Poincare Journal of Analysis & Applications
Vol. 8, No. 1(II) (2021), 39-52

ON THE SPECTRAL PROPERTIES OF NON- SELF-ADJOINT ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL OPERATORS

REZA ALIZADEH[†] AND ALI SAMERIPOUR

Date of Receiving : 22. 11. 2020
Date of Revision : 20. 05. 2021
Date of Acceptance : 30. 08. 2021

Abstract. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. In this paper, we will study the spectral properties of degenerate non- selfadjoint elliptic systems of differential operator P acting in Hilbert space $H_\ell = L^2(\Omega)^\ell$ and defined by $(Pu)(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x)a_{ij}(x)Q(x)u'_{x_i}(x)\right)'_{x_j}$, with Dirichlet-type boundary conditions, indeed this paper was written in continuing on earlier our papers [10], [14], the paper is sufficiently more general than the paper [10]. Here $c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j}$ $\quad (s = (s_1,\dots,s_n) \in \mathbf{C}^n, \quad x \in \Omega), \quad \rho(x) = \operatorname{dist}\{x, \partial\Omega\}, \quad a_{ij}(x) = \overline{a_{ji}(x)} \in C^2(\overline{\Omega}), \quad 0 \le \alpha < 1$. Suppose that $Q(x) \in C^2(\overline{\Omega},\ End\ \mathbf{C}^\ell)$ such that for each $x \in \overline{\Omega}$ the matrix function $Q(x)$ has non-zero simple eigenvalues $\mu_j(x) \in C^2(\overline{\Omega})$ $(1 \le j \le \ell)$ arranged in the complex plane such that: $\mu_1(x),\dots,\mu_\ell(x) \in \mathbf{C}\backslash\Phi$, where $\Phi = \{z \in \mathbf{C} : \ |arg\ z| \le \varphi\},\ \varphi \in (0,\pi)$.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ (i.e., $\partial\Omega \in C^\infty$). We introduce the weighted Sobolev space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$ as the space of complex value functions $u(x)$ defined on Ω with the finite norm:

$$|u|_+ = \left(\sum_{i=1}^{n}\int_\Omega \rho^{2\alpha}(x)|u'_{x_i}(x)|^2dx + \int_\Omega |u(x)|^2dx\right)^{1/2}.$$

We denote by $\overset{\circ}{\mathcal{H}}$ the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}$ with respect to the above norm. i.e., $\overset{\circ}{\mathcal{H}}$ is the closure of $C_0^\infty(\Omega)$ in $W^2_{2,\alpha}(\Omega)$. The notion $C_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in Ω. In this paper, we investigate the

2010 *Mathematics Subject Classification.* 35JXX, 35PXX .
Key words and phrases. Resolvent, Eigenvalues, Non-selfadjoin elliptic differential operators, spectrum .
Communicated by: Sohrab Azimpour
[†] *Corresponding author.*

spectral properties, in particular we estimate the resolvent of a non-selfadjoint elliptic differential operator of type

$$(Pu)(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x)a_{ij}(x)Q(x)u'_{x_i}(x)\right)'_{x_j} \tag{1.1}$$

acting in Hilbert space $H_\ell = L^2(\Omega)^\ell$ with Dirichlet-type boundary conditions. Here $\rho(x) = \text{dist}\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $a_{ij}(x) = \overline{a_{ji}(x)}$ $(i, j = 1, \dots, n)$, $a_{ij}(x) \in C^2(\overline{\Omega})$ $(i, j = 1, \dots, n)$, and the functions $a_{ij}(x)$ satisfies the uniformly elliptic condition, i.e., there exists $c > 0$ such that:

$$c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j} \quad (s = (s_1, \dots, s_n) \in \mathbf{C}^n, \ \ x \in \Omega).$$

Furthermore, suppose that $Q(x) \in C^2(\overline{\Omega},\ End\ \mathbf{C}^\ell)$ such that for each $x \in \overline{\Omega}$ the matrix function $Q(x)$ has non-zero simple eigenvalues $\mu_j(x) \in C^2(\overline{\Omega})$ $(1 \le j \le \ell)$ arranged in the complex plane in the following way:

$$\mu_1(x), \dots, \mu_\ell(x) \in \mathbf{C}\backslash\Phi,$$

where $\Phi = \{z \in \mathbf{C} :\ |arg\ z| \le \varphi\}$, $\varphi \in (0, \pi)$. (i.e., the eigenvalues $\mu_j(x)$ of $Q(x)$ lie on the complex plane and outside of the closed angle Φ).

For a closed extension of the operator P with respect to space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$ above, we need to extend its domain to the closed domain

$$D\,(P) = \left\{ y \in \overset{\circ}{\mathcal{H}}_\ell \cap W^2_{2,\ loc}(\Omega)^\ell :\ \sum_{i,j=1}^{n} \left(\rho^{2\alpha}a_{ij}Q{y'}_{x_i}\right)_{x_j} \in H_\ell \right\},$$

(see [8]) where the local space $W^2_{2,\ loc}(\Omega)$ is the functions $u(x)$ $(x \in \Omega)$ in this form $W^2_{2,loc}(\Omega) = \{u(x) :\ \sum_{i=0}^{2} \int_J |u^{(i)}(x)|^2 dx < \infty, \quad J \subset \Omega, \quad \text{open}\}$. Here, and in the sequel the value of the function arg $z \in (-\pi, \pi]$, and $\|T\|$ denotes the norm of the bounded operator $T : H \longrightarrow H$.

