly we^*-continuous but neither contra we^*-continuous nor contra $w\beta$-continuous.

Theorem 4.11. *Let Y be a locally indiscrete space. A function $f : X \to Y$ is contra we^*-continuous if and only if f is slightly we^*-continuous.*

Proof. $(\Rightarrow)$: Let $V \in CO(Y)$.

$$\left.\begin{array}{r} V \in CO(Y) \Rightarrow V \in C(Y) \\ f \text{ is contra } we^*\text{-continuous} \end{array}\right\} \Rightarrow \ f^{-1}[V] \in we^*O(X).$$

$(\Leftarrow)$: Let Y be locally indiscrete and $V \in C(Y)$.

$$\left.\begin{array}{r} V \in C(Y) \\ Y \text{ is locally indiscrete} \end{array}\right\} \Rightarrow \left.\begin{array}{c} V \in CO(Y) \\ f \text{ is slightly } we^*\text{-continuous} \end{array}\right\} \Rightarrow f^{-1}[V] \in we^*O(X).$$

Definition 4.12. A function $f : X \to Y$ is said to be weakly we^*-continuous if for every $x \in X$ and every $V \in O(Y, f(x))$, there exists $U \in we^*O(X, x)$ such that $f[U] \subseteq cl(V)$.

Theorem 4.13. *If a function $f : X \to Y$ is contra we^*-continuous, then f is weakly we^*-continuous.*

Proof. Let $x \in X$ and $V \in C(Y, f(x))$.

$$\left.\begin{array}{r} V \in C(Y, f(x)) \Rightarrow V = cl(V) \\ f \text{ is contra } we^*\text{-continuous} \end{array}\right\} \overset{\text{Theorem 4.4(3)}}{\Rightarrow} (\exists U \in we^*O(X,x))(f[U] \subseteq cl(V)).$$

Definition 4.14. The we^*-frontier of a subset A of a space X, denoted by $we^*\text{-}Fr(A)$, is defined by $we^*\text{-}Fr(A) := we^*\text{-}cl(A) \setminus we^*\text{-}int(A)$.

Theorem 4.15. *Let $f : X \to Y$ be a function. The set of all points $x \in X$ at which $f : X \to Y$ is not contra we^*-continuous is identical with the union of all the we^*-frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. Let $A := \{x \in X | f \text{ is not contra } we^*\text{-continuous at } x\}$ and $x \in A$.

$$\begin{array}{lll} x \in A & \Rightarrow & f \text{ is not contra } we^*\text{-continuous at } x \\ & \Rightarrow & (\exists F \in C(Y, f(x)))(\forall U \in we^*O(X,x))(f[U] \nsubseteq F) \\ & \Rightarrow & (\exists F \in C(Y, f(x)))(\forall U \in we^*O(X,x))(U \cap \left(X \setminus f^{-1}[F]\right) \neq \emptyset) \\ & \Rightarrow & (x \in f^{-1}[F])(x \in we^*\text{-}cl(X \setminus f^{-1}[F]) = X \setminus we^*\text{-}int(f^{-1}[F])) \\ & \Rightarrow & x \in we^*\text{-}Fr(f^{-1}[F]) \end{array}$$

Then we have

$$A \subseteq \bigcup \left\{we^*\text{-}Fr(f^{-1}[F]) | F \in C(Y, f(x))\right\} \ldots (1)$$

Let $x \notin A$.

$$\left.\begin{array}{r} x \notin A \Rightarrow f \text{ is contra } we^*\text{-continuous at } x \\ F \in C(Y, f(x)) \end{array}\right\} \Rightarrow (\exists U \in we^*O(X,x))(f[U] \subseteq F)$$

$$\begin{array}{l} \Rightarrow (\exists U \in we^*O(X,x))(U \subseteq f^{-1}[F]) \\ \Rightarrow x \in we^*\text{-}int(f^{-1}[F]) \\ \Rightarrow x \notin we^*\text{-}Fr(f^{-1}[F]) \\ \Rightarrow x \notin \bigcup \left\{we^*\text{-}Fr(f^{-1}[F]) | F \in C(Y, f(x))\right\} \end{array}$$

