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DENUMERABLY MANY POSITIVE SOLUTIONS FOR ITERATIVE SYSTEM OF FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS WITH RS-INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we consider an iterative system of fractional order boundary value problems with Riemann-Stieltjes integral boundary conditions. By applying Krasnosellski's cone fixed point theorem on a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions to the boundary value problem.

1. Introduction

Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. There are many techniques to solve a differential equations, see [7, 12, 13, 16–18] and references therein. On the other hand, fractional derivative gives a perfect aid to characterize the memory and hereditary properties of various processes and materials, therefore differential equations of fractional order are being used in modeling of electrical and mechanical properties of various real materials, rock's rheological properties, and in many other areas [3]. In the qualitative theory of (classical and fractional) differential equations, various theorems have been extensively deployed by researchers in establishing the existence and uniqueness of solutions to both the initial

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and boundary value problems. For a detailed study on the existence and uniqueness of solutions of fractional equations, see [1, 2, 4, 5, 8, 10, 11] and references therein.

In [15], Toyoda and Watanabe established the existence and uniquness theorem for fractional order differentil equations with boundary conditions and two fractional order

$$\left.\begin{aligned} \mathtt{D}^{\beta}_{0+}\left(\mathtt{D}^{\alpha}_{0+}x(t)\right) = \mathtt{f}(t, x(t)),\ 0 < t < 1,\\ x(0) = 0,\ x(1) = \mathtt{D}^{\alpha}_{0+}x(0) = 0,\ \mathtt{D}^{\alpha}_{0+}x(1) = 0,\end{aligned}\right\} \tag{1.1}$$

where $1 < \alpha, \beta \leq 2$, $2 < \alpha+\beta < 4$ and $\mathtt{D}^{\star}_{0+}$ is the Riemann-Liouville fractional derivative of order $\star \in \{\alpha, \beta\}$.

Recently, Song and Cui [14] studied the following fractional differential equation with the Riemann-Stieltjes integral boundary conditions

$$\left.\begin{aligned} {}^{\mathtt{C}}\mathtt{D}^{\alpha}_{1-}\mathtt{D}^{\beta}_{0+}x(t) = \mathtt{f}(t, x(t), \mathtt{D}^{\beta+1}_{0+}x(t), \mathtt{D}^{\beta}_{0+}x(t)), 0 < t < 1,\\ x(0) = x'(0) = 0,\ x(1) = \int_0^1 x(t)d\mathtt{A}(t),\qquad\qquad\end{aligned}\right\} \tag{1.2}$$

where ${}^{\mathtt{C}}\mathtt{D}^{\alpha}_{1-}$ is the left Caputo fractional derivative of order $1 < \alpha \leq 2$ and $\mathtt{D}^{\beta}_{0+}$ is the right Riemann-Liouville fractional derivative of order $0 < \beta \leq 1$, $\int_0^1 x(t)d\mathtt{A}(t)$ is the Riemann-Stieltjes integral of x with respect to $\mathtt{A}$, $\mathtt{A} : [0, 1] \to \mathbb{R}$ is a function of bounded variation and $d\mathtt{A}$ is a signed measure and established existence of solutions by the method of coincidence degree theory.

More recently, Prasad, Krushna and Wesen [9] studied the solvability of iterative system of fractional order boundary value problem

$$\left.\begin{aligned} \mathtt{D}^{\mathtt{q}_1}_{0+}x_i(t)\big) + \lambda_i\mathtt{p}_i(t)\mathtt{g}_i(x_{i+1}(t)) = 0,\ 0 < t < 1,\ i = 1, 2, \cdots, n,\\ x_{i+1}(t) = x_1(t),\ 0 < t < 1,\qquad\qquad\\ x_i(0) = \mathtt{D}^{\mathtt{q}_2}_{0+}x_i(0) = 0,\ x_i'(1) - \zeta x_i'(\xi_1) = \vartheta x_i'(\xi_2),\ i = 1, 2, \cdots, n,\end{aligned}\right\} \tag{1.3}$$

where $2 < \mathtt{q}_1 \leq 3$, $0 < \mathtt{q}_1 \leq 1$, $0 < \xi_1 < \xi_2 < 1$, ζ, ϑ are positive constants and $\mathtt{D}^{\star}_{0+}$ is a Riemann-Liouville fractional derivative of order $\star \in \{\mathtt{q}_1, \mathtt{q}_2\}$, by Krasnoselskii's cone fixed point theorem on Banach space.

Inspired by the aforementioned works, in this paper we derive the sufficient conditions for the existence of denumerably many positive solutions for the iterative system of fractional order boundary value problems,

$$\left.\begin{aligned} \mathtt{D}^{\lambda}\left(\mathtt{D}^{\delta}x_i(t)\right) + a(t)\mathtt{g}_i(x_{i+1}(t)) = 0,\ 0 < t < 1,\ i = 1, 2, \cdots, \ell,\\ x_{i+1}(t) = x_1(t),\ 0 < t < 1,\qquad\qquad\end{aligned}\right\} \tag{1.4}$$

$$\left.\begin{aligned} x_i(0) = 0,\ x_i(1) = \int_0^1 x_i(s)d\Lambda(s),\ i = 1, 2, \cdots, \ell,\\ \mathtt{D}^{\delta}x_i(0) = 0,\ \mathtt{D}^{\delta}x_i(1) = \int_0^1 x_i(s)d\Lambda(s),\ i = 1, 2, \cdots, \ell,\end{aligned}\right\} \tag{1.5}$$

where $0 < \lambda, \delta < 1$, $1 < \lambda + \delta < 2$ and $\mathtt{D}^{\star}$ is a Riemann-Liouville fractional derivative of order $\star \in \{\lambda, \delta\}$, $a(t) = \prod_{j=1}^{n} a_j(t)$ and $a_j(t) \in L^{p_j}[0, 1](p_j \geq 1)$ has a singularity in $(0, 1/2)$, $\int_0^1 x_i(s)d\Lambda(s)$ denotes Riemann-Stieltjes integral of $x_i(s)$ with respect to Λ, $\Lambda : [0, 1] \to \mathbb{R}$ is a function of bounded variation and $d\Lambda$ is a signed measure. Moreover, the results of this paper extend and generalize several results existing in the literature [9, 14, 15] in some of the following aspects: iterative system, RS-integral boundary conditions, singularities and denumerably many positive solutions.

