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PERTURBATION OF P-APPROXIMATE SCHAUDER FRAMES FOR SEPARABLE BANACH SPACES

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Abstract. Paley-Wiener theorems for frames for Hilbert spaces, Banach frames, Schauder frames and atomic decompositions for Banach spaces are known. In this paper, we derive Paley-Wiener theorem for p-approximate Schauder frames for separable Banach spaces. We show that our result gives Paley-Wiener theorem for frames for Hilbert spaces.

1. Introduction

About a century old theorem of Paley and Wiener states that sequences which are close to orthonormal bases for Hilbert spaces are Riesz bases (see Chapter 1, Theorem 13 in [21] and [1]). Since frames are generalizations of Riesz bases, we naturally ask whether a sequence which is close to a frame is a frame? Recall that a sequence $\{\tau_n\}_n$ in a separable Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is said to be a frame for $\mathcal{H}$ if there exist $a, b > 0$ such that

$$a\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \tau_n\rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

Constants a and b are called as lower and upper frame bounds, respectively [12]. First Paley-Wiener theorem (also known as perturbation theorem) of a frame for a Hilbert space is due to Christensen, in 1995, which states as follows.

Theorem 1.1. *[8] Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$ with bounds a and b. If $\{\omega_n\}_n$ in $\mathcal{H}$ satisfies*

$$c := \sum_{n=1}^{\infty} \|\tau_n - \omega_n\|^2 < a,$$

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then it is a frame for $\mathcal{H}$ with bounds $a\left(1-\sqrt{\frac{c}{a}}\right)^2$ *and* $b\left(1+\sqrt{\frac{c}{b}}\right)^2$.

In a short time after the derivation of Theorem 1.1, Christensen himself generalized Theorem 1.1 and obtained the following result.

Theorem 1.2. *[9] Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$ with bounds a and b. If $\{\omega_n\}_n$ in $\mathcal{H}$ is such that there exist $\alpha, \gamma \geq 0$ with $\alpha + \frac{\gamma}{\sqrt{a}} < 1$ and*

$$\left\|\sum_{n=1}^{m} c_n(\tau_n-\omega_n)\right\| \leq \alpha \left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m}|c_n|^2\right)^{\frac{1}{2}}, \quad \forall c_1,\dots,c_m \in \mathbb{K}, m=1,\dots,$$

then it is a frame for $\mathcal{H}$ with bounds $a\left(1-(\alpha+\frac{\gamma}{\sqrt{a}})\right)^2$ *and* $b\left(1+(\alpha+\frac{\gamma}{\sqrt{b}})\right)^2$.

Casazza and Christensen extended the Theorem 1.2 further in 1997, and obtained the next theorem.

Theorem 1.3. *[6] Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$ with bounds a and b. If $\{\omega_n\}_n$ in $\mathcal{H}$ is such that there exist $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha + \frac{\gamma}{\sqrt{a}}, \beta\} < 1$ and*

$$\left\|\sum_{n=1}^{m} c_n(\tau_n-\omega_n)\right\| \leq \alpha \left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m}|c_n|^2\right)^{\frac{1}{2}} + \beta\left\|\sum_{n=1}^{m} c_n\omega_n\right\|,$$
$$\forall c_1,\dots,c_m \in \mathbb{K}, m=1,\dots,$$

then it is a frame for $\mathcal{H}$ with bounds $a\left(1-\frac{\alpha+\beta+\frac{\gamma}{\sqrt{a}}}{1+\beta}\right)^2$ *and* $b\left(1+\frac{\alpha+\beta+\frac{\gamma}{\sqrt{b}}}{1-\beta}\right)^2$.

After the developments of theories of Banach frames, Schauder frames and atomic decompositions for separable Banach spaces (see [2–4, 14]) Paley-Wiener theorems are derived for Banach frames, Schauder frames and atomic decompositions (see [7,11,16,17,19,22]). In this paper, we derive Paley-Wiener theorem for p-ASFs (Theorem 2.6). We show that our result gives Theorem 1.3 for Hilbert spaces (Remark 2.7).

2. Paley-Wiener theorem for p-approximate Schauder frames

Let $\mathcal{X}$ be a separable Banach space and $\mathcal{X}^*$ be its dual. In the sequel, $\{e_n\}_n$ denotes the standard Schauder basis for $\ell^p(\mathbb{N})$, $p \in [1,\infty)$. We now recall the definition of approximate Schauder frames for separable Banach spaces.