Then we have

$$\bigcup \left\{we^*\text{-}Fr(f^{-1}[F]) | F \in C(Y, f(x))\right\} \subseteq A \ldots (2)$$

$$(1), (2) \Rightarrow A = \bigcup \left\{we^*\text{-}Fr(f^{-1}[F]) | F \in C(Y, f(x))\right\}.$$

Lemma 4.16. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f, defined by $g(x) := (x, f(x))$ for every $x \in X$. If g is contra we^*-continuous, then f is contra we^*-continuous.*

Proof. Let $U \in O(Y)$.

$$\left.\begin{array}{r} U \in O(Y) \Rightarrow X \times U \in O(X \times Y) \\ g \text{ is contra } we^*\text{-continuous} \end{array}\right\} \Rightarrow g^{-1}[X \times U] = f^{-1}[U] \in we^*C(X).$$

Theorem 4.17. *Let $f, g : X \to Y$ be two functions. If f is contra we^*-continuous, g is contra wa-continuous and Y is Urysohn, then $E := \{x | f(x) = g(x)\} \in we^*C(X)$.*

Proof. Let $x \notin E$.

$$\left.\begin{array}{r} x \notin E \Rightarrow f(x) \neq g(x) \\ Y \text{ is Urysohn} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists V \in O(Y, f(x)))(\exists W \in O(Y, g(x)))(cl(V) \cap cl(W) = \emptyset) \\ f \text{ is contra } we^*\text{-continuous and } g \text{ is contra } wa\text{-continuous} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (M := f^{-1}[cl(V)] \in we^*O(X,x))(N := g^{-1}[cl(W)] \in waO(X,x)) \\ U := M \cap N \end{array}\right\} \overset{\text{Lemma 3.4(2)}}{\Rightarrow}$$

$\Rightarrow (U \in we^*O(X,x))(f[U] \cap g[U] \subseteq f[M] \cap g[N] \subseteq cl(W) \cap cl(V) = \emptyset)$
$\Rightarrow (U \in we^*O(X,x))(U \cap E = \emptyset)$.

Definition 4.18. The graph $G(f)$ of a function $f : X \to Y$ is said to be contra we^*-closed if for each $(x,y) \notin G(f)$, there exist $U \in we^*O(X,x)$ and $V \in C(Y,y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.19. *The graph $G(f)$ of a function $f : X \to Y$ is contra we^*-closed if and only if for each $(x,y) \notin G(f)$, there exist $U \in we^*O(X,x)$ and $V \in C(Y,y)$ such that $f[U] \cap V = \emptyset$.*

Proof. Straightforward.

Lemma 4.20. *Let $f : X \to Y$ be a function. If f is contra we^*-continuous and Y is Urysohn, then $G(f)$ is we^*-closed in $X \times Y$.*

Proof. Let $(x,y) \notin G(f)$.

$$\left.\begin{array}{r}(x,y) \notin G(f) \Rightarrow y \neq f(x)\\ Y \text{ is Urysohn}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists M \in O(Y,f(x)))(\exists N \in O(Y,y))(cl(M) \cap cl(N) = \emptyset)\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists U \in we^*O(X,x))(cl(N) \in C(Y,y)))(f[U] \cap cl(N) \subseteq cl(M) \cap cl(N) = \emptyset)\\ V := cl(N)\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in we^*O(X,x))(\exists V \in C(Y,y))(f[U] \cap V = \emptyset).$$

Lemma 4.21. [11] *Let $G(f)$ be a graph of a function $f : X \to Y$, for any subsets $A \subseteq X$ and $B \subseteq Y$, we have $f[A] \cap B = \emptyset$ if and only if $(A \times B) \cap G(f) = \emptyset$.*