Throughout the paper, we assume the following conditions hold:

($\mathcal{H}_1$) $\mathtt{g}_i : [0, +\infty) \to [0, +\infty)$ is continuous,

($\mathcal{H}_2$) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \frac{1}{2}$,

$$\lim_{r\to\infty} t_r = t^* < \frac{1}{2}, \lim_{t\to t_r} a_j(t) = +\infty, \ r =\in \mathbb{N}, \ j = 1, 2, 3, \cdots, n$$

and each $a_j(t)$ does not vanish identically on any subinterval of $[0, 1]$. Moreover, there exists $a_j^* > 0$ such that

$$a_j^* < a_j(t) < \infty \text{ a.e. on } [0, 1].$$

($\mathcal{H}_3$) Λ is nondecreasing and bounded variation function such that $0 < \Delta_\lambda < 1$ and $0 < \Delta_\delta < 1$, where

$$\Delta_\star = \int_0^1 s^{\star-1} d\Lambda(s), \ \star \in \{\lambda, \delta\}.$$

The rest of the paper is organized in the following fashion. In Section 2, we estimate bounds for the kernel which are useful in our main results. In Section 3, we establish a criteria for the existence of denumerably many positive solutions for the boundary value problem (1.4)–(1.5) by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem on a Banach space. In Section 3, as an application, we provide an example to demonstrate our obtained results. Finally, we give concluding remarks of the paper.

2. Kernel and Its Bounds

In this section, we construct kernel to the homogeneous boundary value problem corresponding to (1.4)–(1.5), and establish certain lemmas for the bounds of the kernel.

Lemma 2.1. *Let $y \in \mathcal{C}[0, 1]$. Then the boundary value problem*

$$\mathtt{D}_{0+}^{\delta} x_1(t) + y(t) = 0, \ 0 < t < 1, \tag{2.1}$$

$$x_1(0) = 0, \ x_1(1) = \int_0^1 x_1(\tau) d\Lambda(\tau), \tag{2.2}$$

has a unique solution

$$x_1(t) = \int_0^1 \aleph_\delta(t, \tau) y(\tau) d\tau, \tag{2.3}$$

where

$$\aleph_\delta(t, \tau) = \mathcal{K}_\delta(t, \tau) + \frac{t^{\delta-1}}{1 - \Delta_\delta} \mathcal{G}_\delta(\tau),$$

$$\mathcal{K}_\delta(t, \tau) = \frac{1}{\Gamma(\delta)} \begin{cases} t^{\delta-1}(1-\tau)^{\delta-1} - (t-\tau)^{\delta-1}, & \tau \le t, \\ t^{\delta-1}(1-\tau)^{\delta-1}, & t \le \tau, \end{cases}$$

and

$$\mathcal{G}_\delta(\tau) = \int_0^1 \mathcal{K}_\delta(\tau_1, \tau) d\Lambda(\tau_1).$$

Proof. It can be established by simple algebraic calculations, so we omit details here. □

Lemma 2.2. *The function $\mathcal{K}_\delta(t, \tau)$ has the following properties:*

(i) $\mathcal{K}_\delta(t,\tau)$ *is nonnegative and continuous on* $[0,1]\times[0,1]$,
(ii) $\mathcal{K}_\delta(t,\tau)\le \mathcal{K}_\delta(\tau,\tau)$ *for* $t,\tau\in[0,1]$,
(iii) *there exists* $\sigma\in(0,\frac{1}{2})$ *such that* $\sigma^{\delta-1}\mathcal{K}_\delta(\tau,\tau)\le\mathcal{K}_\delta(t,\tau)$ *for* $t\in[\sigma,1-\sigma],\tau\in[0,1]$.

Proof. It is easy to establish the result (i). It can be seen that $\mathcal{K}_\delta(t,\tau)$ is decreasing with respect to t for $\tau\le t$ and increasing with respect to t for $t\le s$. So, by this monotonicity of $\mathcal{K}_\delta(t,\tau)$, we have

$$\mathcal{K}_\delta(t,\tau)\le\mathcal{K}_\delta(\tau,\tau)=\frac{[\tau(1-\tau)]^{1-\delta}}{\Gamma(\delta)},\ \tau\in(0,1).$$

This proves (ii). Now we prove (iii). For $\tau\le t$, we have

$$\frac{\mathcal{K}_\delta(t,\tau)}{\mathcal{K}_\delta(\tau,\tau)}=\frac{t^{\delta-1}(1-\tau)^{\delta-1}-(t-\tau)^{\delta-1}}{\tau^{\delta-1}(1-\tau)^{\delta-1}}\ge\frac{t^{\delta-1}(1-\tau)^{\delta-1}}{\tau^{\delta-1}(1-\tau)^{\delta-1}}\ge 1,$$

and

$$\frac{\mathcal{K}_\delta(t,\tau)}{\mathcal{K}_\delta(\tau,\tau)}=\frac{t^{\delta-1}(1-\tau)^{\delta-1}}{\tau^{\delta-1}(1-\tau)^{\delta-1}}\ge\frac{t^{\delta-1}}{\tau^{\delta-1}}\ge\sigma^{\delta-1}.$$

This completes the proof. □

Lemma 2.3. *Let* $\mathcal{G}^*_\delta(\tau)=\mathcal{K}_\delta(\tau,\tau)+\frac{1}{1-\Delta_\delta}\mathcal{G}_\delta(\tau)$. *The kernel* $\aleph_\delta(t,\tau)$ *has the following properties:*

(i) $\aleph_\delta(t,\tau)$ *is nonnegative and continuous on* $[0,1]\times[0,1]$,
(ii) $\aleph_\delta(t,\tau)\le\mathcal{G}^*_\delta(\tau)$ *for* $t,\tau\in[0,1]$,
(iii) *there exists* $\sigma\in(0,\frac{1}{2})$ *such that* $\sigma^{\delta-1}\mathcal{G}^*_\delta(\tau)\le\aleph_\delta(t,\tau)$ *for* $t\in[\sigma,1-\sigma],\tau\in[0,1]$.

