Poincare Journal of Analysis & Applications
Vol. 8, No. 1(II) (2021), 25-38

MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES CONDITIONALLY SHARE TWO SETS

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Date of Receiving	:	15. 07. 2020
Date of Revision	:	10. 02. 2021
Date of Acceptance	:	03. 05. 2021

Abstract. In the paper, with the aid of relaxed sharing hypothesis, we study the uniqueness of meromorphic (entire) functions whose derivatives share a finite set. The results in this paper will improve a number of theorems earlier obtained by Meng-Hu [11] and Meng [10]. Two examples have been exhibited by us to show that the conclusions in our results actually occur.

1. Introduction and Definitions

Let us denote by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Throughout the paper by a meromorphic function we shall always mean a meromorphic function in the complex plane $\mathbb{C}$. We adopt the usual notations of Nevanlinna theory as explained in [8]. By E and I we denote any set of finite and infinite linear measure respectively of $0 < r < \infty$. For any non-constant meromorphic function $h(z)$ we define $S(r,h)$ by $S(r,h) = o(T(r,h))\ where\ r \longrightarrow \infty, r \notin E$.

Let f be a non-constant meromorphic function, $a \in \overline{\mathbb{C}}$ and p be a positive integer. We denote by $E(a,f)$ the set of zeros of $f(z)-a$(counting multiplicity) and by $E_{p)}(a,f)$ the set of zeros of $f(z)-a$ with multiplicity $\leq p$(counting multiplicity).

Let $S \subset \mathbb{C}$. Set

$$E_{p)}(S,f) = \bigcup_{a \in S} E_{p)}(a,f).$$

Then for two non-constant meromorphic functions f and g we say that f, g share the set S truncated p if $E_{p)}(S,f) = E_{p)}(S,g)$. If $p = \infty$, we define $E_{p)}(S,f) = E(S,f) = E_f(S)$.

The inception of set sharing problem in the realm of the theory of meromorphic function was due to the following famous "Gross Question" {see [7]}.

2010 *Mathematics Subject Classification.* 30D35.
Key words and phrases. Meromorphic function, weighted sharing, shared set, truncated multiplicity.
Communicated by. Subhash S. Bhoosnurmath
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Question A. Can one find two (or possibly even one) finite sets $S_j(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical?

Yi [13] gave a positive answer to the *Question A*. He proved

Theorem A. [13] Let f and g be two non-constant entire functions, $n \geq 5$ a positive integer and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0$ is a constant satisfying $a^{2n} \neq 1$. Let $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

In course of time, on the basis of *Question A*, the set sharing problem was divided into two categories. In one respect, researchers became involved to explore the sets for which two meromorphic (entire) functions share the same becomes identical. On the other hand, the sets, which do not satisfy this property, also got priority. For these second category of range sets, the natural inquisition was to find out the possible forms of the functions. This is an age old problem. Throughout this paper, our main investigations will be oriented around the second aspect.

Next we give some definitions.

Definition 1.1. [1] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)(\overline{N}(r, a; f \mid\geq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid\leq p)(\overline{N}(r, a; f \mid\leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.2. [1] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.3. [1] Let $a \in \mathbb{C} \cup \{\infty\}$ and let f and g be two non-constant meromorphic functions. If z_0 be an a-point of f with multiplicity p and an a-point of g with multiplicity q, then we denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g, where $p > q$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g, where $p = q = 1$ and by $\overline{N}_E^{(2}(r, a; f)$ the counting function of those a-points of f and g, where $p = q \geq 2$, each points in this counting function is counted only once. In the same way we can define $\overline{N}_L(r, a; g), N_E^{1)}(r, a; g)$ and $\overline{N}_E^{(2}(r, a; g)$. We also define $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

In 2001, at the time of investigating *Question A*, Fang [6] proved the following series of results.

Theorem B. [6] Let f and g be two non-constant entire functions, $n(\geq 5)$, k be two positive integers and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b, c\}$, where a, b, c are non-zero finite distinct constants satisfying $a^2 \neq bc$, $b^2 \neq ac$, $c^2 \neq ab$. Let $E_f(S_1) = E_g(S_1)$ and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then $f \equiv g$.

Theorem C. [6] Let f and g be two non-constant entire functions, $n(\geq 5)$, k be two positive integers and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b\}$, where a,b are two non-zero finite distinct constants. Let $E_f(S_1) = E_g(S_1)$ and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then one of the following cases must occurs: (1) $f \equiv g$; (2) $b = -a$, $f = e^{cz+d}$, $g = te^{-cz-d}$, where c, d, t are three constants, $t^n = 1$ and $(-1)^k tc^{2k} = a^2$; (3) $f = e^{cz+d}$, $g = te^{-cz-d}$, where c, d, t are three constants, $t^n = 1$ and $(-1)^k tc^{2k} = ab$; (4) $b = -a$, $f \equiv -g$.

Theorem D. [6] Let f and g be two non-constant entire functions, $n(\geq 5)$, k be two positive integers and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0, \infty$. Let $E_f(S_1) = E_g(S_1)$ and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then one of the following cases must occurs: (1) $f \equiv g$; (2) $b = -a$, $f = e^{cz+d}$, $g = te^{-cz-d}$, where c, d, t are three constants, $t^n = 1$ and $(-1)^k tc^{2k} = a^2$.