Theorem 4.22. *Let $f : X \to Y$ be an we^*-continuous function. Then the following properties hold:*

(1) *If Y is T_1, then $G(f)$ is contra we^*-closed in $X \times Y$,*
(2) *If Y is T_2, then $G(f)$ is contra we^*-closed in $X \times Y$.*

Proof. (1) Let $(x,y) \notin G(f)$.

$$\left.\begin{array}{r}(x,y) \notin G(f) \Rightarrow y \neq f(x)\\ Y \text{ is } T_1\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists W \in O(Y,f(x)))(y \notin W)\\ f \text{ is } we^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r}(\exists U \in we^*O(X,x))(f[U] \subseteq W)(y \in Y \setminus W)\\ V := Y \setminus W\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in we^*O(X,x))(V \in C(Y,y))(f[U] \cap V = \emptyset).$$

(2) It follows from (1).

Definition 4.23. A space X is said to be we^*-T_1 space if for every pair of distinct points x and y of X, there exist $U \in we^*O(X,x)$ and $V \in we^*O(X,y)$ such that $y \notin U$ and $x \notin V$.

Theorem 4.24. *Let $f : X \to Y$ have a contra we^*-closed graph. If f is injective, then X is we^*-T_1.*

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x,y \in X)(x \neq y)\\ f \text{ is injective}\end{array}\right\} \Rightarrow \left.\begin{array}{r}f(x) \neq f(y) \Rightarrow (x, f(x)) \notin G(f)\\ G(f) \text{ is contra } we^*\text{-closed}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in we^*O(X,x))(\exists F \in C(Y,f(x)))(f[U]\cap F=\emptyset)\\ V:=f^{-1}[F]\end{array}\right\}\Rightarrow$$

$\Rightarrow (\exists U \in we^*O(X,x))(\exists V \in we^*O(X,y))(y \notin U)(x \notin V)$.

Remark 4.25. The composition of two contra we^*-continuous function need not be contra we^*-continuous as shown by the following example.

Example 4.26. Let $(\mathbb{R},\tau_{coc})$ and $Y := \{a,b\}$ with the topology $\sigma := \{\emptyset, Y, \{a\}\}$ and $\rho := \{\emptyset, Y, \{b\}\}$. Let $f : (\mathbb{R},\tau_{coc}) \to (Y,\sigma)$ be the function defined by

$$f(x) = \left\{\begin{array}{ccc} b & , & x \in \mathbb{R}\setminus\mathbb{Q} \\ a & , & x \in \mathbb{Q}\end{array}\right.$$

and $g : (Y,\sigma) \to (Y,\rho)$ be the identity function. Then f and g are contra we^*-continuous, but $g \circ f$ is not contra we^*-continuous.

Definition 4.27. A function $f : X \to Y$ is said to be we^*-irresolute if the inverse image of every we^*-open set in Y is we^*-open in X.

Theorem 4.28. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the following properties hold:*

(1) *If f is contra we^*-continuous and g is contra continuous, then $g \circ f$ is contra we^*-continuous.*
(2) *If f is contra we^*-continuous and g is continuous, then $g \circ f$ is contra we^*-continuous.*
(3) *If f is we^*-irresolute and g is contra we^*-continuous, then $g \circ f$ is contra we^*-continuous.*

Proof. Straightforward.

Definition 4.29. A function $f : X \to Y$ is said to be we^*-open if $f[V]$ is we^*-open in Y for every we^*-open set V of X.

Theorem 4.30. *Let $f : X \to Y$ be an we^*-irresolute surjection and we^*-open function and let $g : Y \to Z$ be a function. Then $g \circ f$ is contra we^*-continuous if and only if g is contra we^*-continuous.*

Proof. $(\Rightarrow)$: Let $F \in C(Z)$.

$$\left.\begin{array}{r} F \in C(Z) \\ g\circ f \text{ is contra } we^*\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(g\circ f)^{-1}[F] \in we^*O(X) \\ f \text{ is an } we^*\text{-open surjection}\end{array}\right\}\Rightarrow$$

$\Rightarrow g^{-1}[F] = f[f^{-1}[g^{-1}[F]]] \in we^*O(Y)$.