Proof. (i) and (ii) are obvious. To prove (iii), let $\sigma\in(0,\frac{1}{2})$ and $\tau\in[0,1]$. Then from Lemma 2.2, we have

$$\aleph_\delta(t,\tau)=\mathcal{K}_\delta(t,\tau)+\frac{t^{\delta-1}}{1-\Delta_\delta}\mathcal{G}_\delta(\tau)\ge\sigma^{\delta-1}\mathcal{K}_\delta(\tau,\tau)+\frac{\sigma^{\delta-1}}{1-\Delta_\delta}\mathcal{G}_\delta(\tau)=\sigma^{\delta-1}\mathcal{G}^*_\delta(\tau).$$ □

Lemma 2.4. *Let* $z\in\mathcal{C}[0,1]$. *Then the boundary value problem*

$$\mathtt{D}^\lambda\big(\mathtt{D}^\delta x_1(t)\big)+z(t)=0,\ 0<t<1,\tag{2.4}$$

$$\left.\begin{aligned} x_1(0)=\mathtt{D}^\delta x_1(0)=0,\\ x_1(1)=\mathtt{D}^\delta x_1(1)=\int_0^1 x_1(\tau)d\Lambda(\tau)\end{aligned}\right\}\tag{2.5}$$

has a unique solution

$$x_1(t)=\int_0^1\aleph_\delta(t,\tau)\left[\int_0^1\aleph_\lambda(\tau,s)z(s)ds\right]d\tau,\tag{2.6}$$

where $\aleph_\delta(t,\tau)$ *is defined in Lemma 2.1 and*

$$\aleph_\lambda(t,\tau)=\mathcal{K}_\lambda(t,\tau)+\frac{t^{\lambda-1}}{1-\Delta_\lambda}\mathcal{G}_\lambda(\tau),$$

$$\mathcal{K}_\lambda(t,\tau)=\frac{1}{\Gamma(\lambda)}\begin{cases} t^{\lambda-1}(1-\tau)^{\lambda-1}-(t-\tau)^{\lambda-1}, & \tau\le t,\\ t^{\lambda-1}(1-\tau)^{\lambda-1}, & t\le\tau,\end{cases}$$

and

$$\mathcal{G}_\lambda(\tau) = \int_0^1 \mathcal{K}_\lambda(\tau_1, \tau) d\Lambda(\tau_1).$$

Proof. Let $y_1(t) = \mathtt{D}^\delta x_1(t)$ for $0 < t < 1$. Then the boundary value problem

$$\mathtt{D}^\lambda\big(\mathtt{D}^\delta x_1(t)\big) + z(t) = 0,\ 0 < t < 1,$$

$$\mathtt{D}^\delta x_1(0) = 0,\ \mathtt{D}^\delta x_1(1) = \int_0^1 x_1(\tau) d\Lambda(\tau)$$

is equivalent to the problem

$$\left.\begin{aligned} \mathtt{D}^\lambda y_1(t) + z(t) = 0,\ 0 < t < 1, \\ y_1(0) = 0,\ y_1(1) = \int_0^1 y_1(\tau) d\Lambda(\tau). \end{aligned}\right\} \tag{2.7}$$

By Lemma 2.1, the boundary value problem (2.7) has unique solution

$$y_1(t) = \int_0^1 \aleph_\lambda(t, \tau) z(\tau) d\tau.$$

That is

$$\mathtt{D}^\delta x_1(t) = \int_0^1 \aleph_\lambda(t, \tau) z(\tau) d\tau. \tag{2.8}$$

Again by Lemma 2.1, the differential equation (2.8) with boundary conditions

$$x_1(0) = 0 \text{ and } x_1(1) = \int_0^1 x_1(\tau) d\Lambda(\tau)$$

has a unique solution

$$x_1(t) = \int_0^1 \aleph_\delta(t, \tau) \left[\int_0^1 \aleph_\lambda(\tau, s) z(s) ds\right] d\tau.$$

This completes the proof. □

Lemma 2.5. *The function $\mathcal{K}_\lambda(t, \tau)$ has the following properties:*

(i) $\mathcal{K}_\lambda(t, \tau)$ *is nonnegative and continuous on* $[0, 1] \times [0, 1]$,
(ii) $\mathcal{K}_\lambda(t, \tau) \le \mathcal{K}_\lambda(\tau, \tau)$ *for* $t, \tau \in [0, 1]$,
(iii) *there exists* $\sigma \in (0, \frac{1}{2})$ *such that* $\sigma^{\delta-1}\mathcal{K}_\lambda(\tau, \tau) \le \mathcal{K}_\lambda(t, \tau)$ *for* $t \in [\sigma, 1-\sigma], \tau \in [0, 1]$.

Lemma 2.6. *Let* $\mathcal{G}^*_\lambda(\tau) = \mathcal{K}_\lambda(\tau, \tau) + \frac{1}{1-\Delta_\lambda}\mathcal{G}_\lambda(\tau)$. *The kernel* $\aleph_\lambda(t, \tau)$ *has the following properties:*

(i) $\aleph_\lambda(t, \tau)$ *is nonnegative and continuous on* $[0, 1] \times [0, 1]$,
(ii) $\aleph_\lambda(t, \tau) \le \mathcal{G}^*_\lambda(\tau)$ *for* $t, \tau \in [0, 1]$,
(iii) *there exists* $\sigma \in (0, \frac{1}{2})$ *such that* $\sigma^{\delta-1}\mathcal{G}^*_\lambda(\tau) \le \aleph_\lambda(t, \tau)$ *for* $t \in [\sigma, 1-\sigma], \tau \in [0, 1]$.