Next we require the following well known definition of weighted sharing of values or sets appeared in 2001 in the literature.

Definition 1.4. [9] Let k be a non-negetive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_f(a, k) = E_g(a, k)$, we say f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively. The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

Definition 1.5. [9] Let $S \subset \mathbb{C}$ and k be a non-negative integer or ∞. We set $E_f(S, k) = \bigcup_{a \in S} E_f(a, k)$ we say that f, g share the set S with weight k if $E_f(S, k) = E_g(S, k)$. Clearly $E_f(S) = E_f(S, \infty)$.

In 2009, under the purview of more general shared sets and the notion of weighted sharing, Meng-Hu [11] investigated *Question A* and improved the results of Fang [6] as follows:

Theorem E. [11] Let f and g be two non-constant meromorphic functions, n, k be two positive integers. Let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are non-zero distinct constants. Let $E_f(S_1, u) = E_g(S_1, u)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$. Further
(i) if $n \geq 5$ and $u = 2$ or
(ii) if $n \geq 5$ and $u = 1$ or
(iii) if $n \geq 8$ and $u = 0$;
then one of the following cases must occurs: (1) $f \equiv tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$ where t is a constant satisfying $t^n = 1$; (2) $f = de^{cz}$, $g = \frac{t}{d}te^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_m}\}$, where c, d, t are non-zero constants and $t^n = 1$.

In 2013, using the notion of truncated sharing, Meng [10] improved the above results in the following manner.

Theorem F. [10] Let f and g be two non-constant meromorphic functions, n, k be two positive integers. Let S_i, $i = 1, 2$ be given as in Theorem E. Let $E_{v)}(S_1, f) = E_{v)}(S_1, g)$

and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$. Further
(i) if $n \geq 5$ and $v = 3$ or
(ii) if $n \geq 5$ and $v = 2$;
then the conclusions of *Theorem E* holds

In this paper, we consider the case of meromorphic functions in the above results and prove the following theorems which will improve all the stated results.

Theorem 1.1. Let f and g be two non-constant meromorphic functions, n, k be two positive integers and $r \in \mathbb{C}, r \neq 0$. Let $S_1 = \{z : z^n = r\}$ and $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are non-zero finite distinct constants. Suppose $E_f(S_1, u) = E_g(S_1, u)$, $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$ and let f, g share $(\infty, 0)$. Further
(i) if $n \geq 6$ and $u = 2$ or
(ii) if $n \geq 7$ and $u = 1$ or
(iii) if $n \geq 11$ and $u = 0$;
then one of the following cases must occurs:
(1) $f \equiv sg$, $\{a_1, a_2, \ldots, a_m\} = s\{a_1, a_2, \ldots, a_m\}$ where s is a constant satisfying $s^n = 1$;
(2) $f = de^{cz}$, $g = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_m}\}$, where c, d, t are non-zero constants and $t^n = r^2$.

With the notion of truncated sharing we also prove the following theorem analogous to *Theorem 1.1.*

Theorem 1.2. Under the same situation as in *Theorem 1.1*, let $E_{v)}(S_1, f) = E_{v)}(S_1, g)$, $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$ and let f, g share $(\infty, 0)$. Further
(i) if $n \geq 6$ and $v = 3$ or
(ii) if $n \geq 7$ and $v = 2$ or
(iii) if $n \geq 10$ and $v = 1$;
then one of the conclusions of *Theorem 1.1* holds.

When $m = 1$, we can easily deduce the following Corollary from *Theorem 1.1.*

Corollary 1.1. Let f and g be two non-constant meromorphic functions, n, k be two positive integers and $r \in \mathbb{C}, r \neq 0$. Let $S_1 = \{z : z^n = r\}$ and $S_2 = \{a\}$, where a be non-zero finite constant. Let $E_f(S_1, u) = E_g(S_1, u)$, $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$ and let f, g share $(\infty, 0)$. Further
(i) if $n \geq 6$, $u = 2$ or
(ii) if $n \geq 7$, $u = 1$ or
(iii) if $n \geq 11$, $u = 0$;
then one of the following cases must occurs:
(1) $f \equiv g$;
(2) $f = de^{cz}$, $g = \frac{t}{d}e^{-cz}$, $a = (-1)^k c^{2k} t \frac{1}{a}$, where c, d, t are non-zero constants and $t^n = r^2$.

Again for $m \geq 2$ we have following Corollary from *Theorem 1.1*:

Corollary 1.2. Let f and g be two non-constant meromorphic functions, n, k be two positive integers and $r \in \mathbb{C}, r \neq 0$. Let S_i, $i = 1, 2$ be given as in *Theorem 1.1.* Let $E_f(S_1, u) = E_g(S_1, u)$, $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$ and let f, g share $(\infty, 0)$. Further
(i) if $n \geq 6$, $u = 2$ or

(ii) if $n \geq 7$, $u = 1$ or
(iii) if $n \geq 11$, $u = 0$;
then one of the following cases must occurs:
(1) $f \equiv sg$, $s^l = 1$, where $l \mid n$ and $l \mid m$;
(2) $f = de^{cz}$, $g = \frac{t}{d}e^{-cz}$ and one of the cases below occurs:
(I) $b_1^2 = (-1)^k c^{2k} t$, $b_2 b_3 = b_4 b_5 = \ldots = b_{m-1} b_m = (-1)^k c^{2k} t$, when m is odd, where $\{b_1, b_2, b_3, \ldots, b_m\} = \{a_1, a_2, a_3, \ldots, a_m\}$,
(II) $b_1^2 = (-1)^k c^{2k} t = b_2^2$ or $b_1 b_2 = (-1)^k c^{2k} t$ and $b_3 b_4 = b_5 b_6 = \ldots = b_{m-1} b_m = (-1)^k c^{2k} t$, when m is even, where $\{b_1, b_2, b_3, b_4, \ldots, b_m\} = \{a_1, a_2, a_3, a_4, \ldots, a_m\}$.
Here c, d, t are non-zero constants and $t^n = r^2$.