$(\Leftarrow)$: Let $F \in C(Z)$.

$$\left.\begin{array}{r} F \in C(Z) \\ g \text{ is contra } we^*\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}g^{-1}[F] \in we^*O(Y) \\ f \text{ is } we^*\text{-irresolute}\end{array}\right\}\Rightarrow$$

$\Rightarrow (g\circ f)^{-1}[F] = f^{-1}[g^{-1}[F]] \in we^*O(X)$.

Definition 4.31. A function $f : X \to Y$ is said to be perfectly we^*-continuous if the inverse of every open set in Y is we^*-clopen in X.

Theorem 4.32. *Let $f : X \to Y$ be a function. Then the followings are equivalent:*

(1) *f is perfectly we^*-continuous;*

(2) *f is we^*-continuous and contra we^*-continuous.*

Remark 4.33. We have the following diagram from definitions given above.

$$
\begin{array}{ccccc}
\text{contra continuous} & \rightarrow & \text{contra } w\text{-continuous} & \rightarrow & \text{contra } w\beta\text{-continuous} \\
\downarrow & & \downarrow & \swarrow & \\
\text{contra } e^*\text{-continuous} & \rightarrow & \text{contra } we^*\text{-continuous} & \leftarrow & \text{perfect } we^*\text{-continuous} \\
 & & \downarrow & & \downarrow \\
\text{slightly } we^*\text{-continuous} & \leftarrow & \text{weakly } we^*\text{-continuous} & \leftarrow & we^*\text{-continuous}
\end{array}
$$

5. Some Fundamental Properties

Definition 5.1. A space X is said to be we^*-connected if X can not be expressed as the disjoint union of two non-empty we^*-open sets.

Theorem 5.2. *Let $f : X \to Y$ be a contra we^*-continuous surjection. If X is we^*-connected, then Y is connected.*

Proof. Suppose that Y is not connected.

$$
\left.\begin{array}{r} Y \text{ is not connected } \Rightarrow (\exists U, V \in O(Y) \setminus \{\emptyset\})(U \cap V = \emptyset)(U \cup V = Y) \\ f \text{ is contra } we^*\text{-continuous surjection} \end{array}\right\} \Rightarrow
$$

$\Rightarrow (f^{-1}[U], f^{-1}[V] \in we^*C(X) \setminus \{\emptyset\})(f^{-1}[U \cap V] = \emptyset)(f^{-1}[U \cup V] = X)$

$\Rightarrow (f^{-1}[U], f^{-1}[V] \in we^*C(X) \setminus \{\emptyset\})(f^{-1}[U] \cap f^{-1}[V] = \emptyset)(f^{-1}[U] \cup f^{-1}[V] = X)$

$\Rightarrow (f^{-1}[U], f^{-1}[V] \in we^*O(X) \setminus \{\emptyset\})(f^{-1}[U] \cap f^{-1}[V] = \emptyset)(f^{-1}[U] \cup f^{-1}[V] = X)$

This means X is not we^*-connected which is a contradiction.

Theorem 5.3. *If every contra we^*-continuous function from a space X into any T_0-space Y is constant, then X is we^*-connected.*

Proof. Suppose that X is not we^*-connected and every contra we^*-continuous function from X into any T_0-space Y is constant. Since X is not we^*-connected, there exists a proper nonempty we^*-clopen subset A of X. Look at the example 4.26, f is not constant and f is contra we^*-continuous such that Y is T_0. This is a contradiction. Hence X must be we^*-connected.

We recall that a space X is called hyperconnected if the closure of every nonempty open set is the entire set X. It is well known that every hyperconnected space is connected but not conversely.