To get a feeling for the history of the subject under study, refer to our earlier papers [10], [14]. Indeed this paper was written in continuing on earlier our papers, the paper is sufficiently more general than earlier our papers, which here, we obtain the resolvent estimate of the operator P, that satisfying the special and general conditions, The paper consists of four sections. Section 1 is devoted to introduction. In Section 2, we have Theorem 2.1 on the resolvent estimate of the differential operator P, acting in H in the certain case (i.e., in this case, we will study Theorem 2.1 under assumption (2.2)). In Section 3, we have Theorem 3.1 the resolvent estimate of the differential operator P, acting in H in the general case (i.e., in this case, we will study Theorem 3.1 in contrast to Theorem 2.1. in other words, Theorem 3.1 does not include assumption (2.2) of Theorem 2.1). In Section 4, we have Theorem 4.1 on the resolvent estimate of the differential operator in H_ℓ.

2. The Resolvent Estimate of Degenerate Elliptic Differential Operators in H in Some Special Cases

Theorem 2.1. *Let $A = P$ in (1.1), i.e., assume that the operator A acting in Hilbert space $H = L^2(\Omega)$ with Dirichlet-type boundary conditions, and the sector Φ be defined as in Section 1. Let the complex function $q(x)$ satisfies the following conditions*

$$q(x) \in C^1(\bar{\Omega}),\ q(x) \in \mathbb{C}\backslash\Phi,\ (\forall x \in \bar{\Omega}), \tag{2.1}$$

$$|arg\{q(x_1)q^{-1}(x_2)\}| \leq \frac{\pi}{8}, \quad (\forall\ x_1, x_2 \in \bar{\Omega}). \tag{2.2}$$

Then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse $(A - \lambda I)^{-1}$ exists and is continuous in H, and the following estimates are valid

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}\ (\lambda \in \Phi,\ |\lambda| > C_\Phi), \tag{2.3}$$

$$\|\rho^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1}\|\ \leq M'_\Phi |\lambda|^{-\frac{1}{2}}\ (\lambda \in \Phi,\ |\lambda| > C_\Phi), \tag{2.4}$$

$$for\, i = 1, \ldots, n$$

where M_Φ, $C_\Phi > 0$ are sufficiently large numbers depending on Φ.

Proof. Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.3). So, as in Section 1 for a closed extension the operator A (for more explain see chapter 6 of [8]), we need to extend its domain to the closed set

$$D(A) = \{u \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\mathrm{loc}}(\Omega) : \quad Au \in H\}.$$

Let the operator A, now satisfy (2.1), (2.2). Then there exists a complex number $Z \in \mathbb{C}$ (noticed that we can take $Z = e^{i\gamma}$, for a fix real $\gamma \in (-\pi, \pi]$), such that:

$$c' \leq Re\{Zq(x)\},\quad c'|\lambda| \leq -Re\{Z\lambda\}, \quad c' > 0\ (\forall\ x \in \bar{\Omega},\ \lambda \in \Phi). \tag{2.5}$$

In view of the uniformly elliptic condition, we have

$$c|s|^2 = c\sum_{i=1}^{n} |s_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j},\ (c > 0,\ s = (s_1, \ldots, s_n) \in \mathbb{C}^n,\ x \in \Omega)$$

take $s_i = y'_{x_i}$ implies that $c\sum\limits_{i=1}^{n} |y'_{x_i}(x)|^2 \leq \sum\limits_{i,j=1}^{n} a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}$. From this, and according to $c' \leq Re\{Zq(x)\}$ in (2.4), we then multiply these two positive relations with each other implies that

$$c_1\sum_{i=1}^{n} |y'_{x_i}(x)|^2 \leq ReZq(x)\sum_{i,j=1}^{n} a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}; y \in D(A).$$

Multiply both sides of the latter relation by the positive term $\rho^{2\alpha}(x)$, and then integrate from both sides, we will have

$$c_1\sum_{i=1}^{n}\int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2dx \leq ReZ\sum_{i,j=1}^{n}\int_\Omega \rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx.$$

Now by applying the integration by parts, and using Dirichlet-type condition, the right sides of the latter relation without multiple ReZ becomes:

$$\sum_{i,j=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx$$
$$= -\sum_{i,j=1}^{n}\int_{\Omega}(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x))'_{x_j}\overline{y}(x)dx$$
$$= (-\sum_{i,j=1}^{n}(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x))'_{x_j}\ ,\ y(x)) = (Ay,y). \tag{2.6}$$

Notice that $(Ay)(x) = -\sum\limits_{i,j=1}^{n}\left(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'{}_{x_i}(x)\right)'{}_{x_j}$.

Here, the the symbol (,) denotes the inner product in H.

