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PROPERTIES OF θ-I-COMPACT SETS IN IDEAL TOPOLOGICAL SPACES

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(Dedicated to Professor Maximilian Ganster on the occasion of his retirement)

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Abstract. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be θ-I-compact relative to X if for every cover $\mathcal{U}$ of A by θ-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \setminus \cup\, \mathcal{U}_0 \in I$. We obtain several properties of these sets. Moreover, for a function $f : (X, \tau, I) \to (Y, \sigma)$ we define an ideal $J_I = \{B \subset Y : f^{-1}(B) \in I\}$ and by using ideals $f(I)$ [7] and J_I on Y we obtain several preservation theorems.

1. Introduction

In 1967, Newcomb [7] introduced the notion of compactness modulo an ideal. Rančin [9] and Hamlett and Janković [2] further investigated this notion and obtained some more properties of compactness modulo an ideal. The present authors [4] introduced and studied compactness via ideals called θ-I-compactness. In this paper, we define a subset A of an ideal topological space (X, τ, I) to be θ-I-compact relative to X if for every cover $\mathcal{U}$ of A by θ-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \setminus \cup\, \mathcal{U}_0 \in I$. We obtain several properties of these sets. And also, we define and investigate two kinds of strong forms of "θ-I-compact relative to X". Moreover, for a function $f : (X, \tau, I) \to (Y, \sigma)$ we define an ideal $J_I = \{B \subset Y : f^{-1}(B) \in I\}$ and by using ideals $f(I)$ [7] and J_I on Y we obtain several preservation theorms.

2. Preliminaries

The notion of ideals has been introduced in [6] and [10] and further investigated in [5].

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Definition 2.1. A nonempty collection I of subsets of a set X is called an *ideal* on X [6], [10] if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I).

Lemma 2.2. [7] *For a function $f : (X, \tau) \to (Y, \sigma)$ and ideals I and J, the following properties hold:*
(1) if f is surjective and I is an ideal on X, then $f(I) = \{f(A) : A \in I\}$ is an ideal on Y,
(2) if f is injective and J is an ideal on Y, then $f^{-1}(J) = \{f^{-1}(B) : B \in J\}$ is an ideal on X.

Lemma 2.3. *Let (X, τ, I) be an ideal topological space, $f : (X, \tau, I) \to (Y, \sigma)$ a function and $J_I = \{B \subset Y : f^{-1}(B) \in I\}$. Then the following properties hold:*
(1) J_I is an ideal on Y,
(2) if f is injective, then $I \subset f^{-1}(J_I)$,
(3) if f is surjective, then $J_I \subset f(I)$,
(4) if f is bijective, then $J_I = f(I)$.

Proof. (1) (i) Let $A, B \in J_I$, then $f^{-1}(A), f^{-1}(B) \in I$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \in I$. Therefore, $A \cup B \in I$. (ii) Let $A \subset B$ and $B \in J_I$, then $f^{-1}(A) \subset f^{-1}(B) \in I$. Hence $f^{-1}(A) \in I$ and $A \in J_I$. Therefore, J_I is an ideal on Y.
(2) Since f is injective, for any $A \in I, f^{-1}(f(A)) = A \in I$ and $f(A) \in J_I$. Therefore, $A \in f^{-1}(J_I)$ and $I \subset f^{-1}(J_I)$.
(3) For any $B \in J_I, f^{-1}(B) \in I$. Since f is surjective, $B = f(f^{-1}(B)) \in f(I)$ and hence $J_I \subset f(I)$.
(4)The proof is obvious by (2) and (3). □

Let (X, τ) be a topological space and A be a subset of X. The closure of A is denoted by $\mathrm{Cl}(A)$ and the interior of A is denoted by $\mathrm{Int}(A)$. A point $x \in X$ is called a *θ-cluster point* of a subset A of X if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be *θ-closed* [11]. The complement of a θ-closed set is said to be *θ-open*. The union of all θ-open sets contained in A is called the *θ-interior* of A and is denoted by $\mathrm{Int}_\theta(A)$. It is shown in [11] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of X and $\mathrm{Cl}_\theta(S)$ is closed in X for every subset S of X. For a topological space (X, τ), the family of all θ-open sets of (X, τ) forms a topology for X which is weaker than τ. This topology is usually denoted by τ_θ.