Remark 5.4. A contra we^*-continuous surjection do not necessarily preserve hyperconnectedness as shown by the following example.

Example 5.5. Let $X := \mathbb{R}$ with the topologies $\tau := \{\emptyset, X\}$ and $\sigma := \{A | 0 \in A\} \cup \{\emptyset\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. The space (X, τ) is a hyperconnected space and the function f is contra we^*-continuous surjection but (X, σ) is not hyperconnected.

Definition 5.6. A topological space X is said to be:

(1) we^*-T_2 if for every two distinct points x and y in X, there exist $U \in we^*O(X, x)$ and $V \in we^*O(X, y)$ such that $U \cap V = \emptyset$.

(2) weakly Hausdorff [21] if each element of X is an intersection of some regular closed sets of X.

(3) ultra normal [17] (resp. we^*-normal) if each pair of non-empty disjoint closed sets can be seperated by disjoint clopen (resp. we^*-open) sets.

Theorem 5.7. *Let $f : X \to Y$ be a contra we^*-continuous injection. Then the following properties holds:*

(1) *If Y is weakly Hausdorff, then X is we^*-T_2,*
(2) *If Y is a Urysohn space, then X is we^*-T_2,*
(3) *If Y is ultra normal and f is closed, then X is we^*-normal.*

Proof. (1) Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\ f \text{ is injection}\end{array}\right\}\Rightarrow \left.\begin{array}{c}f(x)\neq f(y)\\ Y \text{ is weakly Hausdorff}\end{array}\right\}\Rightarrow$$
$$\Rightarrow (\exists \mathcal{A}_1 \subseteq RC(Y,f(x))(\{f(x)\}=\bigcap \mathcal{A}_1)(\exists \mathcal{A}_2 \subseteq RC(Y,f(y))(\{f(y)\}=\bigcap \mathcal{A}_2)$$
$$\left.\begin{array}{r}\Rightarrow (\exists A\in RC(Y,f(x)))(\exists B\in RC(Y,f(y)))(A\cap B=\emptyset)\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\}\Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (f^{-1}[A]\in we^*O(X,x))(f^{-1}[B]\in we^*O(X,y))(f^{-1}[A]\cap f^{-1}[B]=\emptyset)\\ (U:=f^{-1}[A])(V:=f^{-1}[B])\end{array}\right\}\Rightarrow$$
$$\Rightarrow (U\in we^*O(X,x))(V\in we^*O(X,y))(U\cap V=\emptyset).$$

(2) Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\ f \text{ is injection}\end{array}\right\}\Rightarrow \left.\begin{array}{c}f(x)\neq f(y)\\ Y \text{ is Urysohn}\end{array}\right\}\Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists E\in O(Y,f(x)))(\exists F\in O(Y,f(y)))(cl(E)\cap cl(F)=\emptyset)\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\}\Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (f^{-1}[cl(E)]\in we^*O(X,x))(f^{-1}[cl(F)]\in we^*O(X,y))(f^{-1}[cl(E)]\cap f^{-1}[cl(F)]=\emptyset)\\ (U:=f^{-1}[cl(E)])(V:=f^{-1}[cl(F)])\end{array}\right\}\Rightarrow$$
$$\Rightarrow (U\in we^*O(X,x))(V\in we^*O(X,y))(U\cap V=\emptyset).$$

(3) Let $E, F \in C(X)$ and $E \cap F = \emptyset$.

$$\left.\begin{array}{r}(E,F\in C(X))(E\cap F=\emptyset)\\ f \text{ is a closed injection}\end{array}\right\}\Rightarrow \left.\begin{array}{r}(f[E],f[F]\in C(Y))(f[E]\cap f[F]=\emptyset)\\ Y \text{ is ultra normal}\end{array}\right\}\Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists A\in CO(Y,f(E)))(\exists B\in CO(Y,f(F)))(A\cap B=\emptyset)\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\}\Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (f^{-1}[A]\in we^*O(X,E))(f^{-1}[B]\in we^*O(X,F))(f^{-1}[A]\cap f^{-1}[B]=\emptyset)\\ (U:=f^{-1}[A])(V:=f^{-1}[B])\end{array}\right\}\Rightarrow$$
$$\Rightarrow (U\in we^*O(X,E))(V\in we^*O(X,F))(U\cap V=\emptyset).$$