Definition 2.1. [13,20] Let $\{\tau_n\}_n$ be a sequence in $\mathcal{X}$ and $\{f_n\}_n$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be an approximate Schauder frame (ASF) for $\mathcal{X}$ if

$$S_{f,\tau} : \mathcal{X} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^{\infty} f_n(x)\tau_n \in \mathcal{X} \tag{2.1}$$

is a well-defined bounded linear, invertible operator.

Following [18], real $a, b > 0$ satisfying

$$a\|x\| \leq \left\|\sum_{n=1}^{\infty} f_n(x)\tau_n\right\| \leq b\|x\|, \quad \forall x \in \mathcal{X}$$

are called as lower ASF bound and upper ASF bound, respectively. There is a particular case of ASFs studied by the authors of this paper which contains many important properties of frames for Hilbert spaces (see [18]).

Definition 2.2. [18] An ASF $(\{f_n\}_n, \{\tau_n\}_n)$ for $\mathcal{X}$ is said to be a p-ASF, $p \in [1, \infty)$ if both the maps

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N}) \text{ and}$$
$$\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau \{a_n\}_n := \sum_{n=1}^{\infty} a_n \tau_n \in \mathcal{X}$$

are well-defined bounded linear operators.

It is observed from the Definition 2.2 that the invertibility condition is not imposed neither for the operator θ_f nor for θ_τ. It may happen that both θ_f and θ_τ are not invertible but $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\mathcal{X}$, which is illustrated in the following example.

Example 2.3. Let $p \in [1, \infty)$, $R : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ be the right shift operator and $L : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ be the left shift operator defined by

$$R(a_n)_{n=1}^{\infty} := (0, a_1, a_2, \dots), \quad L(a_n)_{n=1}^{\infty} := (a_2, a_3, \dots).$$

Then $LRx = x$ for all $x \in \ell^p(\mathbb{N})$. Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define

$$f_n := \zeta_n R, \quad \tau_n := Le_n, \quad \forall n \in \mathbb{N}.$$

Then

$$\sum_{n=1}^{\infty} f_n(x)\tau_n = \sum_{n=1}^{\infty} (\zeta_n R)(x)(Le_n) = L\left(\sum_{n=1}^{\infty} \zeta_n(Rx)e_n\right) = LRx = x, \quad \forall x \in \ell^p(\mathbb{N}),$$

$$\begin{aligned}
\theta_f(\ell^p(\mathbb{N})) &= \{\theta_f(a_n)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})\} \\
&= \{(f_m(a_n)_{n=1}^{\infty})_{m=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})\} \\
&= \{(\zeta_m(R(a_n)_{n=1}^{\infty}))_{m=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})\} \\
&= \{(\zeta_m(0, a_1, a_2, \dots))_{m=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})\} \\
&= \{(0, a_1, a_2, \dots) : (a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ker}(\theta_\tau) &= \{(a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N}) : \theta_\tau(a_n)_{n=1}^{\infty} = 0\} \\
&= \left\{(a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N}) : \sum_{n=1}^{\infty} a_n \tau_n = 0\right\} \\
&= \left\{(a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N}) : \sum_{n=1}^{\infty} a_n Le_n = 0\right\} \\
&= \{(a_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N}) : L(a_n)_{n=1}^{\infty} = 0\} \\
&= \{(a, 0, 0, \dots)_{n=1}^{\infty} : a \in \mathbb{K}\}.
\end{aligned}$$

Therefore θ_f is not surjective and θ_τ is not injective. Hence neither θ_f nor θ_τ is invertible operator but $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\ell^p(\mathbb{N})$.

It is clear that a p-approximate Schauder frame is an approximate Schauder frame. The following example shows that the set of all p-approximate Schauder frames is strictly contained in the set of all approximate Schauder frames.