We note that an ℓ-tuple $(x_1, x_2, \cdots, x_\ell)$ is a solution of (1.4)–(1.5) if and only if

$$x_1(t) = \int_0^1 \aleph_\delta(t, \tau_1)\Big[\int_0^1 \aleph_\lambda(\tau_1, \tau_2) a(\tau_2) \mathtt{g}_1\Big[\int_0^1 \aleph_\delta(\tau_2, \tau_3)\Big[\int_0^1 \aleph_\lambda(\tau_3, \tau_4) a(\tau_4) \mathtt{g}_2 \cdots$$
$$\mathtt{g}_{\ell-1}\Big[\int_0^1 \aleph_\delta(\tau_{2\ell-2}, \tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1}, \tau_{2\ell}) a(\tau_{2\ell}) \mathtt{g}_\ell(x_1(\tau_{2\ell})) d\tau_{2\ell}\Big] d\tau_{2\ell-1} \cdots d\tau_2\Big] d\tau_1$$

and

$$x_i(t)=\int_0^1 \aleph_\delta(t,\tau)\left[\int_0^1 \aleph_\lambda(\tau,s)a(s)\mathrm{g}_\ell(x_{\ell+1}(\mathrm{s}))d\mathrm{s}\right]d\tau,\ i=2,3,\cdots,\ell,$$
$$x_{\ell+1}(t)=x_1(t),\ 0<t<1.$$

We denote the Banach space $\mathcal{C}([0,1],\mathbb{R})$ by $\mathcal{B}$ with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$. For $\sigma\in(0,1/2)$, the cone $\mathcal{P}_\sigma\subset\mathcal{B}$ is defined by

$$\mathcal{P}_\sigma=\Big\{x\in\mathcal{B}:x(t)\geq 0,\ \min_{t\in[\sigma,\,1-\sigma]}x(t)\geq\sigma^{\delta-1}\|x\|\Big\},$$

For any $x_1\in\mathcal{P}_\sigma$, define an operator $\mathscr{F}:\mathcal{P}_\sigma\to\mathcal{B}$ by

$$\begin{aligned}(\mathscr{F}x_1)(t)=&\int_0^1\aleph_\delta(t,\tau_1)\Big[\int_0^1\aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathrm{g}_1\Big[\int_0^1\aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1\aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathrm{g}_2\cdots\\&\mathrm{g}_{\ell-1}\Big[\int_0^1\aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathrm{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1.\end{aligned}$$

Lemma 2.7. *For each $\sigma\in(0,1/2)$, $\mathscr{F}(\mathcal{P}_\sigma)\subset\mathcal{P}_\sigma$ and $\mathscr{F}:\mathcal{P}_\sigma\to\mathcal{P}_\sigma$ is completely continuous.*

Proof. Let $\sigma\in(0,1/2)$. Since $\mathrm{g}_i(x_{i+1}(\tau))$ is nonnegative for $\tau\in[0,1]$, $x_1\in\mathcal{P}_\sigma$. Since $\aleph_\delta(t,\tau)$, $\aleph_\lambda(t,\tau)$ are nonnegative for all $t,\tau\in[0,1]$, it follows that $\mathscr{F}(x_1(t))\geq 0$ for all $t\in[0,1]$, $x_1\in\mathcal{P}_\sigma$ Now, by Lemma 2.3 and 2.6, we have

$$\begin{aligned}&\min_{t\in[\sigma,1-\sigma]}(\mathscr{F}x_1)(t)\\
&=\min_{t\in[\sigma,1-\sigma]}\bigg\{\int_0^1\aleph_\delta(t,\tau_1)\Big[\int_0^1\aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathrm{g}_1\Big[\int_0^1\aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1\aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathrm{g}_2\cdots\\
&\qquad\mathrm{g}_{\ell-1}\Big[\int_0^1\aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathrm{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1\bigg\}\\
&\geq\sigma^{\delta-1}\int_0^1\mathcal{G}_\delta^*(\tau_1)\Big[\int_0^1\aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathrm{g}_1\Big[\int_0^1\aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1\aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathrm{g}_2\cdots\\
&\qquad\mathrm{g}_{\ell-1}\Big[\int_0^1\aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathrm{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1\\
&\geq\sigma^{\delta-1}\max_{t\in[0,1]}\bigg\{\int_0^1\aleph_\delta(t,\tau_1)\Big[\int_0^1\aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathrm{g}_1\Big[\int_0^1\aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1\aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathrm{g}_2\cdots\\
&\qquad\mathrm{g}_{\ell-1}\Big[\int_0^1\aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathrm{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1\bigg\}\\
&\geq\sigma^{\delta-1}\max_{t\in[0,1]}|\mathscr{F}x_1(t)|.\end{aligned}$$

Thus $\mathscr{F}(\mathcal{P}_\sigma)\subset\mathcal{P}_\sigma$. Therefore, the operator $\mathscr{F}$ is completely continuous by standard methods and by the Arzela-Ascoli theorem. □

3. Denumerably Many Positive Solutions

In this paper, we establish the existence of denumerably many positive solutions for the boundary value problem (1.4)–(1.5) by utilizing following theorems.

Theorem 3.1 (Krasnoselskii's [6]). *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{B}$ and Ψ_1, Ψ_2 are open sets with $0 \in \Psi_1, \overline{\Psi}_1 \subset \Psi_2$. Let $\mathcal{F} : \mathcal{E} \cap (\overline{\Psi}_2 \backslash \Psi_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{F}x\| \leq \|x\|$, $x \in \mathcal{E} \cap \partial\Psi_1$, *and* $\|\mathcal{F}x\| \geq \|x\|$, $x \in \mathcal{E} \cap \partial\Psi_2$, *or*
(b) $\|\mathcal{F}x\| \geq \|x\|$, $x \in \mathcal{E} \cap \partial\Psi_1$, *and* $\|\mathcal{F}x\| \leq \|x\|$, $x \in \mathcal{E} \cap \partial\Psi_2$.

Then $\mathcal{F}$ has a fixed point in $\mathcal{E} \cap (\overline{\Psi}_2 \backslash \Psi_1)$.