When $m = 1$ from *Theorem 1.2* we have the following corollary:

Corollary 1.3. Under the same situation as in *Corollary 1.1*, let $E_{v)}(S_1, f) = E_{v)}(S_1, g)$, $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$ and let f, g share $(\infty, 0)$. Further
(i) if $n \geq 6$, $v = 3$ or
(ii) if $n \geq 7$, $v = 2$ or
(iii) if $n \geq 10$, $v = 1$;
then conclusion of *Corollary 1.1* holds.

Again for $m \geq 2$ in *Theorem 1.2* we have next corollary:

Corollary 1.4. Under the same situation as in *Corollary 1.2*, let $E_{v)}(S_1, f) = E_{v)}(S_1, g)$, $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$ and let f, g share $(\infty, 0)$. Further
(i) if $n \geq 6$, $v = 3$ or
(ii) if $n \geq 7$, $v = 2$ or
(iii) if $n \geq 10$, $v = 1$;
then conclusion of *Corollary 1.2* holds.

Now we give an example which shows that case (1) in above corollaries holds.

Example 1. Let a, b be two non-zero complex constants. Let $S_1 = \{z : z^n = a\}$ and $S_2 = \{z : z^m = b\}$. Let $f(z)$, $g(z)$ be any two meromorphic functions such that $f(z) = sg(z)$, where $s^l = 1$, the integer l divides n and m. Since $f^n = s^n g^n = g^n$ and $(f^{(k)})^m = s^m (g^{(k)})^m = (g^{(k)})^m$, then $E_f(S_1, \infty) = E_g(S_1, \infty)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$.

The next example shows that case (2) in above corollaries also holds. Here we have assumed $\omega = e^{\frac{2\pi i}{n}}$.

Example 2. Let a, b be any two non-zero complex constants. Also let $S_1 = \{z : z^n = r\}$ and when m be odd, choose

$$\begin{aligned} S_2 \quad = \quad & \left\{ \frac{a}{c_1}, bc_1, \frac{a}{c_2}, bc_2, \ldots, \frac{a}{c_{\frac{m-1}{2}}}, bc_{\frac{m-1}{2}}, \sqrt{ab}; c_i \in \mathbb{C}\backslash\{0\}, i = 1, 2, \ldots, \frac{m-1}{2} \right\} \quad \textit{or} \\ & \left\{ \frac{a}{c_1}, bc_1, \frac{a}{c_2}, bc_2, \ldots, \frac{a}{c_{\frac{m-1}{2}}}, bc_{\frac{m-1}{2}}, -\sqrt{ab}; c_i \in \mathbb{C}\backslash\{0\}, i = 1, 2, \ldots, \frac{m-1}{2} \right\}; \end{aligned}$$

otherwise when m be even

$$\begin{aligned} S_2 &= \left\{\frac{a}{c_1}, bc_1, \frac{a}{c_2}, bc_2, \ldots, \frac{a}{c_{\frac{m-2}{2}}}, bc_{\frac{m-2}{2}}, \sqrt{ab}, -\sqrt{ab}; c_i \in \mathbb{C}\backslash\{0\}, i = 1, 2, \ldots, \frac{m-2}{2}\right\} \\ & \quad or \ \left\{\frac{a}{c_1}, bc_1, \frac{a}{c_2}, bc_2, \ldots, \frac{a}{c_{\frac{m}{2}}}, bc_{\frac{m}{2}}; c_i \in \mathbb{C}\backslash\{0\}, i = 1, 2, \ldots, \frac{m}{2}\right\}. \end{aligned}$$

Set $f(z) = \frac{1}{\sqrt{(-ab)^{(k-1)}}}\ e^{\sqrt{-ab}z}$, $g(z) = \frac{(-1)^{(k-1)}}{\sqrt{(-ab)^{(k-1)}}}\ e^{-\sqrt{-ab}z}$. Choosing c, d, t as stated in the Corollary 1.2 as: $c = \sqrt{-ab}$, $d = \frac{1}{\sqrt{(-ab)^{(k-1)}}}$, $t = \frac{1}{(ab)^{(k-1)}}$, we have $t^n = r^2$; i.e., $r^2 = \frac{1}{(ab)^{n(k-1)}}$; i.e., $r = \pm\frac{1}{\sqrt{(ab)^{n(k-1)}}}$. Let us suppose $\frac{1}{\sqrt{(ab)^{n(k-1)}}} = \frac{1}{|\sqrt{(ab)^{n(k-1)}}|}e^{i\theta}$, $-\pi < \theta \leq \pi$.