Example 2.4. Let $\mathcal{X} = \mathbb{K}$. Define $\tau_n \coloneqq \frac{1}{n^2}$, $f_n(x) = x, \forall x \in \mathbb{K}$, $\forall n \in \mathbb{N}$. Then $\sum_{n=1}^\infty f_n(x)\tau_n = \frac{\pi^2}{6}x$, $\forall x \in \mathbb{K}$. Therefore $(\{f_n\}_n, \{\tau_n\}_n)$ is an approximate Schauder frame for $\mathcal{X}$. Let $x \in \mathbb{K}$ be non-zero. Then for every $p \in [1, \infty)$,

$$\sum_{n=1}^m |f_n(x)|^p = m|x|^p \to \infty \quad \text{as} \quad m \to \infty.$$

Thus $\{f_n(x)\}_n \notin \ell^p(\mathbb{N})$ for any $p \in [1, \infty)$ and hence $(\{f_n\}_n, \{\tau_n\}_n)$ is not a p-ASF for any $p \in [1, \infty)$.

In order to derive Paley-Wiener theorem for p-ASFs, we need a generalization of result of Hilding [15].

Theorem 2.5. *[5, 6] Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $U : \mathcal{X} \to \mathcal{Y}$ be a bounded invertible operator. If a bounded linear operator $V : \mathcal{X} \to \mathcal{Y}$ is such that there exist $\alpha, \beta \in [0, 1)$ with*

$$\|Ux - Vx\| \leq \alpha\|Ux\| + \beta\|Vx\|, \quad \forall x \in \mathcal{X},$$

then V is invertible and

$$\frac{1-\alpha}{1+\beta}\|Ux\| \leq \|Vx\| \leq \frac{1+\alpha}{1-\beta}\|Ux\|, \quad \forall x \in \mathcal{X}$$
$$\frac{1-\beta}{1+\alpha}\frac{1}{\|U\|}\|y\| \leq \|V^{-1}y\| \leq \frac{1+\beta}{1-\alpha}\|U^{-1}\|\|y\|, \quad \forall y \in \mathcal{Y}.$$

Theorem 2.6. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. Assume that a collection $\{\tau_n\}_n$ in $\mathcal{X}$ and a collection $\{g_n\}_n$ in $\mathcal{X}^*$ are such that there exist $r, s, t, \alpha, \beta, \gamma \geq 0$ with $\max\{\alpha + \gamma\|\theta_f S_{f,\tau}^{-1}\|, \beta, s\} < 1$ and*

$$\left\|\sum_{n=1}^m (f_n - g_n)(x)e_n\right\| \leq r\left\|\sum_{n=1}^m f_n(x)e_n\right\| + t\|x\| + s\left\|\sum_{n=1}^m g_n(x)e_n\right\|, \quad \forall x \in \mathcal{X}, m = 1, \dots, \tag{2.2}$$

$$\left\|\sum_{n=1}^m c_n(\tau_n - \omega_n)\right\| \leq \alpha\left\|\sum_{n=1}^m c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^m |c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^m c_n\omega_n\right\|, \quad \forall c_1, \dots, c_m \in \mathbb{K}, m = 1, \dots. \tag{2.3}$$

Then $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$ with bounds

$$\frac{1 - (\alpha + \gamma\|\theta_f S_{f,\tau}^{-1}\|)}{(1+\beta)\|S_{f,\tau}^{-1}\|} \quad \text{and} \quad \left(\frac{1+\alpha}{1-\beta}\|\theta_\tau\| + \frac{\gamma}{1-\beta}\right)\left(\frac{1+r}{1-s}\|\theta_f\| + \frac{t}{1-s}\right).$$

Proof. For $m = 1, \dots,$ for each $x \in \mathcal{X}$ and for every $c_1, \dots, c_m \in \mathbb{K}$,

$$\begin{aligned}\left\|\sum_{n=1}^{m} g_n(x)e_n\right\| &\leq \left\|\sum_{n=1}^{m}(f_n - g_n)(x)e_n\right\| + \left\|\sum_{n=1}^{m} f_n(x)e_n\right\| \\ &\leq (1+r)\left\|\sum_{n=1}^{m} f_n(x)e_n\right\| + s\left\|\sum_{n=1}^{m} g_n(x)e_n\right\| + t\|x\|\end{aligned}$$

and

$$\begin{aligned}\left\|\sum_{n=1}^{m} c_n\omega_n\right\| &\leq \left\|\sum_{n=1}^{m} c_n(\tau_n - \omega_n)\right\| + \left\|\sum_{n=1}^{m} c_n\tau_n\right\| \\ &\leq (1+\alpha)\left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m}|c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{m} c_n\omega_n\right\|.\end{aligned}$$