Notice that the above equality in (2.6) obtains by the well known theorem of the m-sectorial operators which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for more explanation see the well known Theorem 2.1, chapter 6 of [8]). This why we extend the domain of the operator A to the closed domain in space $\mathcal{H}$ above. Therefore

$$c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'{}_{x_i}(x)|^2dx \le ReZ(Ay\ ,\ y)$$

from (2.4) we have: $c'|\lambda| \le -Re\{Z\lambda\}$, $c' > 0$, $\forall\lambda \in \Phi$. Multiply this inequality by $\int_{\Omega}|y(x)|^2\,dx = (y\ ,\ y) = \|y\|^2 > 0$. It follows that

$$c'|\lambda|\int_{\Omega}|y(x)|^2\,dt \le -Re\{Z\lambda\}(y\ ,\ y).$$

From this and the above inequality we will have

$$\begin{aligned} c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'_{x_i}(x)|^2dx + c'|\lambda|\int_{\Omega}|y(x)|^2dx &\le Re\{Z(Ay,y) - Z\lambda(y,y)\} \\ &= Re\{Z((A-\lambda I)y,y)\} \\ &\le \|Z\|\|y\|\|(A-\lambda I)y\| \\ &= \|y\|\|(A-\lambda I)y\|; \quad (2.7) \end{aligned}$$

i.e.,

$$c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'_{x_i}(x)|^2dx + c'|\lambda|\int_{\Omega}|y(x)|^2\,dx \le \|y\|\ \|(A\ -\ \lambda\ I)y\|.$$

Since $c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'_{x_i}(x)|^2dx$ is positive. We will have either $c'|\lambda|\|y(x)\|^2 = |\lambda|\int_{\Omega}|y(x)|^2\,dx \le \|y\|\|(A\ -\lambda\ I)y\|$. Or

$$|\lambda|\|y(x)\| \le M_{\Phi}\|(A\ -\lambda\ I)y\|. \tag{2.8}$$

This inequality ensures that the operator $(A\ -\ \lambda\ I)$ is one to one, which implies that $Ker(A\ -\ \lambda\ I) = 0$. Therefore the inverse operator $(A -\ \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (2.3) of Theorem 2.1. To prove (2.3), we set $y=(A-\lambda I)^{-1}f,\ f\in H$, (2.8) implies that

$$|\lambda|\int_\Omega |(A-\lambda I)^{-1}f|^2dx \le M_\Phi \left\|(A-\lambda I)^{-1}f\right\| \ \left\|(A-\lambda I)(A-\lambda I)^{-1}f\right\|.$$

Since $(A-\lambda I)(A-\lambda I)^{-1}f=I(f)=f$. Then

$$|\lambda|\int_\Omega |(A-\lambda I)^{-1}f|^2dx \le M_\Phi \left\|(A-\lambda I)^{-1}f\right\| \ \|f\|.$$

So

$$|\lambda|\left\|(A-\lambda I)^{-1}(f)\right\|^2 \le M_\Phi \left\|(A-\lambda I)^{-1}(f)\right\| \|f\|.$$

Which this implies that $|\lambda|\left\|(A-\lambda I)^{-1}(f)\right\| \le M_\Phi\|f\|$. Because we know that $\lambda\neq 0$ then $\left\|(A-\lambda I)^{-1}(f)\right\| \le M_\Phi|\lambda|^{-1}\|f\|$; i.e., $\|(A-\lambda I)^{-1}\|\le M_\Phi|\lambda|^{-1}$. This estimate completes the proof of the assertion of Theorem 2.1 together with the estimate (2.3). Now, we start to prove the estimate (2.4) of Theorem 2.1. As in the above argument, we drop the positive term $c'|\lambda|\int_\Omega |y(x)|^2\,dx$ from

$$c_1\sum_{i=1}^n\int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2dx + c'|\lambda|\int_\Omega |y(x)|^2\,dx \le \|y\|\ \|(A\ -\ \lambda\ I)y\|.$$

It follows that

$$c_1\sum_{i=1}^n\int_\Omega \rho^{2\alpha}(x)|{y'}_{x_i}(x)|^2dx \le \|y\|\ \ \|(A-\lambda I)y\|.$$

Equivalently

$$c_1\|\rho^\alpha\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}f\|^2 \le \|y\|\ \|(A\ -\ \lambda\ I)y\|.$$

Set $y=(A-\lambda I)^{-1}f,\ f\in H$ in the latter relation, and proceeding by similar calculation as in the proof (2.3) we then obtain:

$$c_1\|\rho^\alpha\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}f\|^2 \le \|(A-\lambda I)^{-1}f\|\ \|(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f\|.$$

Since $(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f=I(f)=f$, then

$$c_1\|\rho^\alpha\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}f\|^2 \le \|(A-\lambda I)^{-1}\|\|f\|^2,$$

consequently, by (2.3) this implies that

$$c_1\|\rho^\alpha\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}f\|^2 \le M_\Phi|\lambda|^{-1}\|f\|^2$$

to this end we will have

$$\|\rho^\alpha\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}\| \le M'_\Phi|\lambda|^{-\frac{1}{2}}. \tag{2.7}$$

Thus, here the proof of the estimate (2.4) is finished; i.e., this completes the proof of Theorem 2.1. $\square$

Now let the condition (2.2) does not hold. Then we will have the following statement:

3. The Resolvent Estimate of Some Classes of Degenerate Elliptic Differential Operators in H

In this section, we will derive a new general theorem by dropping the assumption (2.2) from Theorem 2.1 in Section 2.

Theorem 3.1. *As in Section 1, let Φ be some closed sector with vertex at 0 in the complex plane (for more explain see [8]), let the complex function $q(x)$ satisfies*

$$q(x) \in C^1(\overline{\Omega}),\ q(x) \in \mathbf{C}\backslash\Phi,\ (\forall x \in \overline{\Omega}). \tag{3.1}$$

Then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse $(A - \lambda I)^{-1}$ exists and is continuous in H, and the following estimates holds:

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1},\ (\lambda \in \Phi,\ |\lambda| > C_\Phi) \tag{3.2}$$

where M_Φ, $C_\Phi > 0$ are sufficiently large numbers depending on Φ.