Definition 2.4. An ideal topological space (X, τ, I) is said to be *θ-I-compact* [4] if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of X by θ-open sets of X, there exists a finite subset Δ_0 of Δ such that $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$.

Definition 2.5. Let (X, τ) be a topological space. A subset A of X is said to be *θ-compact relative to* X [3] if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of A by θ-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$.

Definition 2.6. A topological space (X, τ) is said to be *θ-compact* [3] if the set X is θ-compact relative to X.

Definition 2.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
(1) *quasi θ-continuous* [8] if $f^{-1}(V)$ is θ-open in (X, τ) for every θ-open set V in (Y, σ),
(2) *M-θ-open* [1] $f(U)$ is θ-open in (Y, σ) for every θ-open set U of (X, τ).

3. θ-I-compact sets

Definition 3.1. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be *θ-I-compact relative to* X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by θ-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$.

Remark 3.2. Let (X, τ, I) be an ideal topological space. If $I = \{\emptyset\}$, then "θ-I-compact relative to X" coincides with "θ-compact relative to X".

Theorem 3.3. *Let (X, τ, I) be an ideal topological space. For a subset A of X, the following properties are equivalent:*
(1) A is θ-I-compact relative to X;
(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of θ-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) = \emptyset$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) \in I$.

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of θ-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) = \emptyset$. Then, we have $A \subset X \setminus (\cap\{F_\alpha : \alpha \in \Delta\}) = \cup\{X \setminus F_\alpha : \alpha \in \Delta\}$. Since $X \setminus F_\alpha$ is θ-open for each $\alpha \in \Delta$, by (1) there exists a finite subset Δ_0 of Δ such that $A \setminus (\cup\{X \setminus F_\alpha : \alpha \in \Delta_0\}) \in I$. Therefore, we have

$$A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) = A \cap [X \setminus \cup\{(X \setminus F_\alpha : \alpha \in \Delta_0\})] = A \setminus (\cup\{X \setminus F_\alpha : \alpha \in \Delta_0\}) \in I.$$

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by θ-open sets of X. Then $A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) = \emptyset$. Since $X \setminus U_\alpha$ is θ-closed for each $\alpha \in \Delta$, by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) \in I$. Therefore, we have

$$A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\}) = A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I.$$

This shows that A is θ-I-compact relative to X. $\square$

Corollary 3.4. (Theorem 2.1 of [4]) For an ideal topological space (X, τ, I), the following properties are equivalent:
(1) (X, τ, I) is θ-I-compact;
(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of θ-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} = \emptyset$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_0\} \in I$.

Definition 3.5. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *θIg-closed* if $\mathrm{Cl}_\theta(A) \subset U$ whenever $A \setminus U \in I$ and U is θ-open,
(2) *θg-closed* if $\mathrm{Cl}_\theta(A) \subset U$ whenever $A \subset U$ and U is θ-open.

Theorem 3.6. *Let (X, τ, I) be an ideal topological space and A, B subsets of X such that $A \subset B \subset \mathrm{Cl}_\theta(A)$, then the following properties are hold:*

(1) if A is θ-I-compact relative to X and θIg-closed, then B is θ-compact relative to X,

(2) if B is θ-I-compact relative to X and θg-closed, then A is θ-I-compact relative to X.

Proof. (1): Suppose that A is θ-I-compact relative to X and θIg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of B by θ-open sets of X. Then $\{U_\alpha : \alpha \in \Delta\}$ is a cover of A by θ-open sets of X. Since A is θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Since A is θIg-closed, $\mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{Cl}_\theta(A)$, we have $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, B is θ-compact relative to X.

(2): Suppose that B is θ-I-compact relative to X and θg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by θ-open sets of X. Since A is θg-closed, we have $B \subset \mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Since $A \subset B$, $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Therefore, A is θ-I-compact relative to X.