Theorem 3.2. *(Hölder's) Let $f \in L^{\mathtt{p}_j}[0,1]$ with $\mathtt{p}_j > 1$, for $j = 1, 2, \cdots, n$ and $\sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} = 1$. Then $\prod_{j=1}^{n} f_j \in L^1[0,1]$ and $\left\|\prod_{j=1}^{n} f_i\right\|_1 \leq \prod_{j=1}^{n} \|f_j\|_{\mathtt{p}_j}$. Further, if $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$. Then $fg \in L^1[0,1]$ and $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.*

Consider the following three possible cases for $a \in L^{p_i}[0,1]$:

$$\sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} < 1, \ \sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} = 1, \ \sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} > 1.$$

Firstly, we seek denumerably many positive solutions for the case $\sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} < 1$.

Theorem 3.3. *Assume that $(\mathcal{H}_1) - (\mathcal{H}_3)$ hold, let $\{\sigma_k\}_{k=1}^{\infty}$ be such that $t_{k+1} < \sigma_k < t_k$, $k = 1, 2, 3, \cdots$. Let $\{\mathtt{R}_k\}_{k=1}^{\infty}$ and $\{\mathtt{r}_k\}_{k=1}^{\infty}$ be such that*

$$\mathtt{R}_{k+1} < \sigma_k^{\delta-1}\mathtt{r}_k < \mathtt{L}\mathtt{r}_k < \mathtt{R}_k, \ k \in \mathbb{N},$$

where

$$\mathtt{L} = \max\left\{\left[\sigma_1^{\delta+\lambda-2}\prod_{j=1}^{n} a_j^* \int_{\sigma_1}^{1-\sigma_1} \mathcal{G}_\delta^*(\tau)d\tau\Big[\int_{\sigma_1}^{1-\sigma_1} \mathcal{G}_\lambda^*(\tau)d\tau\Big]\right]^{-1}, \ 1\right\}.$$

Further, assume that $\mathtt{g}_i$ satisfies

($\mathcal{J}_1$) $\mathtt{g}_i(x(t)) \leq \mathtt{A}_1\mathtt{R}_k$ *for all* $0 \leq x(t) \leq \mathtt{R}_k$, $t \in [0,1]$,
where $\mathtt{A}_1 < \left[\|\mathcal{G}_\lambda^*\|_{\mathtt{q}}\prod_{j=1}^{n}\|a_j\|_{\mathtt{p}_j}\int_0^1 \mathcal{G}_\delta^*(\tau)d\tau\right]^{-1}$.
($\mathcal{J}_2$) $\mathtt{g}_i(x(t)) \geq \mathtt{L}\mathtt{r}_k$ *for all* $\sigma_k^{\delta-1}\mathtt{r}_k \leq x(t) \leq \mathtt{r}_k$, $t \in [\sigma_k, 1-\sigma_k]$.

Then the iterative system (1.4)–(1.5) has denumerably many solutions $\{(x_1^{[k]}, x_2^{[k]}, \cdots, x_\ell^{[k]})\}_{k=1}^{\infty}$ such that $x_i^{[k]}(t) \geq 0$ on $(0,1)$, $i = 1, 2, \cdots, \ell$ and $k \in \mathbb{N}$.

Proof. Consider the sequences $\{\Psi_{1,k}\}_{k=1}^{\infty}$ and $\{\Psi_{2,k}\}_{k=1}^{\infty}$ of open subsets of $\mathcal{B}$ defined by

$$\Psi_{1,k} = \{x \in \mathcal{B} : \|x\| < \mathtt{R}_k\}, \ \Psi_{2,k} = \{x \in \mathcal{B} : \|x\| < \mathtt{r}_k\}.$$

Let $\{\sigma_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $t^* < t_{k+1} < \sigma_k < t_k < \frac{1}{2}$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathcal{P}_{\sigma_k}$ by

$$\mathcal{P}_{\sigma_k} = \left\{ x \in \mathcal{B} : x(t) \geq 0 \text{ and } \min_{t\in[\sigma_k,\, 1-\sigma_k]} x(t) \geq \sigma_k^{\delta-1}\|x\| \right\}.$$

Let $x_1 \in \mathcal{P}_{\sigma_k} \cap \partial\Psi_{1,k}$. Then,

$$x_1(\tau) \leq \mathtt{R}_k = \|x_1\|$$

for all $\tau \in [0,1]$. By $(\mathcal{J}_1)$ and $0 < \tau_{2\ell-2} < 1$, we have

$$\begin{aligned}
\int_0^1 \aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}&\\
\leq \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\int_0^1 \mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}&\\
\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\int_0^1 \mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})d\tau_{2\ell}\Big]d\tau_{2\ell-1}.&
\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\dfrac{1}{\mathtt{q}} + \displaystyle\sum_{j=1}^{n} \dfrac{1}{\mathtt{p}_j} = 1$. By the first part of Theorem 3.2, we have

$$\begin{aligned}
\int_0^1 \aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}&\\
\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\Big[\int_0^1 \mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})d\tau_{2\ell}\Big]&\\
\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\|\mathcal{G}_\lambda^*\|_{\mathtt{q}}\, \|a\|_{\mathtt{p}_j}&\\
\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\|\mathcal{G}_\lambda^*\|_{\mathtt{q}} \prod_{j=1}^{n} \|a_j\|_{\mathtt{p}_j} \leq \mathtt{R}_k.&
\end{aligned}$$