Proof. Let us (2.2) does not satisfy. To prove the assertion of theorem 3.1, we construct the functions
$\varphi_1(x), \ldots, \varphi_m(x),\ \ q_1(x), \ldots, q_m(x)$ so that each one of the functions
$q_1(x), \ldots, q_m(x)$ $(x \in \overline{\Omega})$, as the function $q(x)$ in Theorem 2.1 satisfies (2.2). Therefore, let

$$\varphi_1(x), \ldots, \varphi_m(x),\ \ q_1(x), \ldots, q_m(x) \in C_0^\infty(\Omega),$$

satisfy

$$0 \leq \varphi_r(x),\ \ r = 1, \ldots, m,\ \ \varphi_1^2(x) + \ldots + \varphi_m^2(x) \equiv 1 \quad (x \in \overline{\Omega})$$

$$\frac{d}{dt}\varphi_r(x) \in C_0^\infty(\Omega),\ q_r(x) = q(x),\ \forall x \in supp\ \varphi_r$$

$$q_r(x) \in \mathbf{C}\backslash\Phi,\ (\forall x \in \overline{\Omega}),\ r = 1, \ldots, m.$$

$$|\arg\{q_r(x_1)q_r^{-1}(x_2)\}| \leq \frac{\pi}{8}, \quad (\forall\ x_1, x_2 \in supp\ \varphi_r), \quad r = 1, \ldots, m.$$

In view of Theorem 2.1, and by (2.3), and (2.4), set $A_r = A$ in the definition of the differential operator, implies that

$$A_r u(x) = -\sum_{i,j=1}^n \left(\rho^{2\alpha}(x) a_{ij}(x) q_r(x) u'_{x_i}(x)\right)'_{x_j} \text{ acting in } H \ \text{ where}$$

$$D(A_r) = \{u \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\ loc}(\Omega) :\ A_r u \in H\}.$$

Due to the assertion of Theorem 2.1, for $0 \neq \lambda \in \Phi$ the inverse $(A_r - \lambda I)^{-1}$ exists and is continuous in space $H = L^2\,(\Omega)$, and satisfies

$$\|(A_r - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}, \quad \|\rho^\alpha \frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\| \leq M'_\Phi |\lambda|^{-\frac{1}{2}}\ (\lambda \in \Phi,\ |\lambda| > C_\Phi),$$
$$(0 \neq \lambda \in \Phi). \tag{3.3}$$

Let us introduce

$$G(\lambda) = \sum_{r=1}^m \varphi_r (A_r - \lambda I)^{-1} \varphi_r, \tag{3.4}$$

Here φ_r is the multiplication operator in H by the function $\varphi_r(x)$. Consequently, easily we will have

$$\begin{aligned}(A-\lambda I)G(\lambda) &= I+\rho^{2\alpha-1}(x)\sum_{r=1}^{m}\beta_r(x)(A_r-\lambda I)^{-1}\varphi_r \\ &+ \rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{i_r}(x)\frac{\partial}{\partial x_i}(A_r-\lambda I)^{-1}\varphi_r \quad (3.5)\end{aligned}$$

where $\beta_r, \gamma_{i_r} \in L_\infty(\Omega)$, supp β_r and supp γ_{i_r} are contained in supp φ_r.

First we prove that $G(\lambda)u \in D(A_r), \quad (\forall u \in \mathcal{H})$. Now from the definition of $D(A_r)$ it follows that

$$\varphi_r(A_r-\ \lambda)^{-1}\varphi_r \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\ \mathrm{loc}}(\Omega), \qquad (r=1,2,\ldots,m).$$

Therefore from this implies that $G(\lambda)u \in D(A_r)$;

$$\begin{aligned}(A-\lambda I)G(\lambda)u &= A(G(\lambda)u)-\lambda G(\lambda)u \\ &= -\sum_{i,j=1}^{n}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(G(\lambda)u)'_{x_i}\}'_{x_j}-\lambda G(\lambda)u \\ &= -\sum_{i,j=1}^{n}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x) \\ &\quad (\sum_{r=1}^{m}\varphi_r(A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_j}-\lambda G(\lambda)u \\ &= -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}((A_r-\lambda I)^{-1}) \\ &\quad \varphi_r u)\}'_{x_j}-\lambda G(\lambda)u \\ &\quad -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)((A_r-\lambda I)^{-1})\varphi_r u))'{}_{x_i}\}'{}_{x_j}.\end{aligned}$$