First choose $r = \frac{1}{\sqrt{(ab)^{n(k-1)}}}$, then $S_1 = \left\{\frac{1}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i\theta}{n}}\omega^j;\ j = 0, 1, \ldots, (n-1)\right\}$. Let $f(z) = \frac{1}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i\theta}{n}}\omega^p$, $0 \leq p \leq n-1$; i.e., $\frac{1}{\sqrt{(-ab)^{(k-1)}}}\ e^{\sqrt{-ab}z} = \frac{1}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i\theta}{n}}\omega^p$; i.e., $e^{\sqrt{-ab}z} = \frac{\sqrt{(-ab)^{(k-1)}}}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i\theta}{n}}\omega^p$. As $\frac{1}{t} = (ab)^{(k-1)} = \frac{1}{r^{\frac{2}{n}}} =| \sqrt{(ab)^{(k-1)}}\ |^2\ e^{-\frac{2i\theta}{n}}\omega^{-2q}$, $q = 0, 1, \ldots, (n-1)$, we have

$$\begin{aligned} g(z) &= \frac{(-1)^{(k-1)}}{\sqrt{(-ab)^{(k-1)}}}\ e^{-\sqrt{-ab}z} = \frac{|\ \sqrt{(ab)^{(k-1)}}\ |}{(ab)^{(k-1)}}\ e^{-\frac{i\theta}{n}}\omega^{-p} \\ &= \frac{|\ \sqrt{(ab)^{(k-1)}}\ |}{|\ \sqrt{(ab)^{(k-1)}}\ |^2}\ e^{-\frac{i\theta}{n}}\omega^{-p}e^{\frac{2i\theta}{n}}\omega^{2q} = \frac{1}{|\ \sqrt{(ab)^{(k-1)}}\ |}\ e^{\frac{i\theta}{n}}\omega^{(2q-p)}. \end{aligned}$$

Clearly $1 - n \leq 2q - p \leq 2n - 2$.

Next suppose $r = -\frac{1}{\sqrt{(ab)^{n(k-1)}}}$, then $S_1 = \left\{\frac{1}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i(\pi+\theta)}{n}}\omega^j; j = 0, 1, \ldots, (n-1)\right\}$. Let $f(z) = \frac{1}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i(\pi+\theta)}{n}}\omega^p$, $0 \leq p \leq n-1$. In a similar manner as above in this case also we can show that $g(z) = \frac{1}{|\sqrt{(ab)^{(k-1)}}|}\ e^{\frac{i(\pi+\theta)}{n}}\omega^{(2q-p)}$, where as usual $1 - n \leq 2q - p \leq 2n - 2$. We note that

$$\omega^{1-n+j} = \begin{cases} \omega^{j+1}, & \text{if } 0 \leq j \leq n-1, \\ \omega^{j-(n-1)}, & \text{if } n \leq j \leq 2n-1, \\ \omega^{j-(2n-1)}, & \text{if } 2n \leq j \leq 3n-3. \end{cases}$$

It follows that, $E_f(S_1, \infty) = E_g(S_1, \infty)$.

For the sharing of the second set, we have $g^{(k)}(z) = \frac{a}{c_i}$; $i = 1, 2, \ldots, M$ where $M = \frac{m-1}{2}$, when m is odd; or $\frac{m-2}{2}$ or $\frac{m}{2}$, when m is even. So $-\sqrt{-ab}\ e^{-\sqrt{-ab}z} = \frac{a}{c_i}$; i.e., $e^{-\sqrt{-ab}z} = -\frac{a}{c_i\sqrt{-ab}}$; i.e., $e^{-\sqrt{-ab}z} = \frac{\sqrt{-ab}}{bc_i}$; iff $f^{(k)}(z) = \sqrt{-ab}\ e^{\sqrt{-ab}z} = bc_i$. Similrly $g^{(k)}(z) = bc_i$ iff $f^{(k)}(z) = \frac{a}{c_i}$. Also $g^{(k)}(z) = \sqrt{ab}$ or $-\sqrt{ab}$ iff $f^{(k)}(z) = \sqrt{ab}$ or $-\sqrt{ab}$. Therefore $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$ as follows

$$F = \frac{f^n}{r}, \quad G = \frac{g^n}{r}. \tag{2.1}$$

Let H, V be the following functions:

$$H = \left(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}\right) - \left(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}\right) \quad \textit{and} \tag{2.2}$$

$$V = \frac{F^{'}}{F(F-1)} - \frac{G^{'}}{G(G-1)}. \tag{2.3}$$

Lemma 2.1. [12] Let f be a non-constant meromorphic function and let

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.2. [3] Let $E_{v)}(1, F) = E_{v)}(1, G)$ and let $H \not\equiv 0$, then

$$N(r, 1; F \mid= 1) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 2.3. [2] Let $E_{v)}(1, F) = E_{v)}(1, G)$ and let F and G share (∞, k). Also let $H \not\equiv 0$. Then

$$\begin{aligned} N(r, \infty; H) \quad \leq \quad & \overline{N}(r, 0; F \mid\geq 2) + \overline{N}(r, 0; G \mid\geq 2) + \overline{N}_*(r, \infty; F, G) \\ & + \overline{N}(r, 1; F \mid\geq v+1) + \overline{N}(r, 1; G \mid\geq v+1) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'). \end{aligned}$$