6. Covering Properties

Definition 6.1. A topological space X is said to be:

(1) strongly S-closed [7] if every closed cover of X has a finite subcover,
(2) we^*-compact if every we^*-open cover of X has a finite subcover,
(3) mildly we^*-compact if every we^*-clopen cover of X has a finite subcover.

A subset A of a space X is said to be we^*-compact relative to X if for any cover $\{U_i | i \in I\}$ of A by we^*-open sets of X, there exists a finite subset I_0 of I such that $A \subseteq \bigcup\{U_i | i \in I_0\}$.

Theorem 6.2. *Let $f : X \to Y$ be a contra we^*-continuous surjection and $T \subseteq X$. If T is we^*-compact relative to X, then $f[T]$ is strongly S-closed in Y.*

Proof. Let $\mathcal{A} \subseteq C(X)$ and $f[T] \subseteq \bigcup \mathcal{A}$.

$$\left.\begin{array}{r}(\mathcal{A} \subseteq C(X))(f[T] \subseteq \bigcup \mathcal{A})\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subseteq we^*O(X))(T \subseteq \bigcup \mathcal{B})\\ T \text{ is } we^*\text{-compact relative to } X\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(T \subseteq \bigcup \mathcal{B}^*)\\ (\mathcal{A}^* := \{f[B] | B \in \mathcal{B}^*\})(f \text{ is surjective})\end{array}\right\} \Rightarrow (\mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(f[T] \subseteq \bigcup \mathcal{A}^*).$$

Definition 6.3. A subset A of a space X is said to be we^*-regular open if $A = we^*\text{-}int(we^*\text{-}cl(A))$. The complement of an we^*-regular open set is called an we^*-regular closed set.

Definition 6.4. An open cover $\{U_i | i \in I\}$ of a space X is called an we^*-regular cover if for every $i \in I$, there exist a non-empty we^*-regular closed subset C_i of X such that $C_i \subseteq U_i$ and $X = \bigcup_{i\in I} we^*\text{-}int(C_i)$.

Definition 6.5. A topological space X is said to be weakly we^*-regular Lindelöf if every we^*-regular cover $\{U_i | i \in I\}$ of X has a countable subset $\{i_n | n \in \mathbb{N}\} \subseteq I$ such that $X = we^*\text{-}cl\left(\bigcup_{n\in\mathbb{N}} U_{i_n}\right)$.

Theorem 6.6. *Let (X, τ) and (Y, σ) be two topological spaces and let $f : X \to Y$ be a function. If (X, τ) is weakly we^*-regular Lindelöf space and f is both contra we^*-continuous and continuous, then $(f[X], \sigma_{f[X]})$ is Lindelöf.*

Proof. Let $\mathcal{B} \subseteq \sigma_{f[X]}$ and $f[X] = \bigcup \mathcal{B}$.

$$\left.\begin{array}{r}\left(\mathcal{B} \subseteq \sigma_{f[X]}\right)(f[X] = \bigcup \mathcal{B})\\ f \text{ is contra } we^*\text{-continuous and continuous}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\mathcal{A} := \{f^{-1}[B] | B \in \mathcal{B}\} \subseteq we^*CO(X))\,(X = \bigcup \mathcal{A})\\ (X, \tau) \text{ is weakly } we^*\text{-regular Lindelöf}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| \le \aleph_0)\left(X = we^*\text{-}cl\left(\bigcup_{A\in\mathcal{A}^*} \mathcal{A}\right)\right)\\ \mathcal{B}^* := \{B | f^{-1}[B] \in \mathcal{A}^*\}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\bigcup \mathcal{B}^* \in \tau)(\mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| \le \aleph_0)\left(X = we^*\text{-}cl\left(\bigcup_{B\in\mathcal{B}^*} f^{-1}[B]\right) = we^*\text{-}cl\left(f^{-1}[\bigcup \mathcal{B}^*]\right)\right)\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| \le \aleph_0)(X = we^*\text{-}cl(f^{-1}[\bigcup \mathcal{B}^*]) = f^{-1}[\bigcup \mathcal{B}^*])$$
$$\Rightarrow (\mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| \le \aleph_0)(f[X] = \bigcup \mathcal{B}^*).$$