□

Corollary 3.7. Let (X, τ, I) be an ideal topological space. If A is θIg-closed and $A \subset B \subset \mathrm{Cl}_\theta(A)$, then the following properties are equivalent:

(1) A is θ-I-compact relative to X;

(2) B is θ-I-compact relative to X.

Theorem 3.8. *Let (X, τ, I) be an ideal topological space. If subsets A and B of X are θ-I-compact relative to X, then $A \cup B$ is θ-I-compact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be any cover of $A \cup B$ by θ-open sets of X. Then $\mathcal{U}$ is a cover of A and B by θ-open sets of X. Since A and B are θ-I-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ and subsets I_A and I_B of I such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\} \cup I_A$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\} \cup I_B$. Hence we have $A \cup B \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \cup (I_A \cup I_B)$. Since I is an ideal, we have $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \in I$. This shows that $A \cup B$ is θ-I-compact relative to X. □

Theorem 3.9. *Let (X, τ, I) be an ideal topological space, and A, B be subsets of X. If A is θ-I-compact relative to X and B is θ-closed, then $A \cap B$ is θ-I-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by θ-open sets of X. Then $A \setminus B \subset X \setminus B$ and $X \setminus B$ is θ-open. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus B\}$ is a cover of A by θ-open sets of X. Since A is θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset (\cup\{U_\alpha : \alpha \in \Delta_0\}) \cup \{X \setminus B\} \cup I_0$, where $I_0 \in I$. Then we have

$$(A \cap B) \subset (\cup\{U_\alpha \cap B : \alpha \in \Delta_0\}) \cup (I_0 \cap B) \subset \cup\{U_\alpha : \alpha \in \Delta_0\} \cup I_0.$$

Therefore, $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subseteq I_0 \in I$. This shows that $A \cap B$ is θ-I-compact relative to X. □

Corollary 3.10. If an ideal topological space (X, τ, I) is θ-I-compact and B is θ-closed, then B is θ-I-compact relative to X.

Theorem 3.11. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a surjective quasi θ-continuous function and A is θ-I-compact relative to X, then f(A) is θ-f(I)-compact relative to Y.*

Proof. Suppose that A is θ-I-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(A)$ by θ-open sets of Y. Since f is quasi θ-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a cover of A by θ-open sets of X. Since A is θ-I-compact relative to X, there exists a finte subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in I$. Hence we have $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \cup I_0$, where $I_0 \in I$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(I_0)$. Therefore, we have $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in f(I)$. This shows that $f(A)$ is θ-$f(I)$-compact relative to Y. □

Corollary 3.12. (Theorem 2.8 of [4]) If $f : (X, \tau, I) \to (Y, \sigma)$ is a surjective quasi θ-continuous function and (X, τ, I) is θ-I-compact, then $(Y, \sigma, f(I))$ is θ-$f(I)$-compact.

Theorem 3.13. *Let $f : (X, \tau) \to (Y, \sigma, J)$ be an M-θ-open bijective function. If B is θ-J-compact relative to Y, then $f^{-1}(B)$ is θ-$f^{-1}(J)$-compact relative to X.*

Proof. Since $f^{-1} : (Y, \sigma, J) \to (X, \tau)$ is a quasi-continuous bijection, by Theorem 3.11 the proof is obvious. □

Corollary 3.14. (Theorem 2.10 of [4]) Let $f : (X, \tau) \to (Y, \sigma, J)$ be an M-θ-open bijective function and (Y, σ, J) is θ-J-compact, then $(X, \tau, f^{-1}(J))$ is θ-$f^{-1}(J)$-compact.