Lemma 2.4. Let F and G be two non-constant meromorphic functions and let $E_{v)}(1, F) = E_{v)}(1, G)$, where $1 \leq v < \infty$. Then

$$\begin{aligned} & \overline{N}(r, 1; F) + \overline{N}(r, 1; G) - N_E^{1)}(r, 1; F) \\ & + \left(\frac{v}{2} - \frac{1}{2}\right) \{\overline{N}(r, 1; F \mid\geq v+1) + \overline{N}(r, 1; G \mid\geq v+1)\} \leq \frac{1}{2}\{N(r, 1; F) + N(r, 1; G)\}. \end{aligned}$$

Proof. The lemma can be proved in the line of proof of *Lemma 2.2* [4]. So we omit the details. □

Lemma 2.5. [13] Let H be defined as above. If $H \equiv 0$ and

$$\limsup_{r \to \infty} \frac{\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)}{T(r)} < 1, \quad r \in I,$$

where I is the set with infinite linear measure and $T(r) = max\{T(r,F), T(r,G)\}$, then $FG \equiv 1$ or $F \equiv G$.

Lemma 2.6. [2] Let F and G share $(\infty, 0)$ and $V \equiv 0$. Then $F \equiv G$.

Lemma 2.7. [2] Let H be defined as above. If $E_{3)}(1,F) = E_{3)}(1,G)$ and F, G share (∞, k), then one of the following cases occurs:

$$(i) T(r,F) + T(r,G) \leq 2\{N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G)\} + S(r,F) + S(r,G),$$

(ii) $F \equiv G$,

(iii) $FG \equiv 1$.

Lemma 2.8. Let F and G be defined as above and let $E_{v)}(1,F) = E_{v)}(1,G)$, where $1 \leq v < \infty$. Then

$$\overline{N}(r,1;F \mid\geq v+1) \leq \frac{1}{v}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + S(r,f).$$

Proof. The lemma can be proved in the line of proof of *Lemma 2.4* [3]. So we omit the details. □

Lemma 2.9. [5] Let F, G be two non-constant meromorphic functions such that they share $(1,u)$, $(\infty,0)$ and $H \not\equiv 0$. Now the following holds:

(i) if $u \geq 2$ then

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G),$$

(ii) if $u = 1$ then

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G),$$

(iii) if $u = 0$ then

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G).$$

Lemma 2.10. [2] Let V be defined as above and $V \not\equiv 0$. If F and G share (∞,k) and $E_{v)}(1,F) = E_{v)}(1,G)$, then

$$(nk+n-1)\overline{N}(r,\infty;f \mid\geq k+1) = (nk+n-1)\overline{N}(r,\infty;F \mid\geq nk+n) \leq \tfrac{v+1}{v}[\overline{N}(r,0;f) + \overline{N}(r,0;g)] + \tfrac{2}{v}\overline{N}(r,\infty;f) + S(r,f) + S(r,g).$$

Lemma 2.11. Let V be defined as above and $V \not\equiv 0$. If F and G share $(1,u)$ and (∞,k), then

$$(nk+n-1)\overline{N}(r,\infty;f \mid\geq k+1) = (nk+n-1)\overline{N}(r,\infty;F \mid\geq nk+n) \leq \tfrac{u+2}{u+1}[\overline{N}(r,0;f) + \overline{N}(r,0;g)] + \tfrac{2}{u+1}\overline{N}(r,\infty;f) + S(r,f) + S(r,g).$$

Proof. In the line of proof of *Lemma 2.10*, we can prove the lemma. So we omit the details. □

Lemma 2.12. [8] Let f be a non-constant meromorphic function, n be a positive integer and ψ be a function of the form $\psi = f^n + Q$, where Q is a differential polynomial of degree $\leq n-1$. If

$$N(r,\infty;f) + N(r,0;\psi) = S(r,f)$$

then $\psi = (f+\alpha)^n$, where α is a meromorphic function with $T(r,\alpha) = S(r,f)$, determined by the term of degree $n-1$ in Q.

3. Proofs of the theorems

Proof of Theorem 1.1. From $E_f(S_1,u) = E_g(S_1,u)$, we deduce that F, G share $(1,u)$ and also we have F, G share $(\infty,0)$. Now by *Lemma 2.1*, we get

$$T(r,F) = nT(r,f) + S(r,f), \qquad T(r,G) = nT(r,g) + S(r,g). \tag{3.1}$$

Let $H \not\equiv 0$. Since F, G share $(\infty,0)$, then $\overline{N}_*(r,\infty;F,G) \leq \overline{N}(r,\infty;F)$ and $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;G)$. Now we consider the following cases:

Case-1.1. For $u=2$, by (i) of *Lemma 2.9*, we have

$$\begin{aligned}
&T(r,F) + T(r,G)\\
&\leq 2\{N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\} + S(r,F) + S(r,G)\\
&\leq 4\{\overline{N}(r,0;F) + \overline{N}(r,0;G)\} + 6\overline{N}(r,\infty;F) + S(r,F) + S(r,G)\\
&\leq 4\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + 6\overline{N}(r,\infty;f) + S(r,f) + S(r,g).
\end{aligned}$$

Since $H \not\equiv 0$, then $F \not\equiv G$. It follows from *Lemma 2.6* that $V \not\equiv 0$. So by *Lemma 2.11* for $k=0$ and $u=2$, we have

$$\left(n - \frac{5}{3}\right)\overline{N}(r,\infty;f) \leq \frac{4}{3}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + S(r,f) + S(r,g).$$

Hence from above and from (3.1), we get

$$\begin{aligned}
&n\{T(r,f) + T(r,g)\}\\
&\leq 4\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + \frac{24}{3n-5}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + S(r,f) + S(r,g),
\end{aligned}$$

which contradicts $n \geq 6$.