Theorem 6.7. *Let $f : X \to Y$ be a function. Then the following properties hold:*

(1) *If f is a contra we^*-continuous surjection and X is we^*-compact, then Y is strongly S-closed.*
(2) *If f is a perfectly we^*-continuous surjection and X is midly we^*-compact, then Y is strongly S-closed.*

Proof. (1) Let $\mathcal{A} \subseteq C(Y)$ and $Y = \bigcup \mathcal{A}$.

$$\left.\begin{array}{r}(\mathcal{A} \subseteq C(Y))(Y = \bigcup \mathcal{A})\\ f \text{ is contra } we^*\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subseteq we^*O(X))(X = \bigcup \mathcal{B})\\ X \text{ is } we^*\text{-compact}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \bigcup \mathcal{B}^*)\\ (\mathcal{A}^* := \{f[A] | A \in \mathcal{B}^*\})(f \text{ is surjective})\end{array}\right\} \Rightarrow (\mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(Y = \bigcup \mathcal{A}^*).$$

(2) Let $\mathcal{A} \subseteq C(Y)$ and $Y = \bigcup \mathcal{A}$.

$$\left.\begin{array}{r} (\mathcal{A} \subseteq C(Y))(Y = \bigcup \mathcal{A}) \\ f \text{ is perfectly } we^*\text{-continuous} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{r} (\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subseteq we^*CO(X))(X = \bigcup \mathcal{B}) \\ X \text{ is midly } we^*\text{-compact} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists \mathcal{B}^* \subseteq \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \bigcup \mathcal{B}^*) \\ (\mathcal{A}^* := \{f[A] | A \in \mathcal{B}^*\})(f \text{ is surjective}) \end{array}\right\} \Rightarrow (\mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(Y = \bigcup \mathcal{A}^*).$$

7. Strongly contra we^*-closed graphs

Definition 7.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be strongly contra we^*-closed if for each $(x, y) \notin G(f)$, there exist $U \in we^*O(X, x)$ and $V \in RC(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 7.2. *The following properties are equivalent for a graph $G(f)$ of a function $f : X \to Y$:*

(1) *$G(f)$ is strongly contra we^*-closed;*
(2) *For every point $(x, y) \notin G(f)$, there exist $U \in we^*(X, x)$ and $V \in RC(Y, y)$ such that $f[U] \cap V = \emptyset$.*

Proof. (1) $\Rightarrow$ (2) : Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r} (x, y) \notin G(f) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists U \in we^*O(X, x))(\exists V \in RC(Y, y))((U \times V) \cap G(f) = \emptyset)$$

$$\Rightarrow (\exists U \in we^*O(X, x))(\exists V \in RC(Y, y))(f[U] \cap V = \emptyset).$$

(2) $\Rightarrow$ (1) : Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r} (x, y) \notin G(f) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists U \in we^*O(X, x))(\exists V \in RC(Y, y))(f[U] \cap V = \emptyset)$$

$$\Rightarrow (\exists U \in we^*O(X, x))(\exists V \in RC(Y, y))((U \times V) \cap G(f) = \emptyset).$$

Theorem 7.3. *Let $f : X \to Y$ be a function such that Y is a Urysohn space. Then the following properties hold:*