We write the second term as

$$L_1+L_2=-\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}a_{ij}q_r\varphi_r((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_j}$$

where

$$L_1=-\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\varphi_r\}'_{x_i}\{\rho^{2\alpha}a_{ij}q_r((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\},$$

and

$$
\begin{aligned}
L_2 &= -\sum_{i,j=1}^{n}\sum_{r=l}^{m}\{\varphi_r\}.\{\rho^{2\alpha}a_{ij}q_r((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_j} \\
&= \sum_{r=1}^{m}\{\varphi_r\}.(-\sum_{i,j=1}^{n}\{\rho^{2\alpha}a_{ij}q_r((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_j}) \\
&= \sum_{r=1}^{m}\{\varphi_r\}(A_r\{(A_r-\lambda I)^{-1}\varphi_r u\}) \\
&= \sum_{r=1}^{m}\{\varphi_r\}((A_r-\lambda I)+\lambda I)\{(A_r-\lambda I)^{-1}\varphi_r u\}) \\
&= \sum_{r=1}^{m}\{\varphi_r\}I\varphi_r u+\lambda\sum_{r=1}^{m}\varphi_r(A_r-\lambda I)^{-1}\varphi_r=\sum_{r=1}^{m}{\varphi_r}^2u+\lambda G(\lambda)u=u+\lambda G(\lambda)u.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(A-\lambda I)G(\lambda)u &= -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i} \\
&\quad ((A_r-\lambda I)^{-1}\varphi_r u)\}'_{x_j}+L_1+L_2-\lambda G(\lambda)u \\
&= -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}((A_r-\lambda I)^{-1}\varphi_r u)'_{x_j} \\
&\quad -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_i} \\
&\quad ((A_r-\lambda I)^{-1}\varphi_r u)+L_1+L_2-\lambda G(\lambda)u
\end{aligned}
$$

and since

$$
\begin{aligned}
&-\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}((A_r-\lambda I)^{-1}\varphi_r u)'_{x_j} \\
&=-\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i},
\end{aligned}
$$

therefore we have

$$
\begin{aligned}
(A-\lambda I)G(\lambda)u &= -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i} \\
&- \sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_i}((A_r-\lambda I)^{-1}\varphi_r u) \\
&\quad -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\varphi_r\}'_{x_i}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\}+u \\
&\quad +\lambda G(\lambda)u-\lambda G(\lambda)u.
\end{aligned}
$$

Then

$$
\begin{aligned}
(A-\lambda I)G(\lambda)u &= -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\} \\
&\quad ((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i} \\
&\quad -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_i} \\
&\quad ((A_r-\lambda I)^{-1}\varphi_r u) \\
&\quad -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\varphi_r\}'_{x_i}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x) \\
&\quad ((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\}+u.
\end{aligned}
$$

By factoring out $\rho^{2\alpha}$ we will have

$$
\begin{aligned}
(A-\lambda I)G(\lambda)u &= \rho^{2\alpha}(x)(-\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i} \\
&\quad +a_{ij}(x)q_r(x)\{\varphi_r\}'_{x_i}((A_r-\lambda I)^{-1}\varphi_r u)'_{x_i}\} \\
&-\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_i}((A_r-\lambda I)^{-1}\varphi_r u)+u.
\end{aligned}
$$

Since $i=1,\ldots,n$ and $j=1,\ldots,n$ are dummy variables, so implies that

$$
\begin{aligned}
(A-\lambda I)G(\lambda)u &= u+\rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{i_r}(x)\frac{\partial}{\partial x_i}(A_r-\lambda I)^{-1}\varphi_r \\
&\quad -\sum_{i,j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_i} \\
&\quad ((A_r-\lambda I)^{-1}\varphi_r u).
\end{aligned}
$$

Now since

$$
\begin{aligned}
\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_j} &= \frac{\partial}{\partial x_j}(\rho^{2\alpha}(x))(a_{ij}(x)q_r(x)(\varphi_r)'_{x_i} \\
&\quad +\rho^{2\alpha}(x)(a_{ij}(x)q_r(x)(\varphi_r)'_{x_i})'_{x_j}.
\end{aligned}
$$

$$
|\frac{\partial}{\partial x_j}\rho^{2\alpha}(x)|\leq M_\Phi\rho^{2\alpha-1}(x)\ \ ;\quad \rho^{2\alpha}(x)=\rho^{2\alpha-1}(x)\rho(x)
$$

Therefore the above equality is equivalent to

$$
\rho^{2\alpha-1}(x)\beta'_r+\rho^{2\alpha-1}(x)\rho(x)\beta''_r.
$$

To the end, we will have

$$
\begin{aligned}
(A-\lambda I)G(\lambda) &= I+\rho^{2\alpha-1}(x)\sum_{r=1}^{m}\beta_r(x)(A_r-\lambda I)^{-1}\varphi_r \\
&+\ \rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{i_r}(x)\frac{\partial}{\partial x_i}(A_r-\lambda I)^{-1}\varphi_r.
\end{aligned}
$$

Now the proof of (3.5) is completed. Let us take the right side of (3.5) equals to $I+T(\lambda)$ Thus, we will have

$$(A-\lambda I)G(\lambda)=I+T(\lambda). \tag{3.6}$$

Now according to section 2 if we put $A=A_r$ for $r=1,\ldots,m$ in (2.2) we will have

$$\|(A_r-\lambda I)^{-1}\|\leq M1_S|\lambda|^{-1},\ \|\rho^\alpha\frac{\partial}{\partial x_i}(A_r-\lambda I)^{-1}\|\leq M'_\Phi\|\lambda\|^{-\frac{1}{2}}.$$

Owing to the definition of $T(\lambda)$ in the (3.5) easily it follows that

$$\|T(\lambda)\|\leq M_\Phi|\lambda|^{-\frac{1}{2}}(\lambda\in\Phi,\ \ |\lambda|>1). \tag{3.7}$$