Hence

$$\left\|\sum_{n=1}^{m} g_n(x)e_n\right\| \leq \frac{1+r}{1-s}\left\|\sum_{n=1}^{m} f_n(x)e_n\right\| + \frac{t}{1-s}\|x\|, \quad \forall x \in \mathcal{X}, m = 1, \dots$$

and

$$\left\|\sum_{n=1}^{m} c_n\omega_n\right\| \leq \frac{1+\alpha}{1-\beta}\left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \frac{\gamma}{1-\beta}\left(\sum_{n=1}^{m}|c_n|^p\right)^{\frac{1}{p}}, \quad \forall c_1, \dots, c_m \in \mathbb{K}, m = 1, \dots.$$

Therefore θ_g and θ_ω are well-defined bounded linear operators with

$$\|\theta_g\| \leq \frac{1+r}{1-s}\|\theta_f\| + \frac{t}{1-s}, \quad \|\theta_\omega\| \leq \frac{1+\alpha}{1-\beta}\|\theta_\tau\| + \frac{\gamma}{1-\beta}.$$

Now Equation (2.3) gives

$$\left\|\sum_{n=1}^{\infty} c_n(\tau_n - \omega_n)\right\| \leq \alpha\left\|\sum_{n=1}^{\infty} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{\infty}|c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{\infty} c_n\omega_n\right\|, \quad \forall \{c_n\}_n \in \ell^p(\mathbb{N}).$$

That is,

$$\|\theta_\tau\{c_n\}_n - \theta_\omega\{c_n\}_n\| \leq \alpha\|\theta_\tau\{c_n\}_n\| + \gamma\left(\sum_{n=1}^{\infty}|c_n|^p\right)^{\frac{1}{p}} + \beta\|\theta_\omega\{c_n\}_n\|, \quad \forall \{c_n\}_n \in \ell^p(\mathbb{N}). \tag{2.4}$$

By taking $\{c_n\}_n = \{f_n(S_{f,\tau}^{-1}x)\}_n = \theta_f S_{f,\tau}^{-1}x$ in Equation (2.4), we get

$$\|\theta_\tau\theta_f S_{f,\tau}^{-1}x - \theta_\omega\theta_f S_{f,\tau}^{-1}x\| \leq \alpha\|\theta_\tau\theta_f S_{f,\tau}^{-1}x\| + \gamma\left(\sum_{n=1}^{\infty}|f_n(S_{f,\tau}^{-1}x)|^p\right)^{\frac{1}{p}} + \beta\|\theta_\omega\theta_f S_{f,\tau}^{-1}x\|, \quad \forall x \in \mathcal{X}.$$

That is,

$$\begin{aligned}\|x - S_{g,\omega}S_{f,\tau}^{-1}x\| &\leq \alpha\|x\| + \gamma\|\theta_f S_{f,\tau}^{-1}x\| + \beta\|S_{g,\omega}S_{f,\tau}^{-1}x\| \\ &\leq (\alpha + \gamma\|\theta_f S_{f,\tau}^{-1}\|)\|x\| + \beta\|S_{g,\omega}S_{f,\tau}^{-1}x\|, \quad \forall x \in \mathcal{X}.\end{aligned}$$

Since $\max\{\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|,\beta\}<1$, we can use Theorem 2.5 to get the operator $S_{g,\omega}S_{f,\tau}^{-1}$ is invertible and

$$\|(S_{g,\omega}S_{f,\tau}^{-1})^{-1}\|\leq \frac{1+\beta}{1-(\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|)}.$$

Hence the operator $S_{g,\omega}=(S_{g,\omega}S_{f,\tau}^{-1})S_{f,\tau}$ is invertible. Therefore $(\{g_n\}_n,\{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$. We get the frame bounds from the following calculations:

$$\|S_{g,\omega}^{-1}\|\leq \|S_{f,\tau}^{-1}\|\|S_{f,\tau}S_{g,\omega}^{-1}\|\leq \frac{\|S_{f,\tau}^{-1}\|(1+\beta)}{1-(\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|)} \quad \text{and}$$
$$\|S_{g,\omega}\|\leq \|\theta_\omega\|\|\theta_g\|\leq \left(\frac{1+\alpha}{1-\beta}\|\theta_\tau\|+\frac{\gamma}{1-\beta}\right)\left(\frac{1+r}{1-s}\|\theta_f\|+\frac{t}{1-s}\right).$$