Theorem 3.15. *If $f : (X, \tau, I) \to (Y, \sigma)$ is an injective quasi θ-continuous function and A is θ-I-compact relative to X, then $f(A)$ is θ-J_I-compact relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(A)$ by θ-open sets of Y. Then $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ and $f^{-1}(V_\alpha)$ is θ-open in X for each $\alpha \in \Delta$. Since A is θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in I$. Hence $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \cup I_0$, where $I_0 \in I$. Therefore, we have $f(A) \subset \cup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_0\} \cup f(I_0) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(I_0)$. Since f is injective, $f^{-1}(f(I_0)) = I_0 \in I$ and $f(I_0) \in J_I$. Consequently, we obtain $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in J_I$. This shows that $f(A)$ is θ-J_I-compact relative to Y. □

4. Strongly θ-I-compact sets

Definition 4.1. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be *strongly θ-I-compact relative to* X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of θ-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$.

Definition 4.2. An ideal topological space (X, τ, I) is said to be *strongly θ-I-compact* if the set X is strongly θ-I-compact relative to X.

Theorem 4.3. *Let (X, τ, I) be an ideal topological space. For a subset A of X, the following properties are equivalent:*

(1) A is strongly θ-I-compact relative to X;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of θ-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in I$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) \in I$.

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of θ-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in I$. Then $A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta\}) = A \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in I$. Since $X \setminus F_\alpha$ is θ-open for each $\alpha \in \Delta$ and A is strongly θ-I-compact relative to X by (1), there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \in I$. This implies that$A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) = A \setminus (X \setminus (\cap\{F_\alpha : \alpha \in \Delta_0\})) = A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \in I$.

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of θ-closed sets of X and also $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) \in I$. Thus by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) \in I$. Therefore, we have $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) \in I$. This shows that A is strongly θ-I-compact relative to X. □

Corollary 4.4. For an ideal topological space (X, τ, I), the following properties are equivalent:

(1) (X, τ, I) is strongly θ-I-compact;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of θ-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in I$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_0\} \in I$.

Theorem 4.5. *Let (X, τ, I) be an ideal topological space. If A is θIg-closed and $A \subset B \subset \mathrm{Cl}_\theta(A)$, then the following properties are equivalent:*

(1) A is strongly θ-I-compact relative to X;

(2) B is strongly θ-I-compact relative to X.

Proof. (1) $\Rightarrow$ (2): Suppose that A is strongly θ-I-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A is strongly θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Since A is θIg-closed, $\mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{Cl}_\theta(A)$, we have $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subset \mathrm{Cl}_\theta(A) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} = \emptyset \in I$. Therefore, B is strongly θ-I-compact relative to X.

(2) $\Rightarrow$ (1): Suppose that B is strongly θ-I-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A is θIg-closed, we have $B \subset \mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is strongly θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Since $A \subset B, A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Therefore, A is strongly θ-I-compact relative to X. □

Theorem 4.6. *Let (X, τ, I) be an ideal topological space. If subsets A and B of X are strongly θ-I-compact relative to X, then $A \cup B$ is strongly θ-I-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$ and $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A and B are strongly θ-I-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ and subsets I_A and I_B of I such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\} \cup I_A$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\} \cup I_B$. Hence we have $(A \cup B) \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \cup (I_A \cup I_B)$. Since I is an ideal, we have $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \in I$. This shows that $A \cup B$ is strongly θ-I-compact relative to X. □

Theorem 4.7. *Let (X, τ, I) be an ideal topological space, and A, B be subsets of X. If A is strongly θ-I-compact relative to X and B is θ-closed, then $A \cap B$ is strongly θ-I-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus B\}$ is a family of θ-open sets of X such that $A \setminus [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \in I$. Since A is strongly θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup[\{U_\alpha : \alpha \in \Delta_0\} \cup \{X \setminus B\}] \cup I_0$, where $I_0 \in I$. Then we have

$$(A \cap B) \subset [\cup\{U_\alpha \cap B : \alpha \in \Delta_0\}] \cup (I_0 \cap B) \subset [\cup\{U_\alpha : \alpha \in \Delta_0\}] \cup I_0.$$

Therefore, $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subset I_0 \in I$. This shows that $A \cap B$ is strongly θ-I-compact relative to X. □

Corollary 4.8. If an ideal topological space (X, τ, I) is strongly θ-I-compact and B is θ-closed, then B is strongly θ-I-compact relative to X.