Since $|\lambda|$ is sufficiently large number easily implies that $\|T(\lambda)\|<\frac{1}{2}<1$, from this and using the well known theorem in the operator theory we conclude that $I+T(\lambda)$ and so $(A-\lambda I)G(\lambda)$ are invertible. Hence, $((A-\lambda I)G(\lambda))^{-1}$ exists and equals to

$$(G(\lambda))^{-1}(A-\lambda I)^{-1}=(I\ +\ T(\lambda))^{-1}, \tag{3.8}$$

By adding $+I$ and $-I$ to the right side of the (3.7) it follows that

$$(G(\lambda))^{-1}(A-\lambda I)^{-1}=(I\ +\ T(\lambda))^{-1}\ \ -I\ \ +I.$$

We now set

$$F(\lambda)=(I\ +\ T(\lambda))^{-1}\ -\ I.$$

Then

$$(G(\lambda))^{-1}(A-\lambda I)^{-1}=I\ +\ F(\lambda).$$

In view of $\|T(\lambda)\|<1$ and (3.7) by estimating the following geometric series for the $F(\lambda)$ we will have

$$\begin{aligned}\|F(\lambda)\|\leq\sum_{i=2}^{+\infty}\|T^k(\lambda)\| &\leq \|T(\lambda)\|^2(1+\|T(\lambda)\|+\|T(\lambda)\|^2+\ldots)\\ &\leq \|T(\lambda)\|^2M_\Phi(1+1/2+1/4+\ldots)\leq 2M_\Phi(M'_\Phi|\lambda|^{-1/2})^2\end{aligned}$$

i.e., $\|F(\lambda)\|\leq 2M1_\Phi|\lambda|^{-1}$. By (3.3) and $\|(A_r-\lambda I)^{-1}\|\leq M1_\Phi|\lambda|^{-1}$, we will have

$$\|G(\lambda)\|=\|\sum_{r=1}^{m}\varphi_r(A_r-\lambda I)^{-1}\varphi_r\|\leq M''_\Phi\|(A_r-\lambda I)^{-1}\|\leq M''_\Phi M1_\Phi|\lambda|^{-1};$$

i.e., $\|G(\lambda)\|\leq M2_\Phi|\lambda|^{-1}$. Now from (3.8) we have

$$(A-\lambda I)^{-1}=G(\lambda)(I\ +\ T(\lambda))^{-1}=G(\lambda)(I+F(\lambda))$$

Therefore

$$\|(A-\lambda I)^{-1}\|=\|G(\lambda)\|\|(I\ +\ F(\lambda))\|\leq M2_\Phi|\lambda|^{-1}\|(1+2M1_\Phi|\lambda|^{-1});$$

i.e., here the assertion of Theorem 3.1 is proved. Therefore to complete the proof Theorem 3.1 we must prove the estimate in (3.2). to the end we have according to latter inequality we have

$$\|(A-\lambda I)^{-1}\|\leq M2_\Phi|\lambda|^{-1}+2M2_\Phi M1_\Phi|\lambda|^{-1}|\lambda|^{-1},$$

and since $|\lambda|^{-1}|\lambda|^{-1}=|\lambda|^{-2}\leq|\lambda|^{-1}$, it follows that

$$\|(A-\lambda I)^{-1}\|\leq M_\Phi|\lambda|^{-1},\ \ (|\lambda|\geq C,\qquad \lambda\in\Phi).$$

This complete the proof of Theorem 3.1. □

4. The Resolvent Estimate of the Differential Operator in H_ℓ

As in Section 1, let the differential operator

$$(Pu)(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x)a_{ij}(x)Q(x)u'_{x_i}(x)\right)'_{x_j}$$

acting in Hilbert space $H_\ell = L^2(\Omega)^\ell$ with Dirichlet-type boundary conditions, and suppose that $Q(x) \in C^2(\overline{\Omega},\ End\,\mathbf{C}^\ell)$ such that for each $x \in \overline{\Omega}$ the matrix function $Q(x)$ has non-zero simple eigenvalues $\mu_j(x) \in C^2(\overline{\Omega})$ $(1 \le j \le \ell)$ arranged in the complex plane in the following way:

$$\mu_1(x), \ldots, \mu_\ell(x) \in \mathbf{C}\backslash\Phi, \tag{4.1}$$

where

$$\Phi = \{z \in \mathbf{C} :\ |arg\ z| \le \varphi\},\ \varphi \in (0, \pi).$$

Furthermore suppose that for $j = 1, \ldots, \ell$ we have

$$\mu_j(x) \in C^1(\overline{\Omega}),\ \mu_j(x) \in \mathbf{C}\backslash\Phi,\ (\forall x \in \overline{\Omega}), \tag{4.1}$$

$$|arg\{\mu_j(x_1)\mu_j^{-1}(x_2)\}| \le \frac{\pi}{8}, \quad (\forall\, x_1, x_2 \in \overline{\Omega}). \tag{4.2}$$

Now according to Theorem 2.1, but here instead of the operator A which acting in the space $H = L^2(\Omega)$, let the operator P acting in the space $H_\ell = L^2(\Omega)^\ell$, now by the assumption of Section 1, we will have the following theorem in the general case:

Theorem 4.1. *Let the assumptions of Section 1, and so assumptions (4.1), (4.2) hold for the operator P as in (1,1), then for sufficiently large in modulus $\lambda \in \Phi$, the inverse $(P-\lambda I)^{-1}$ exists and is continuous in the space $H_\ell = L^2(\Omega)^\ell$ and the following estimate holds;*

$$\|(P - \lambda I)^{-1}\| \le M_\Phi |\lambda|^{-1} \tag{4.2}$$

where M_Φ, $C_\Phi > 0$ is sufficiently large number depending on Φ and $|\lambda| > C_\Phi$.