Case-1.2. For $u=1$, by (ii) of *Lemma 2.9*, we have

$$\begin{aligned}
&T(r,F) + T(r,G)\\
&\leq 2\{N_2(r,0;F) + N_2(r,0;G)\} + \frac{1}{2}\{\overline{N}(r,0;F) + \overline{N}(r,0;G)\} + 7\overline{N}(r,\infty;F)\\
&+S(r,F) + S(r,G)\\
&\leq \frac{9}{2}\{\overline{N}(r,0;F) + \overline{N}(r,0;G)\} + 7\overline{N}(r,\infty;F) + S(r,F) + S(r,G)\\
&\leq \frac{9}{2}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + 7\overline{N}(r,\infty;f) + S(r,f) + S(r,g).
\end{aligned}$$

Since $F \not\equiv G$ we get by *Lemma 2.6* that $V \not\equiv 0$. Now by *Lemma 2.11* for $k=0$ and $u=1$ we have

$$(n-2)\,\overline{N}(r,\infty;f) \leq \frac{3}{2}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + S(r,f) + S(r,g).$$

Hence using (3.1) we get from above,

$$n\{T(r,f)+T(r,g)\}$$
$$\leq \frac{9}{2}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{21}{2n-4}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g),$$

which contradicts $n \geq 7$.

Case-1.3. For $u=0$, by (iii) of *Lemma 2.9*, we have

$$T(r,F)+T(r,G)$$
$$\leq 2\{N_2(r,0;F)+N_2(r,0;G)\}+3\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+12\overline{N}(r,\infty;F)$$
$$+S(r,F)+S(r,G)$$
$$\leq 7\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+12\overline{N}(r,\infty;F)+S(r,F)+S(r,G)$$
$$\leq 7\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+12\overline{N}(r,\infty;f)+S(r,f)+S(r,g).$$

Since $F \not\equiv G$, we get by *Lemma 2.6* that $V \not\equiv 0$. Hence by *Lemma 2.11* for $k=0$ and $u=0$ we have

$$(n-3)\,\overline{N}(r,\infty;f) \leq 2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g).$$

Hence from above, using (3.1), we get

$$n\{T(r,f)+T(r,g)\}$$
$$\leq 7\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{24}{n-3}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g),$$

which contradicts $n \geq 11$.

Therefore $H \not\equiv 0$ can not be possible for all three cases. Thus by *Lemma 2.5*, we have $F \equiv G$ or $FG \equiv 1$, that is $f = sg$, where s is a constant satisfying $s^n = 1$ or $fg = t$, where t is a constant satisfying $t^n = r^2$. Next we consider the following cases:

Case-2.1. Let $f = sg$. Then $f^{(k)} = sg^{(k)}$. So by $E_{f^{(k)}}(S_2,\infty) = E_{g^{(k)}}(S_2,\infty)$, we get
$\{a_1,a_2,\ldots,a_m\} = s\{a_1,a_2,\ldots,a_m\}$.

Case-2.2. Let $fg = t$. Since f, g share $(\infty,0)$, there exists an entire function h such that $f = e^h$ and $g = te^{-h}$. Therefore

$$f^{(i)} = \alpha_i f \quad and \quad g^{(i)} = \beta_i g, \quad i=1,2,\ldots,k, \tag{3.2}$$

where $\alpha_1 = h'$, $\beta_1 = -h'$ and α_k, β_k respectively satisfy the following recurrence formulas

$$\alpha_i = \alpha_1^i + P(\alpha_1), \beta_i = \beta_1^i + Q(\beta_1), i=1,2,\ldots,k, \tag{3.3}$$

where $P(\alpha_1)$ is a differential polynomial in α_1 of degree $k-1$ and $Q(\beta_1)$ is a differential polynomial in β_1 of degree $k-1$.

Without loss of generality we assume that a_1 is not an exceptional value of $f^{(k)}$. Suppose $f^{(k)}(z_0) = a_1$, then $\frac{t}{a_1}\alpha_k(z_0)\beta_k(z_0) = g^{(k)}(z_0) = a_j \in S_2$. Hence we can get points $z_1^1, z_1^2, \ldots; z_2^1, z_2^2 \ldots; \ldots; z_p^1, z_p^2 \ldots$ and values $b_1, b_2, \ldots, b_p \in S_2$, $1 \leq p \leq m$ such that $f^{(k)}(z_i^j) = a_1$, $g^{(k)}(z_i^j) = b_i$, $i=1,2,\ldots,p$; $j=1,2,\ldots$. Then

$$\left(\frac{t}{a_1}\alpha_k(z_i^j)\beta_k(z_i^j) - b_i\right) = 0, \quad for \quad i=1,2,\ldots,p; j=1,2,\ldots. \tag{3.4}$$

Suppose

$$\left(\frac{t}{a_1}\alpha_k\beta_k - b_i\right) \not\equiv 0, \quad for \quad i = 1, 2, \ldots, p.$$