It follows in similar manner (for $0 < \tau_{2\ell-4} < 1$) that

$$\begin{aligned}
&\int_0^1 \aleph_\delta(\tau_{2\ell-4},\tau_{2\ell-3})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-3},\tau_{2\ell-2})a(\tau_{2\ell-2})\\
&\times \mathtt{g}_{\ell-1}\Big[\int_0^1 \aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\Big]d\tau_{2\ell-2}\Big]d\tau_{2\ell-3}\\
&\leq \int_0^1 \aleph_\delta(\tau_{2\ell-4},\tau_{2\ell-3})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-3},\tau_{2\ell-2})a(\tau_{2\ell-2})\mathtt{g}_{\ell-1}(\mathtt{R}_k)d\tau_{2\ell-2}\Big]d\tau_{2\ell-3}\\
&\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-3})\Big[\int_0^1 \mathcal{G}_\lambda^*(\tau_{2\ell-2})a(\tau_{2\ell-2})d\tau_{2\ell-2}\Big]d\tau_{2\ell-3}\\
&\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\|\mathcal{G}_\lambda^*\|_{\mathtt{q}}\, \|a\|_{\mathtt{p}_j}\\
&\leq \mathtt{A}_1\mathtt{R}_k \int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\|\mathcal{G}_\lambda^*\|_{\mathtt{q}} \prod_{j=1}^{n} \|a_j\|_{\mathtt{p}_j} \leq \mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$(\mathscr{F}x_1)(t) = \int_0^1 \aleph_\delta(t,\tau_1)\Big[\int_0^1 \aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathtt{g}_1\Big[\int_0^1 \aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathtt{g}_2\cdots$$
$$\mathtt{g}_{\ell-1}\Big[\int_0^1 \aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1 \le \mathtt{R}_k.$$

Since $\mathtt{R}_k = \|x_1\|$ for $x_1 \in \mathcal{P}_{\sigma_k} \cap \partial\Psi_{1,k}$, we get

$$\|\mathscr{F}x_1\| \le \|x_1\|. \tag{3.1}$$

Let $t \in [\sigma_k, 1-\sigma_k]$. Then,

$$\mathtt{r}_k = \|x_1\| \ge x_1(t) \ge \min_{t\in[\sigma_k,1-\sigma_k]} x_1(t) \ge \sigma_k^{\delta-1}\|x_1\| \ge \sigma_k^{\delta-1}\mathtt{r}_k.$$

By $(\mathscr{J}_2)$ and for $\tau_{2\ell-2} \in [\sigma_k, 1-\sigma_k]$, we have

$$\begin{aligned}
&\int_0^1 \aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\qquad \ge \sigma_k^{\delta-1}\int_0^1 \mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\qquad \ge \sigma_k^{\delta-1}\int_{\sigma_k}^{1-\sigma_k} \mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\sigma_k^{\lambda-1}\int_{\sigma_k}^{1-\sigma_k} \mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\qquad \ge \sigma_k^{\delta-1}\sigma_k^{\lambda-1}\int_{\sigma_k}^{1-\sigma_k} \mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\mathtt{Lr}_k\prod_{j=1}^n a_j^*\int_{\sigma_k}^{1-\sigma_k} \mathcal{G}_\lambda^*(\tau_{2\ell})d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\qquad \ge \mathtt{Lr}_k\sigma_1^{\delta+\lambda-2}\prod_{j=1}^n a_j^*\int_{\sigma_1}^{1-\sigma_1} \mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\Big[\int_{\sigma_1}^{1-\sigma_1} \mathcal{G}_\lambda^*(\tau_{2\ell})d\tau_{2\ell}\Big] \ge \mathtt{r}_k.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$(\mathscr{F}\vartheta_1)(t) = \int_0^1 \aleph_\delta(t,\tau_1)\Big[\int_0^1 \aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathtt{g}_1\Big[\int_0^1 \aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathtt{g}_2\cdots$$
$$\mathtt{g}_{\ell-1}\Big[\int_0^1 \aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1 \le \mathtt{r}_k.$$

Thus, if $x_1 \in \mathcal{P}_{\sigma_k} \cap \partial\Psi_{2,k}$, then

$$\|\mathscr{F}x_1\| \ge \|x_1\|. \tag{3.2}$$

It is evident that $0 \in \Psi_{2,k} \subset \overline{\Psi}_{2,k} \subset \Psi_{1,k}$. From (3.1),(3.2), it follows from Theorem 3.1 that the operator Ψ has a fixed point $x_1^{[k]} \in \mathcal{P}_{\sigma_k} \cap \big(\overline{\Psi}_{1,k}\backslash\Psi_{2,k}\big)$ such that $x_1^{[k]}(t) \ge 0$ on $(0,1)$, and $k \in \mathbb{N}$. Next setting $x_{\ell+1} = x_1$, we obtain denumerably many positive solutions $\{(x_1^{[k]}, x_2^{[k]}, \cdots, x_\ell^{[k]})\}_{k=1}^\infty$ of (1.4)–(1.5) given iteratively by

$$x_i(t) = \int_0^1 \aleph_\delta(t,\tau)\left[\int_0^1 \aleph_\lambda(\tau,s)\mathtt{g}_i(x_{i+1}(s))ds\right]d\tau,\ i = \ell, \ell-1, \cdots, 2, 1.$$

The proof is completed. □

For $\sum_{j=1}^n \frac{1}{\mathtt{p}_j} = 1$, we have the following theorem.

Theorem 3.4. *Assume that* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *hold, let* $\{\sigma_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1}<\sigma_k<t_k$, $k=1,2,3,\cdots$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{r}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1}<\sigma_k^{\delta-1}\mathtt{r}_k<\mathtt{L}\mathtt{r}_k<\mathtt{R}_k,\ k\in\mathbb{N}.$$

Further, assume that $\mathtt{g}_i$ *satisfies* $(\mathcal{J}_2)$ *and*

$(\mathcal{J}_3)$ $\mathtt{g}_i(x(t))\leq \mathtt{A}_2\mathtt{R}_k$ *for all* $0\leq x(t)\leq \mathtt{R}_k$, $t\in[0,1]$,

$$\text{where } \mathtt{A}_2<\min\left\{\left[\|\mathcal{G}_\lambda^*\|_\infty\prod_{j=1}^{n}\|a_j\|_{\mathtt{p}_j}\int_0^1\mathcal{G}_\delta^*(\tau)d\tau\right]^{-1},\mathtt{L}\right\}$$

Then the iterative system (1.4)–(1.5) *has denumerably many solutions* $\{(x_1^{[k]},x_2^{[k]},\cdots,x_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $x_i^{[k]}(t)\geq 0$ *on* $(0,1)$, $i=1,2,\cdots,\ell$ *and* $k\in\mathbb{N}$.