Theorem 4.9. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a bijective quasi θ-continuous function and A is strongly θ-I-compact relative to X, then f(A) is strongly θ-f(I)-compact relative to Y.*

Proof. Suppose that A is strongly θ-I-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in f(I)$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup f(I_0)$ for some $I_0 \in I$. Since f is bijective, $A = f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup I_0$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \in I$. Since f is quasi θ-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of θ-open sets of X. Since A is strongly θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in I$. Hence we have $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \cup I_A$, where $I_A \in I$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(I_A)$. Therefore, we have $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in f(I)$. This shows that $f(A)$ is strongly θ-$f(I)$-compact relative to Y. □

Corollary 4.10. If $f : (X, \tau, I) \to (Y, \sigma)$ is a bijective quasi θ-continuous function and (X, τ, I) is strongly θ-I-compact, then $(Y, \sigma, f(I))$ is strongly θ-$f(I)$-compact.

Theorem 4.11. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a quasi θ-continuous injective function and A is strongly θ-I-compact relative to X, then f(A) is strongly θ-J_I-compact relative to Y.*

Proof. Suppose that A is strongly θ-I-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in J_I$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup J_0$ for some $J_0 \in J_I$. Then $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup f^{-1}(J_0)$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \in I$. Since f is quasi θ-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of θ-open sets of X. Since A is strongly θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in I$. Hence we have $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \subset I_0$, where $I_0 \in I$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(I_0)$. Since f is injective, we have $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in J_I$ and hence $f(A)$ is strongly θ-J_I-compact relative to Y. □

Corollary 4.12. If $f : (X, \tau, I) \to (Y, \sigma)$ is a bijective quasi θ-continuous function and (X, τ, I) is strongly θ-I-compact, then (Y, σ) is strongly θ-J_I-compact.

5. Super θ-I-compact sets

Definition 5.1. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be *super θ-I-compact relative to* X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of θ-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$.

Definition 5.2. An ideal topological space (X, τ, I) is said to be *super θ-I-compact* if the set X is super θ-I-compact relative to X.

Theorem 5.3. *Let (X, τ, I) be an ideal topological space. For a subset A of X, the following properties are equivalent:*

(1) A is super θ-I-compact relative to X;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of θ-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in I$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of θ-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in I$. Then $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is a family of θ-open sets of X. Then $A \setminus \cup\{(X \setminus F_\alpha) : \alpha \in \Delta\} = A \cap [X \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\})] = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in I$. Since $A \setminus \cup\{(X \setminus F_\alpha) : \alpha \in \Delta\} \in I$, by (1) there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{(X \setminus F_\alpha) : \alpha \in \Delta_0\}$. This implies that $A \cap (\{F_\alpha : \alpha \in \Delta_0\}) = \emptyset$.

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of θ-closed sets of X and $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) \in I$. Thus by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) = \emptyset$; hence $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. This shows that (X, τ, I) is super θ-I-compact. □

Corollary 5.4. For an ideal topological space (X, τ, I), the following properties are equivalent:

(1) (X, τ, I) is super θ-I-compact;

(2) for every family $\{F_\alpha : \alpha \in \Delta\}$ of θ-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in I$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_0\} = \emptyset$.

Theorem 5.5. *Let (X, τ, I) be an ideal topological space and A, B subsets of X such that $A \subset B \subset \mathrm{Cl}_\theta(A)$, then the following properties are hold:*

(1) if A is super θ-I-compact relative to X and θg-closed, then, B is super θ-I-compact relative to X,

(2) if A is strongly θ-I-compact relative to X and θIg-closed, then B is super θ-I-compact relative to X,

(3) if B is θ-compact relative to X and A is θIg-closed, then A is super θ-I-compact relative to X.

Proof. (1): Suppose that A is super θ-I-compact relative to X and θg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A is super θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since A is θg-closed, $\mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{Cl}_\theta(A)$, we have $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, B is super θ-I-compact relative to X.