Then in view of *Lemma 2.1* we get

$$\begin{aligned} & \overline{N}(r, a_1; f^{(k)}) \\ \leq & \sum_{i=1}^{p} \overline{N}(r, b_i; \frac{t}{a_1}\alpha_k\beta_k) \\ \leq & \; pT(r, \alpha_k\beta_k) + S(r, f) \\ \leq & \; p\{T(r, \alpha_k) + T(r, \beta_k)\} + S(r, f) \\ \leq & \; pk\{T(r, \alpha_1) + T(r, \beta_1)\} + S(r, f) \\ \leq & \; pk\{T(r, \frac{f'}{f}) + T(r, \frac{g'}{g})\} + S(r, f) \\ \leq & \; S(r, f), \end{aligned}$$

which is a contradiction. Hence we get

$$\left(\frac{t}{a}\alpha_k\beta_k - b_i\right) \equiv 0, \quad for\ some \quad i = 1, 2, \ldots, p, \tag{3.5}$$

which implies that $\alpha_k\beta_k$ is a non-zero constant. If α_1 and β_1 are not constants, then by *Lemma 2.12*, we have

$$\alpha_k = (\alpha_1 + \gamma_1)^k, \quad \beta_k = (\beta_1 + \gamma_2)^k, \tag{3.6}$$

where γ_1, γ_2 are small functions of α_1 and β_1, respectively. Since $\alpha_1 = -\beta_1 = h'$, we conclude that $\alpha_k\beta_k$ can not be constant, which is a contradiction. Hence one of α_1 and β_1 is constant. Thus h is a linear function. Therefore $f(z) = de^{cz}$ and $g(z) = \frac{t}{d}e^{-cz}$, where c, d are non-zero constants. Now from $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, we get $\{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_m}\}$. This completes the proof of *Theorem 1.1.* □

Proof of Theorem 1.2. Here $E_{v)}(S_1, f) = E_{v)}(S_1, g)$ and F, G share $(\infty, 0)$. Now we consider the following cases:

Case-1.1. Let $v = 3$. Suppose $F \not\equiv G$. First we assume that (i) of Lemma 2.7 holds, then

$$\begin{aligned} & T(r, F) + T(r, G) \\ & \leq 2\{N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)\} + S(r, F) + S(r, G) \\ & \leq 4\{\overline{N}(r, 0; F) + \overline{N}(r, 0; G)\} + 6\overline{N}(r, \infty; F) + S(r, F) + S(r, G) \\ & \leq 4\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} + 6\overline{N}(r, \infty; f) + S(r, f) + S(r, g). \end{aligned}$$

Since $F \not\equiv G$ we get by *Lemma 2.6* that $V \not\equiv 0$. Hence by *Lemma 2.10* for $k = 0$ and $v = 3$ we have

$$\left(n - \frac{5}{3}\right)\overline{N}(r, \infty; f) \leq \frac{4}{3}\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} + S(r, f) + S(r, g).$$

Hence from *Lemma 2.1* and from above,

$$n\{T(r,f)+T(r,g)\}$$
$$\leq 4\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{24}{3n-5}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g),$$

which contradicts $n\geq 6$. Therefore we get by *Lemma 2.7*, $FG\equiv 1$. Hence either $F\equiv G$ or $FG\equiv 1$.

Case-1.2. Let $1\leq v\leq 2$ and let $H\not\equiv 0$, then respectively by *Lemmas 2.4, 2.2, 2.3 and 2.8*, we have

$$\frac{1}{2}\{T(r,F)+T(r,G)\}$$
$$\leq N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)$$
$$+\left(\frac{v}{2}-\frac{3}{2}\right)\{\overline{N}(r,1;F\mid\geq v+1)+\overline{N}(r,1;G\mid\geq v+1)\}+S(r,F)+S(r,G)$$
$$\leq 2\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+3\overline{N}(r,\infty;F)+\left(\frac{3-v}{2v}\right)\{\overline{N}(r,0;f)+\overline{N}(r,0;g)$$
$$+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,F)+S(r,G).$$

Now we have the following subcases:

Subcase-1.2.1. Let $v=2$, then from above

$$\frac{1}{2}\{T(r,F)+T(r,G)\}$$
$$\leq \frac{9}{4}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{7}{2}\overline{N}(r,\infty;f)+S(r,f)+S(r,g).$$

Since $H\not\equiv 0$, then $F\not\equiv G$, we get by *Lemma 2.6* that $V\not\equiv 0$. So by *Lemma 2.10* for $k=0$ and $v=2$ we have

$$(n-2)\,\overline{N}(r,\infty;f)\leq \frac{3}{2}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g).$$

Hence from above and from *Lemma 2.1*,

$$\frac{n}{2}\{T(r,f)+T(r,g)\}$$
$$\leq \frac{9}{4}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{21}{4n-8}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g),$$

which contradicts $n\geq 7$.

Subcase-1.2.2. Let $v=1$, then from above

$$\frac{1}{2}\{T(r,F)+T(r,G)\}$$
$$\leq 3\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+5\overline{N}(r,\infty;f)+S(r,f)+S(r,g).$$

Since $F\not\equiv G$ we get by *Lemma 2.6* that $V\not\equiv 0$. So by *Lemma 2.10* for $k=0$ and $v=1$ we have

$$(n-3)\,\overline{N}(r,\infty;f)\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g).$$

Hence from above and from *Lemma 2.1*,

$$\frac{n}{2}\{T(r,f)+T(r,g)\}$$
$$\leq 3\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+\frac{10}{n-3}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g),$$

which contradicts $n \geq 10$.