□

Remark 2.7. Theorem 1.3 is a corollary for Theorem 2.6. In particular, Theorems 1.1 and 1.2 are corollaries for Theorem 2.6. Indeed, let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$. We define

$$f_n:\mathcal{H}\ni h\mapsto f_n(h)\coloneqq\langle h,\tau_n\rangle\in\mathbb{K},\quad \forall n\in\mathbb{N}.$$

Then $\theta_f=\theta_\tau$ and $(\{f_n\}_n,\{\tau_n\}_n)$ is a 2-approximate frame for $\mathcal{H}$. We also define

$$g_n=f_n,\quad \forall n\in\mathbb{N}.$$

Then condition (2.2) holds trivially. Theorem 2.6 now says that $(\{g_n\}_n,\{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$. To prove Theorem 1.3, it now suffices to prove that $\{\omega_n\}_n$ is a frame for $\mathcal{H}$. Since $(\{g_n\}_n,\{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$, it follows that θ_ω is surjective. Theorem 5.4.1 in [10] now says that $\{\omega_n\}_n$ is a frame for $\mathcal{H}$.

Corollary 2.8. Let q be the conjugate index of p. Let $(\{f_n\}_n,\{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. Assume that a collection $\{\tau_n\}_n$ in $\mathcal{X}$ and a collection $\{g_n\}_n$ in $\mathcal{X}^*$ are such that $\sum_{n=1}^\infty\|f_n-g_n\|<\infty$ and

$$\lambda\coloneqq\sum_{n=1}^\infty\|\tau_n-\omega_n\|^q<\frac{1}{\|\theta_f S_{f,\tau}^{-1}\|^q}.$$

Then $(\{g_n\}_n,\{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$ with bounds $\frac{1-\lambda^{1/p}\|\theta_f S_{f,\tau}^{-1}\|}{\|S_{f,\tau}^{-1}\|}$ and $(\|\theta_\tau\|+\lambda^{1/p})(\|\theta_f\|+\sum_{n=1}^\infty\|f_n-g_n\|)$.

Proof. Take $r=0,s=0,t=\sum_{n=1}^\infty\|f_n-g_n\|,\alpha=0,\beta=0,\gamma=\lambda^{1/p}$. Then $\max\{\alpha+\gamma\|\theta_f S_{f,\tau}^{-1}\|,\beta,s\}<1$ and

$$\left\|\sum_{n=1}^m(f_n-g_n)(x)e_n\right\|\leq\left(\sum_{n=1}^m\|f_n-g_n\|\right)\|x\|\leq t\|x\|,\quad \forall x\in\mathcal{X},m=1,\dots,$$
$$\left\|\sum_{n=1}^m c_n(\tau_n-\omega_n)\right\|\leq\left(\sum_{n=1}^m\|\tau_n-\omega_n\|^q\right)^{\frac{1}{q}}\left(\sum_{n=1}^m|c_n|^p\right)^{\frac{1}{p}}\leq\gamma\left(\sum_{n=1}^m|c_n|^p\right)^{\frac{1}{p}},$$
$$\forall c_1,\dots,c_m\in\mathbb{K},m=1,\dots.$$

By using Theorem 2.6 we now get the result. □

We next derive stability result which does not demand maximum condition on parameters α and γ.

Theorem 2.9. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. Assume that a collection $\{\tau_n\}_n$ in $\mathcal{X}$ and a collection $\{g_n\}_n$ in $\mathcal{X}^*$ are such that there exist $r, s, t, \alpha, \beta, \gamma \geq 0$ with $\max\{\beta, s\} < 1$ and*

$$\left\|\sum_{n=1}^{m}(f_n - g_n)(x)e_n\right\| \leq r\left\|\sum_{n=1}^{m} f_n(x)e_n\right\| + t\|x\| + s\left\|\sum_{n=1}^{m} g_n(x)e_n\right\|, \quad \forall x \in \mathcal{X}, m = 1, \dots,$$

$$\left\|\sum_{n=1}^{m} c_n(\tau_n - \omega_n)\right\| \leq \alpha\left\|\sum_{n=1}^{m} c_n\tau_n\right\| + \gamma\left(\sum_{n=1}^{m}|c_n|^p\right)^{\frac{1}{p}} + \beta\left\|\sum_{n=1}^{m} c_n\omega_n\right\|, \quad \forall c_1, \dots, c_m \in \mathbb{K}, m = 1, \dots.$$

Assume that one of the following holds.