(2): Suppose that A is strongly θ-I-compact relative to X and θIg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A is strongly θ-I-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. Since A is θIg-closed, $\mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{Cl}_\theta(A)$, we have $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, B is super θ-I-compact relative to X.

(3): Suppose that B is θ-compact relative to X and A is θIg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A is θIg-closed, we have $B \subset \mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is θ-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $A \subset B$, $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, A is super θ-I-compact relative to X. □

Corollary 5.6. Let (X, τ, I) be an ideal topological space. If A is θIg-closed and $A \subset B \subset \mathrm{Cl}_\theta(A)$, then the following properties are equivalent:

(1) A is super θ-I-compact relative to X;

(2) B is super θ-I-compact relative to X.

Theorem 5.7. *Let (X, τ, I) be an ideal topological space. If subsets A and B of X are super θ-I-compact relative to X, then $A \cup B$ is super θ-I-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$ and $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Since A and B are super θ-I-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\}$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\}$. Hence we have $(A \cup B) \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\}$. This shows that $A \cup B$ is super θ-I-compact relative to X. □

Theorem 5.8. *Let (X, τ, I) be an ideal topological space, and A, B be subsets of X. If A is super θ-I-compact relative to X and B is θ-closed, then $A \cap B$ is super θ-I-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of θ-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in I$. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus B\}$ is a family of θ-open sets of X such that $A \subset [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \cup I_0$, where $I_0 \in I$. Since A is super θ-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset [\cup\{U_\alpha : \alpha \in \Delta_0\}] \cup \{X \setminus B\}$. Then we have $(A \cap B) \subset \cup\{U_\alpha \cap B : \alpha \in \Delta_0\} \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. This shows that $A \cap B$ is super θ-I-compact relative to X. □

Corollary 5.9. If an ideal topological space (X, τ, I) is super θ-I-compact and B is θ-closed, then B is super θ-I-compact relative to X.

Theorem 5.10. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a bijective quasi θ-continuous function and A is super θ-I-compact relative to X, then f(A) is super θ-f(I)-compact relative to Y.*

Proof. Suppose that A is super θ-I-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in f(I)$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup f(I_0)$ for some $I_0 \in I$. Since f is bijective, $A = f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup I_0$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \in I$. Since f is quasi θ-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of θ-open sets of X. Since A is super θ-I-compact relative to X, there exists a finte subset Δ_0 of Δ such that $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Hence we have $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$ and hence $f(A)$ is super θ-$f(I)$-compact relative to Y. □

Corollary 5.11. If $f : (X, \tau, I) \to (Y, \sigma)$ is a bijective quasi θ-continuous function and (X, τ, I) is super θ-I-compact, then $(Y, \sigma, f(I))$ is super θ-$f(I)$-compact.

Theorem 5.12. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a quasi θ-continuous function and A is super θ-I-compact relative to X, then f(A) is super θ-J_I-compact relative to Y.*

Proof. Suppose that A is super θ-I-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of θ-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in J_I$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup J_0$ for some $J_0 \in J_I$. Then $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup f^{-1}(J_0)$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \subset f^{-1}(J_0) \in I$. Since f is quasi θ-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of θ-open sets of X. Since A is super θ-I-compact relative to X, there exists a finte subset Δ_0 of Δ such that $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Hence we have $f(A) \subset \cup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_0\} \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$. Therefore, $f(A)$ is super θ-J_I-compact relative to Y. □

Corollary 5.13. If $f : (X, \tau, I) \to (Y, \sigma)$ is a surjective quasi θ-continuous function and (X, τ, I) is super θ-I-compact, then (Y, σ) is super θ-J_I-compact.

Remark 5.14. We have the following relationships:

$$\begin{array}{ccc} \text{super } \theta\text{-}I\text{-compact relative to } X & \Rightarrow & \text{strongly } \theta\text{-}I\text{-compact relative to } X \\ \Downarrow & & \Downarrow \\ \theta\text{-compact relative to } X & \Rightarrow & \theta\text{-}I\text{-compact relative to } X \end{array}$$

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