Therefore $H \equiv 0$ for last two subcases. Hence by *Lemma 2.5*, we have $F \equiv G$ or $FG \equiv 1$.

Next we can prove the theorem in the line of proof of *Theorem 1.1*. □

Proof of Corollary 1.1. Let $S_2 = \{a\}$. So by the conclusion (1) of *Theorem 1.1* for $m = 1$, we have $f \equiv sg$, $S_2 = sS_2$; that is $a = sa$. Therefore $s = 1$. Hence $f \equiv g$.

Now by the conclusion (2) of *Theorem 1.1*, $f = de^{cz}$, $g = \frac{t}{d}e^{-cz}$ and since S_2 contains one element, hence $a = (-1)^k c^{2k} t \frac{1}{a}$. Thus the *Corollary 1.1* is proved. □

Proof of Corollary 1.2. Let $m \geq 2$. Now by conclusion (1) of *Theorem 1.1*, we get $f \equiv sg$, $S_2 = sS_2$. Then we have the following cases:

Case-1.1. Let $a_i = sa_i$, $i = 1, 2, \ldots, m$. Therefore $s = 1$. Hence $f \equiv g$.

Case-1.2. Let $S_2^{'} = \{b_1, b_2, b_3, \ldots, b_l\} \subseteq \{a_1, a_2, \ldots, a_m\}$. Then without loss of generality we may assume that $b_1 = sb_2, b_2 = sb_3, \ldots, b_{l-1} = sb_l, b_l = sb_1$, where $2 \leq l \leq m$. Clearly we have $s^l = 1$. We now claim that, S_2 will be partitioned into disjoint classes of subsets each containing exactly l elements. Otherwise we get a class of q elements, $S_2^{''} = \{c_1, c_2, \ldots, c_q\}$ of S_2, satisfying $c_1 = sc_2, c_2 = sc_3, \ldots, c_{q-1} = sc_q, c_q = sc_1$ where $q \neq l$. If $q < l$, then noting that $s^q = 1$, from $S_2^{'}$, we get $b_1 = b_{q+1}$ which is a contradiction. So we see that l is the least value for which such a partition $S_2^{'}$ of S_2 exists. If $q > l$, then, since $s^l = 1$, it follows that l divides q. Hence we can again partition $S_2^{''}$ into subclasses each containing exactly l elements. This means that every partition of S_2 contains l elements and l is the least value for a partition like $S_2^{'}$ of S_2. Hence l divides m. Also as $s^n = 1$, l can not be greater than n, since otherwise by the same argument as above, we get $b_1 = b_{n+1}$, a contradiction. Thus l also divides n.

Again by the conclusion (2) of *Theorem 1.1*, we get $f = de^{cz}$, $g = \frac{t}{d}e^{-cz}$ and since $S_2 = \{a_1, a_2, \ldots, a_m\}$, we have $a_i = (-1)^k c^{2k} t \frac{1}{a_j}$, $1 \leq i, j \leq m$. Let $i \neq j \neq k$, where $1 \leq i, j, k \leq m$, if $a_i = (-1)^k c^{2k} t \frac{1}{a_j}$ and $a_j = (-1)^k c^{2k} t \frac{1}{a_k}$, then $a_i = a_k$, a contradiction. Hence if $a_i = (-1)^k c^{2k} t \frac{1}{a_j}$, then the only possibility is $a_j = (-1)^k c^{2k} t \frac{1}{a_i}$, where $1 \leq i, j \leq m$. Let $\{b_1, b_2, b_3, \ldots, b_m\} = \{a_1, a_2, a_3, \ldots, a_m\}$. Then we consider the following cases:

Case-2.1. Let m be odd. Then without loss of generality we consider $b_1 = (-1)^k c^{2k} t \frac{1}{b_1}$ and $b_i = (-1)^k c^{2k} t \frac{1}{b_{i+1}}$ where $m \geq i \geq 2$ and i is even, that means $b_1^2 = (-1)^k c^{2k} t$ and $b_2 b_3 = b_4 b_5 = \ldots = b_{m-1} b_m = (-1)^k c^{2k} t$.

Case-2.2. Let m be even. Then also without loss of generality we consider $b_1 = (-1)^k c^{2k} t \frac{1}{b_1}$, $b_2 = (-1)^k c^{2k} t \frac{1}{b_2}$ or $b_1 = (-1)^k c^{2k} t \frac{1}{b_2}$ and $b_i = (-1)^k c^{2k} t \frac{1}{b_{i+1}}$ where $m \geq i \geq 3$ and i is odd, that means $b_1^2 = b_2^2 = (-1)^k c^{2k} t$ or $b_1 b_2 = (-1)^k c^{2k} t$ and $b_3 b_4 = b_5 b_6 = \ldots = b_{m-1} b_m = (-1)^k c^{2k} t$.

This completes the proof of *Corollary 1.2*. □

Proof of Corollary 1.3. The proof of the corollary is similar to that of *Corollary 1.1*. □

Proof of Corollary 1.4. The proof of the corollary is similar to that of *Corollary 1.2*. □

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