Proof. For a fixed k, let $\Psi_{1,k}$ be as in the proof of Theorem 3.3 and let $x_1\in P_{k_j}\cap\partial\Psi_{2,k}$. Again

$$x_1(\tau)\leq \mathtt{R}_k=\|x_1\|,$$

for all $\tau\in[0,1]$. By $(\mathcal{J}_3)$ and Theorem 3.3,

$$\begin{aligned}
&\int_0^1\aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\quad\leq\int_0^1\mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\int_0^1\mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\quad\leq \mathtt{A}_2\mathtt{R}_k\int_0^1\mathcal{G}_\delta^*(\tau_{2\ell-1})\Big[\int_0^1\mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})d\tau_{2\ell}\Big]d\tau_{2\ell-1}\\
&\quad\leq \mathtt{A}_2\mathtt{R}_k\int_0^1\mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\Big[\int_0^1\mathcal{G}_\lambda^*(\tau_{2\ell})a(\tau_{2\ell})d\tau_{2\ell}\Big]\\
&\quad\leq \mathtt{A}_2\mathtt{R}_k\int_0^1\mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\|\mathcal{G}_\lambda^*\|_\infty\,\|a\|_{\mathtt{p}_j}\\
&\quad\leq \mathtt{A}_2\mathtt{R}_k\int_0^1\mathcal{G}_\delta^*(\tau_{2\ell-1})d\tau_{2\ell-1}\|\mathcal{G}_\lambda^*\|_\infty\prod_{j=1}^{n}\|a_j\|_{\mathtt{p}_j}\leq \mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\mathscr{F}x_1)(t)=&\int_0^1\aleph_\delta(t,\tau_1)\Big[\int_0^1\aleph_\lambda(\tau_1,\tau_2)a(\tau_2)\mathtt{g}_1\Big[\int_0^1\aleph_\delta(\tau_2,\tau_3)\Big[\int_0^1\aleph_\lambda(\tau_3,\tau_4)a(\tau_4)\mathtt{g}_2\cdots\\
&\mathtt{g}_{\ell-1}\Big[\int_0^1\aleph_\delta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_\lambda(\tau_{2\ell-1},\tau_{2\ell})a(\tau_{2\ell})\mathtt{g}_\ell(x_1(\tau_{2\ell}))d\tau_{2\ell}\Big]d\tau_{2\ell-1}\cdots d\tau_2\Big]d\tau_1\leq \mathtt{R}_k.
\end{aligned}$$

Thus,

$$\|\mathscr{F}x_1\|\leq\|x_1\|,$$

for $x_1\in\mathcal{P}_{\sigma_k}\cap\partial\Psi_{1,k}$. Now define $\Psi_{2,k}=\{x_1\in\mathcal{B}:\|x_1\|<\mathtt{r}_k\}$. Let $x_1\in\mathcal{P}_{\sigma_k}\cap\partial\Psi_{2,k}$ and let $\tau\in[\sigma_k,\,1-\sigma_k]$. Then, the argument leading to (3.2) can be done to the present case. Hence, the theorem. □

For $\sum_{j=1}^{n}\frac{1}{\mathtt{p}_j}>1$, we have the following theorem.

Theorem 3.5. *Assume that* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *hold, let* $\{\sigma_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1}<\sigma_k<t_k$, $k=1,2,3,\cdots$ *. Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{r}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1}<\sigma_k^{\delta-1}\mathtt{r}_k<\mathtt{L}\mathtt{r}_k<\mathtt{R}_k,\ k\in\mathbb{N}.$$

Further, assume that $\mathtt{g}_i$ *satisfies* $(\mathcal{J}_2)$ *and*

$(\mathcal{J}_4)$ $\mathtt{g}_i(x(t))\le \mathtt{A}_3\mathtt{R}_k$ *for all* $0\le x(t)\le \mathtt{R}_k$, $t\in[0,1]$,

$$\text{where } \mathtt{A}_3<\min\left\{\left[\|\mathcal{G}_\lambda^*\|_\infty\prod_{j=1}^{n}\|a_j\|_1\int_0^1\mathcal{G}_\delta^*(\tau)d\tau\right]^{-1},\mathtt{L}\right\}.$$

Then the iterative system (1.4)–(1.5) *has denumerably many solutions* $\{(x_1^{[k]},x_2^{[k]},\cdots,x_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $x_i^{[k]}(t)\ge 0$ *on* $(0,1)$, $i=1,2,\cdots,\ell$ *and* $k\in\mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.3. So, we omit details here. □

4. Example

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following fractional order boundary value problem,

$$\left.\begin{array}{c}\mathtt{D}^{\frac{1}{2}}\left(\mathtt{D}^{\frac{1}{2}}x_i(t)\right)+a(t)\mathtt{g}_i(x_{i+1}(t))=0,\ 0<t<1,\ i=1,2,\\ x_3(t)=x_1(t),\ 0<t<1,\end{array}\right\}\tag{4.3}$$

$$\left.\begin{array}{c}x_i(0)=0,\ x_i(1)=\displaystyle\int_0^1x_i(s)d\Lambda(s),\ i=1,2,\\ \mathtt{D}^{\frac{1}{2}}x_i(0)=0,\ \mathtt{D}^{\frac{1}{2}}x_i(1)=\displaystyle\int_0^1x_i(s)d\Lambda(s),\ i=1,2,\end{array}\right\}\tag{4.4}$$

where $a(t)=a_1(t)a_2(t)$ in which

$$a_1(t)=\frac{1}{|t-\frac{1}{4}|}\quad\text{and}\quad a_2(t)=\frac{1}{|t-\frac{1}{3}|},$$

$$\mathtt{g}_1(x)=\mathtt{g}_2(x)=\begin{cases}180\times10^{-4}, & x\in(10^{-4},+\infty),\\ \frac{53\times10^{-(k+2)}-180\times10^{-4k}}{10^{-(4k+2)}-10^{-4k}}(x-10^{-4k})+180\times10^{-4k}, & \\ \qquad x\in\left[10^{-(4k+2)},10^{-4k}\right], & \\ 53\times10^{-(4k+2)}, & x\in\left(\frac{1}{5^{1/2}}\times10^{-(4k+2)},10^{-(4k+2)}\right),\\ \frac{53\times10^{-(4k+2)}-180\times10^{-(4k+4)}}{\frac{1}{5^{1/2}}\times10^{-(4k+2)}-10^{-(4k+4)}}(x-10^{-(4k+4)})+180\times10^{-(4k+4)}, & \\ \qquad x\in\left(10^{-(4k+4)},\frac{1}{5^{1/2}}\times10^{-(4k+2)}\right], & \\ 0, & x=0,\end{cases}$$

$$\Lambda(t)=\begin{cases} t, & t\in[0,1/2)\cup[2/3,5/6),\\ \frac{1}{2}, & t\in[1/2,2/3),\\ \frac{5}{6}, & t\in[5/6,1].\end{cases}$$