Proof. Now by applying the eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ of the matrix function $Q(x)$ we defined the operators $P_1, \ldots, P_\ell$ such that:

$$(P_k u)(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x)a_{ij}(x)\mu_k(x){u'}_{x_i}(x)\right)'_{x_j} \quad (k = 1, \ldots, \ell),$$

where its extension domains are:

$$D(P_k) = \{y \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\ loc}(\Omega) :\ (P_k y)(x) \in H\},$$

which as the operator A in Theorem 2.1, the operators P_k $k = 1, \ldots, \ell$, acting in the space $H = L^2(\Omega)$ (Notice that here the operators P_j are the same of the operator A in Section 3, i.e., for defined the operators P_k, we just change the function $q(x)$ in the operator A by the eigenvalues functions $\mu_k(x)$ $k = 1, \ldots, \ell$ of matrix $Q(x)$).

The conditions which we consider on the eigenvalues $\mu_j(x)$ of the matrix function $Q(x)$ in Section 1 guarantee that one can convert the matrix $Q(x)$ to the diagonal form

$$Q(x) = U(x)\Lambda(x)U^{-1}(x),\ \text{where}\ U(x), U^{-1}(x) \in C^2([0,1], End\,\mathbf{C}^\ell)$$

where

$$\Lambda(x) = diag\{\mu_1(x), \ldots, \mu_\ell(x)\}.$$

Consider space $H_\ell = H \oplus \cdots \oplus H$ (ℓ-times). Put

$$\Gamma(\lambda) = UB(\lambda)U^{-1},$$

where the operator

$$B(\lambda) = diag\{(P_1 - \lambda I)^{-1}, \ldots, (P_\ell - \lambda I)^{-1}\}.$$

acting in the direct sum

$$H_\ell = H \oplus \cdots \oplus H \quad (\ell\text{-times})$$

where $\lambda \in \overline{\Phi}\backslash R_+,\ |\lambda| \geq C_0$ and $(Uu)(x) = U(x)u(x),\ (u \in H_\ell)$. Again easily we will have the following equality

$$(P - \lambda I)\Gamma(\lambda) = I + \rho^{2\alpha-1}(x)q_0(x)B(\lambda)U^{-1} + \rho^{2\alpha}(x)\sum_{i=1}^{n} q_i(x)\frac{\partial}{\partial x_i}B(\lambda)U^{-1}. \qquad (4.5)$$

Here $q_i \in C\left(\bar{\Omega}, End\ \mathbf{C}^\ell\right), i = 0, 1, 2, ..., n$. As in section 3, let us take the right side of (4.5) equals to $I + T'(\lambda)$ Thus, we will have

$$(P - \lambda I)\Gamma(\lambda) = I + T'(\lambda). \qquad (4.6)$$

Now according to section 2 if we put $P_j = A$ for $j = 1, \ldots, \ell$ in (2.2) we will have

$$\|(P_j - \lambda I)^{-1}\| \leq M1_\Phi|\lambda|^{-1},\ \|\rho^\alpha \frac{\partial}{\partial x_i}(P_j - \lambda I)^{-1}\| \leq M'_\Phi\|\lambda\|^{-\frac{1}{2}}.$$

Owing to the definition of $T'(\lambda)$ in the (4.5) easily it follows that

$$\|T'(\lambda)\| \leq M0_\Phi|\lambda|^{-\frac{1}{2}}(\lambda \in \Phi,\ \ |\lambda| > 1). \qquad (4.7)$$

Since $|\lambda|$ is sufficiently large number easily implies that $\|T'(\lambda)\| < \frac{1}{2} < 1$, from this and using the well known theorem in the operator theory we conclude that $I + T'(\lambda)$ and so $(P - \lambda I)\Gamma(\lambda)$ are invertible. Hence, $((P - \lambda I)\Gamma(\lambda))^{-1}$ exists and equals to

$$(\Gamma(\lambda))^{-1}(P - \lambda I)^{-1} = (I + T'(\lambda))^{-1}, \qquad (4.8)$$

By adding $+I$ and $-I$ to the right side of the (4.8) it follows that

$$(\Gamma(\lambda))^{-1}(P - \lambda I)^{-1} = (I + T'(\lambda))^{-1} - I + I.$$

We now set

$$F'(\lambda) = (I + T'(\lambda))^{-1} - I.$$

Then

$$(\Gamma(\lambda))^{-1}(P - \lambda I)^{-1} = I + F'(\lambda).$$

In view of $\|T'(\lambda)\| < 1$ and (3.7) by estimating the following geometric series for the $F'(\lambda)$ we will have

$$\begin{aligned} \|F'(\lambda)\| \leq \sum_{i=2}^{+\infty} \|T'^k(\lambda)\| &\leq \|T'(\lambda)\|^2(1 + \|T'(\lambda)\| + \|T'(\lambda)\|^2 + \ldots) \\ &\leq \|T'(\lambda)\|^2 M0_\Phi(1 + 1/2 + 1/4 + \ldots) \leq 2M0_\Phi(M0_\Phi|\lambda|^{-1/2})^2 \end{aligned}$$

i.e., $\|F'(\lambda)\| \leq 2M1_\Phi|\lambda|^{-1}$. By (3.3) and $\|(P_j - \lambda I)^{-1}\| \leq M1_\Phi|\lambda|^{-1}$, and in view of definition $B(\lambda)$ and $\Gamma(\lambda)$ we will have

$$\|\Gamma(\lambda)\| \leq M2_\Phi|\lambda|^{-1}.$$

Now from (4.8) we have

$$(P-\lambda I)^{-1}=\Gamma(\lambda)(I+T'(\lambda))^{-1}=\Gamma(\lambda)(I+F'(\lambda)).$$