(1) $\sum_{n=1}^{\infty}(\|f_n - g_n\|\|S_{f,\tau}^{-1}\tau_n\| + \|g_n\|\|S_{f,\tau}^{-1}(\tau_n - \omega_n)\|) < 1$.
(2) $\sum_{n=1}^{\infty}(\|f_n - g_n\|\|S_{f,\tau}^{-1}\omega_n\| + \|f_n\|\|S_{f,\tau}^{-1}(\tau_n - \omega_n)\|) < 1$.
(3) $\sum_{n=1}^{\infty}(\|(f_n - g_n)S_{f,\tau}^{-1}\|\|\tau_n\| + \|g_n S_{f,\tau}^{-1}\|\|\tau_n - \omega_n\|) < 1$.
(4) $\sum_{n=1}^{\infty}(\|(f_n - g_n)S_{f,\tau}^{-1}\|\|\omega_n\| + \|f_n S_{f,\tau}^{-1}\|\|\tau_n - \omega_n\|) < 1$.

Then $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$. Moreover, an upper bound is

$$\left(\frac{1+\alpha}{1-\beta}\|\theta_\tau\| + \frac{\gamma}{1-\beta}\right)\left(\frac{1+r}{1-s}\|\theta_f\| + \frac{t}{1-s}\right).$$

Proof. Following the initial lines in the proof of Theorem 2.6, we see that θ_g and θ_ω are well-defined bounded linear operators. We now consider four cases.
Assume (1). Then

$$\begin{aligned}
\left\|x - \sum_{n=1}^{\infty} g_n(x)S_{f,\tau}^{-1}\omega_n\right\| &= \left\|\sum_{n=1}^{\infty} f_n(x)S_{f,\tau}^{-1}\tau_n - \sum_{n=1}^{\infty} g_n(x)S_{f,\tau}^{-1}\omega_n\right\| \\
&\leq \sum_{n=1}^{\infty}\|f_n(x)S_{f,\tau}^{-1}\tau_n - g_n(x)S_{f,\tau}^{-1}\omega_n\| \\
&\leq \sum_{n=1}^{\infty}\left\{\|f_n(x)S_{f,\tau}^{-1}\tau_n - g_n(x)S_{f,\tau}^{-1}\tau_n\| + \|g_n(x)S_{f,\tau}^{-1}\tau_n - g_n(x)S_{f,\tau}^{-1}\omega_n\|\right\} \\
&= \sum_{n=1}^{\infty}\left\{\|(f_n - g_n)(x)S_{f,\tau}^{-1}\tau_n\| + \|g_n(x)S_{f,\tau}^{-1}(\tau_n - \omega_n)\|\right\} \\
&\leq \left(\sum_{n=1}^{\infty}\left\{\|f_n - g_n\|\|S_{f,\tau}^{-1}\tau_n\| + \|g_n\|\|S_{f,\tau}^{-1}(\tau_n - \omega_n)\|\right\}\right)\|x\|.
\end{aligned}$$

Therefore the operator $S_{f,\tau}^{-1}S_{g,\omega}$ is invertible.
Assume (2). Then