(1) *If f is weakly we^*-continuous, then $G(f)$ is strongly contra we^*-closed in $X \times Y$,*
(2) *If f is contra we^*-continuous, then $G(f)$ is strongly contra we^*-closed in $X \times Y$.*

Proof. (1) Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r} (x, y) \notin G(f) \Rightarrow y \neq f(x) \\ Y \text{ is Urysohn} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists M \in O(Y, y))(\exists N \in O(Y, f(x)))(cl(M) \cap cl(N) = \emptyset) \\ f \text{ is weakly } we^*\text{-continuous} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists M \in O(Y, y))(\exists U \in we^*O(X, x))(f[U] \subseteq cl(N))(cl(M) \cap cl(N) = \emptyset) \\ V := cl(M) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in we^*O(X, x))(V \in RC(Y, y))(f[U] \cap V = \emptyset).$$

(2) This is obvious from the fact that every contra we^*-continuous function is weakly we^*-continuous.

Theorem 7.4. *If the collection of all we^*-open sets form a topological space on X and $G(f)$ is contra we^*-closed, the inverse image of a strongly S-closed subspace K of Y is we^*-closed in X.*

Proof. Let $x \notin f^{-1}[K]$.

$$\left.\begin{array}{r} x \notin f^{-1}[K] \Rightarrow (\forall k \in K)((x,k) \notin G(f)) \\ G(f) \text{ is contra } we^*\text{-closed} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U_k \in we^*O(X,x))(\exists V_k \in RC(Y,k))(f[U_k] \cap V_k = \emptyset)$$

$$\left.\begin{array}{r} \Rightarrow (\exists U_k \in we^*O(X,x))(\exists V_k \in C(Y,k))(f[U_k] \cap (V_k \cap K) = \emptyset) \\ (V_k \cap K \in C(K,\sigma_K)(\bigcup_{k\in K}(V_k \cap K) = K) \\ K \text{ is strongly } S\text{-closed} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists K_0 \subseteq K)(|K_0| < \aleph_0)(K \subseteq \bigcup_{k\in K_0} V_k) \\ U := \bigcap_{k\in K_0} U_k \Rightarrow f[U] = f\left[\bigcap_{k\in K_0} U_k\right] \subseteq \bigcap_{k\in K_0} f[U_k] \\ we^*O(X) \text{ is a topology on } X \end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in we^*O(X,x))\left(f[U] \cap K \subseteq \left(\bigcap_{k\in K_0} f[U_k]\right) \cap \left(\bigcup_{k\in K_0} V_k\right) = \emptyset\right)$$

$$\Rightarrow (U \in we^*O(X,x))(U \cap f^{-1}[K] = \emptyset)$$

$$\Rightarrow x \notin we^*\text{-}cl(f^{-1}[K]).$$

Lemma 7.5. [7] *Strong S-closedness is open hereditary.*

Theorem 7.6. *Let Y be a strongly S-closed space. If the collection of all we^*-open sets form a topological space on X and a function $f : X \to Y$ has the contra we^*-closed graph, then f is contra we^*-continuous.*

Proof. Let $U \in O(X)$.

$$U \in O(X) \overset{\text{Lemma 7.5}}{\Rightarrow} (U,\tau_U) \text{ is strongly } S\text{-closed} \overset{\text{Theorem 7.4}}{\Rightarrow} f^{-1}[U] \in we^*C(X).$$

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(Murad ÖZKOÇ) Muğla Sitki Koçman University, Faculty of Science, Department of Mathematics, 48000, Menteşe-Muğla/TURKEY
E-mail address, Corresponding author: `murad.ozkoc@mu.edu.tr; murad.ozkoc@gmail.com`

(Pınar ŞAŞMAZ) Muğla Sitki Koçman University, Graduate School of Natural and Applied Sciences, Mathematics, 48000, Menteşe-Muğla/TURKEY
E-mail address: `pinarsasmaz@posta.mu.edu.tr`