Therefore

$$\left\|(P-\lambda I)^{-1}\right\|=\|\Gamma(\lambda)\|\,\|(I+F'(\lambda))\|\leq M2_{\Phi}|\lambda|^{-1}\|(1+\ 2M1_{\Phi}|\lambda|^{-1}).$$

Therefore

$$\|(P-\lambda I)^{-1}\|\leq M2_{\Phi}|\lambda|^{-1}+2M2_{\Phi}M1_{\Phi}|\lambda|^{-1}|\lambda|^{-1},$$

and since $|\lambda|^{-1}|\lambda|^{-1}=|\lambda|^{-2}\leq|\lambda|^{-1}$, it follows that

$$\|(P-\lambda I)^{-1}\|\leq M_{\Phi}|\lambda|^{-1},\ \ (|\lambda|\geq C,\qquad \lambda\in\Phi).$$

This complete the proof of Theorem 4.1. □

References

[1] M.S. Agranovich, *Elliptic operators on closed manifolds*, Partial Differential Equations - 6, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., **63**, VINITI, Moscow (1990), 5-129 (In Russian).

[2] K.Kh. Boimatov, A.G. Kostyuchenko, *Distribution of eigenvalues of second-order non-selfadjoint differential operators*, Vest. Mosk. Gos. Univ., Ser. I, Mat. Mekh., **45**, No 3 (1990), 24-31 (In Russian).

[3] K.Kh. Boimatov, *Asymptotic behaviour of the spectra of second-order non-selfadjoint systems of differential operators*, Mat. Zametki, **51**, No 4 (1992), 6-16 (In Russian).

[4] K.Kh. Boimatov, *Asymptotics of the spectrum of an elliptic differential operator in the degenerate case*, Dokl. Akad. Nauk SSSR, **243**, No 6 (1978), 1369-1372 (In Russian).

[5] K.Kh. Boimatov, *Separability theorems, weighted spaces and their applications*, *Investigations in the Theory of Differentiable Functions of Many Variables and its Applications. Part 10*, Collection of Articles, Trudy Mat. Inst. Steklov, **170** (1984), 37-76; Proc. Steklov Inst. Math., **170** (1987), 39-81 (In Russian).

[6] K.Kh. Boimatov, *Spectral asymptotics of pseudodifferential operators*, Dokl. Akad. Nauk SSSR, **311**, No 1 (1990), 14-18; *Dokl. Math.*, **41**, No 2 (1990), 196-200 (In Russian).

[7] I.C. Gokhberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, American Mathematical Society, Providence (1969).

[8] T. Kato, Perturbation Theory for Linear Operators, Springer, New York (1966).

[9] M.A. Naymark, Linear Differential Operators, Moscow, Nauka Publ., (1969) (In Russian).

[10] A. Sameripour, K. Seddighi, *Distribution of eigenvalues of nonself-adjoint elliptic systems degenerate on the domain boundary*, Mat. Zametki, **61**, No 3 (1997), 463-467; Math. Notes, **61**, No 3 (1997), 379-384; doi: https://doi.org/10.4213/mzm1524 (In Russian).

[11] A. Sameripour, K. Seddighi, *On the spectral properties of generalized non-selfadjoint elliptic systems of differential operators degenerated on the boundary of domain*, Bull. Iranian Math. Soc., **24**, No 1 (1998), 15-32.

[12] A.A. Shkalikov, *Theorems of Tauberian type on the distribution of zeros of holomorphic functions*, Mat. Sb. (N.S.), **123 (165)**, No 3 (1984), 317-347; *Math. USSR-Sb.*, **51**, No 2 (1985) 315-344 (In Russian).

[13] R.Alizadeh, A.Sameripour, Spectral Properties and Distribution of Eigenvalues of Non selfadjoint Elliptic Differential Operators , International Journal of Applied Mathematics, Vol 34, Issue: 1, 2021. DOI: 10.12732/ijam.v34i1.11

[14] R. Alizadeh, A. Sameripour, *On The Spectral Properties of Non- Self-Adjoint Elliptic Differential Operators in Hilbert space*, Advances in the Theory of Nonlinear Analysis and its Applications 4 (2020) No. 4, 316–320. https://doi.org/10.31197/atnaa.650378

[15] Reza Alizadeh, Ali Sameripour. (2021). *On the Resolvent of a Non-Self-Adjoint Differential Operator in Hilbert Spaces*, Online Mathematics Journal, 03(01), 1–7. DOI: 10.5281/zenodo.4595051. URL http://doi.org/10.5281/zenodo.4595051.

(R. Alizadeh) Department of Mathematics, Lorestan University, Khoramabad, Iran

E-mail address, Corresponding author: `Alizadeh.re@fs.lu.ac.ir`

(A. Sameripour) Department of Mathematics, Lorestan University, Khoramabad, Iran

E-mail address: `asameripour@yahoo.com`