$$
\begin{aligned}
&\left\|x-\sum_{n=1}^\infty g_n(x)S_{f,\tau}^{-1}\omega_n\right\| = \left\|\sum_{n=1}^\infty f_n(x)S_{f,\tau}^{-1}\tau_n-\sum_{n=1}^\infty g_n(x)S_{f,\tau}^{-1}\omega_n\right\|\\
&\leq \sum_{n=1}^\infty \|f_n(x)S_{f,\tau}^{-1}\tau_n-g_n(x)S_{f,\tau}^{-1}\omega_n\|\\
&\leq \sum_{n=1}^\infty \left\{\|f_n(x)S_{f,\tau}^{-1}\tau_n-f_n(x)S_{f,\tau}^{-1}\omega_n\|+\|f_n(x)S_{f,\tau}^{-1}\omega_n-g_n(x)S_{f,\tau}^{-1}\omega_n\|\right\}\\
&= \sum_{n=1}^\infty \left\{\|f_n(x)S_{f,\tau}^{-1}(\tau_n-\omega_n)\|+\|(f_n-g_n)(x)S_{f,\tau}^{-1}\omega_n\|\right\}\\
&\leq \left(\sum_{n=1}^\infty \left\{\|f_n\|\|S_{f,\tau}^{-1}(\tau_n-\omega_n)\|+\|f_n-g_n\|\|S_{f,\tau}^{-1}\omega_n\|\right\}\right)\|x\|.
\end{aligned}
$$

Therefore the operator $S_{f,\tau}^{-1}S_{g,\omega}$ is invertible.
Assume (3). Then

$$
\begin{aligned}
&\left\|x-\sum_{n=1}^\infty g_n(S_{f,\tau}^{-1}x)\omega_n\right\| = \left\|\sum_{n=1}^\infty f_n(S_{f,\tau}^{-1}x)\tau_n-\sum_{n=1}^\infty g_n(S_{f,\tau}^{-1}x)\omega_n\right\|\\
&\leq \sum_{n=1}^\infty \|f_n(S_{f,\tau}^{-1}x)\tau_n-g_n(S_{f,\tau}^{-1}x)\omega_n\|\\
&\leq \sum_{n=1}^\infty \left\{\|f_n(S_{f,\tau}^{-1}x)\tau_n-g_n(S_{f,\tau}^{-1}x)\tau_n\|+\|g_n(S_{f,\tau}^{-1}x)\tau_n-g_n(S_{f,\tau}^{-1}x)\omega_n\|\right\}\\
&= \sum_{n=1}^\infty \left\{\|(f_n-g_n)(S_{f,\tau}^{-1}x)\tau_n\|+\|g_n(S_{f,\tau}^{-1}x)(\tau_n-\omega_n)\|\right\}\\
&\leq \left(\sum_{n=1}^\infty \left\{\|(f_n-g_n)S_{f,\tau}^{-1}\|\|\tau_n\|+\|g_nS_{f,\tau}^{-1}\|\|\tau_n-\omega_n\|\right\}\right)\|x\|.
\end{aligned}
$$

Therefore the operator $S_{g,\omega}S_{f,\tau}^{-1}$ is invertible. Assume (4). Then

$$
\begin{aligned}
&\left\|x-\sum_{n=1}^\infty g_n(S_{f,\tau}^{-1}x)\omega_n\right\| = \left\|\sum_{n=1}^\infty f_n(S_{f,\tau}^{-1}x)\tau_n-\sum_{n=1}^\infty g_n(S_{f,\tau}^{-1}x)\omega_n\right\|\\
&\leq \sum_{n=1}^\infty \|f_n(S_{f,\tau}^{-1}x)\tau_n-g_n(S_{f,\tau}^{-1}x)\omega_n\|\\
&\leq \sum_{n=1}^\infty \left\{\|f_n(S_{f,\tau}^{-1}x)\tau_n-f_n(S_{f,\tau}^{-1}x)\omega_n\|+\|f_n(S_{f,\tau}^{-1}x)\omega_n-g_n(S_{f,\tau}^{-1}x)\omega_n\|\right\}\\
&= \sum_{n=1}^\infty \left\{\|f_n(S_{f,\tau}^{-1}x)(\tau_n-\omega_n)\|+\|(f_n-g_n)(S_{f,\tau}^{-1}x)\omega_n\|\right\}
\end{aligned}
$$

$$\leq \left(\sum_{n=1}^{\infty}\left\{\|f_nS_{f,\tau}^{-1}\|\|\tau_n-\omega_n\|+\|(f_n-g_n)S_{f,\tau}^{-1}\|\|\omega_n\|\right\}\right)\|x\|.$$

Therefore the operator $S_{g,\omega}S_{f,\tau}^{-1}$ is invertible.

Hence in each of assumptions we get that $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-ASF for $\mathcal{X}$. □

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