Let

$$t_k=\frac{31}{64}-\sum_{r=1}^{k}\frac{1}{4(r+1)^4},\ \sigma_k=\frac{1}{2}(t_k+t_{k+1}),\ k=1,2,3,\cdots,$$

then

$$\sigma_1=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$$

and

$$t_{k+1}<\sigma_k<t_k,\ \sigma_k>\frac{1}{5}.$$

Therefore,

$$\sigma_k^{\delta-1}>\frac{1}{5^{1/2}},\ k=1,2,3,\cdots.$$

It is easy to see that

$$t_1=\frac{15}{32}<\frac{1}{2},\ t_j-t_{j+1}=\frac{1}{4(j+2)^4},\ j=1,2,3,\cdots.$$

Since $\sum_{j=1}^{\infty}\frac{1}{j^4}=\frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}$, it follows that

$$t^*=\lim_{r\to\infty}t_r=\frac{31}{64}-\sum_{i=1}^{\infty}\frac{1}{4(i+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5},$$

$$a_1,a_2\in L^p[0,1]\ \text{ for all }\ 0<p<2,\ \text{ so }\ a_1^*=a_2^*=\frac{1}{3}.$$

$$\sigma_1^{\delta+\lambda-2}\prod_{j=1}^{n}a_j^*\int_{\sigma_1}^{1-\sigma_1}\mathcal{G}_\delta^*(\tau)d\tau\Big[\int_{\sigma_1}^{1-\sigma_1}\mathcal{G}_\lambda^*(\tau)d\tau\Big]\approx 0.018975575,$$

$$\begin{aligned}\mathtt{L}&=\max\left\{\left[\sigma_1^{\delta+\lambda-2}\prod_{j=1}^{n}a_j^*\int_{\sigma_1}^{1-\sigma_1}\mathcal{G}_\delta^*(\tau)d\tau\Big[\int_{\sigma_1}^{1-\sigma_1}\mathcal{G}_\lambda^*(\tau)d\tau\Big]\right]^{-1},\ 1\right\}\\ &=\{52.69932393,1\}=52.69932393.\end{aligned}$$

and

$$\|\mathcal{G}_\lambda^*\|_2\approx 12.08024203,\ \int_0^1\mathcal{G}_\delta^*(\tau)d\tau\approx 3.216342800.$$

Next, let $0<\varepsilon<1$ be fixed. Then $a_1,a_2\in L^{1+\varepsilon}[0,1]$. It follows that

$$\|a_1\|_{1+\varepsilon}=\left[-\frac{4^\varepsilon(1+3^{-\varepsilon})}{\varepsilon}\right]^{\frac{1}{1+\varepsilon}}$$

$$\|a_2\|_{1+\varepsilon}=\left[-\frac{3^\varepsilon(1+2^{-\varepsilon})}{\varepsilon}\right]^{\frac{1}{1+\varepsilon}}.$$

So, for $0 < \varepsilon < 1$, we have

$$190.3459262 \leq \left[\|\mathcal{G}^*_{\lambda}\|_{\mathtt{q}} \prod_{j=1}^{n} \|a_j\|_{\mathtt{p}_j} \int_0^1 \mathcal{G}^*_{\delta}(\tau) d\tau \right]^{-1}.$$

Taking $\mathtt{A}_1 = 190$. In addition if we take

$$\mathtt{R}_k = 10^{-4k},\ \mathtt{r}_k = 10^{-(4k+2)},$$

then

$$\mathtt{R}_{k+1} = 10^{-(4k+4)} < \frac{1}{5^{1/2}} \times 10^{-(4k+2)} < \sigma_k^{\delta-1} \mathtt{r}_k$$
$$< \mathtt{r}_k = 10^{-(4k+2)} < \mathtt{R}_k = 10^{-4k},$$

$\mathtt{Lr}_k = 52.69932393 \times 10^{-(4k+2)} < 190 \times 10^{-4k} = \mathtt{A}_1\mathtt{R}_k$, $k = 1, 2, 3, \cdots$, and $\mathtt{g}_1$ and $\mathtt{g}_2$ satisfies the following growth conditions:

$$\mathtt{g}_1(x) = \mathtt{g}_2(x) \leq \mathtt{A}_1\mathtt{R}_k = 190 \times 10^{-4k},\ x \in \left[0, 10^{-4k}\right]$$

$$\mathtt{g}_1(x) = \mathtt{g}_2(x) \geq \mathtt{Lr}_k = 52.69932393 \times 10^{-(4k+2)},\ x \in \left[\frac{1}{5^{1/2}} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

All the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (4.3)–(4.4) has denumerably many positive solutions $\{x^{[k]}\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|x^{[k]}\| \leq 10^{-4k}$ for each $k = 1, 2, 3, \cdots$.

5. Conclusion

The research of fractional order differential equations with Riemann-Stieltjes integral boundary conditions has become a new area of investigation. Moreover, in this paper we considered iterative system with n-singularities. By the use of Krasnoselskii's cone fixed point theorem in a Banach space, sufficient conditions are derived for the existence of denumerably many positive solutions for the problem. An example is presented to illustrate the main results. The conclusion obtained in this paper will be useful in the application point of view. Also, we expect to find some applications in more nonlinear problems.

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