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NONLINEAR QUADRATIC INTEGRAL EQUATIONS USING MÜNTZ-LEGENDRE WAVELETS

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Abstract. In this paper, we introduce a new computational method for solving nonlinear quadratic integral equations. We use the Müntz-Legendre wavelets and construct its operational matrix of integration for solving nonlinear quadratic integral equations. This method reduces the nonlinear quadratic integral equations to a system of nonlinear algebraic equations, which can be solved by using Newton's method. Error analysis of the proposed method is presented. This method is used to solve some nonlinear quadratic integral equations. Numerical experiments show that the proposed technique is accurate and efficient.

1. Introduction

Integral equations arise in many science and technology problems. Integral equations of different types play an important role in many areas of functional analysis and their applications in physics, economics, and other areas. Mathematical physics, including diffraction problems, conformal mapping, scattering in quantum mechanics, water waves, etc., have led to the development of integral equations. Since integral equations are widely used and in many cases no exact solutions are available, researchers, therefore have developed many numerical method to find the approximate the solution of these equations [12, 18, 1, 4, 24]. Expressing solution of these equations as linear combination of of orthogonal or nonorthogonal basis functions, polynomials, and wavelets is a novel algorithm [22, 19, 17, 27, 25].

Mathematical modeling of several real-world problems leads to a special kind of integral equations called quadratic integral equations (QIEs). Many researchers have recently studied QIEs, which has become one of the most enticing and fascinating areas of research for integral equations [2, 6, 9, 11, 5, 3]. QIEs are useful for explaining different real-world events and problems. QIEs, for instance, are applicable in queuing theory,

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radiative transfer theory, neutron transport theory, kinetic gas theory, vehicular traffic theory, plasma physics, biology, and various other applications [7, 21, 15, 13]. Previously, the existence solution and numerical method for solving these integral equations were studied by many authors [8, 10, 20, 29, 14].

Müntz-Legendre wavelets was used by P. Rahimkhani et al. in 2018 [23] to solve fractional pantograph differential equations. The degree of extended sentences is an significant difference among Müntz Legendre-wavelets and the other wavelets. Müntz Legendre-wavelets therefore not only provides good accuracy functions but also covers a broad variety of functions that arise in various models. Due to these properties of Müntz Legendre-wavelets, in this article, we have used the Müntz Legendre-wavelets for solving nonlinear quadratic integral equations.

The objective of the present work is to introduce a new operational matrix of integration of Müntz Legendre-wavelets and solve the nonlinear quadratic integral equations of the following type:

$$y(x) = f(x) + \left(\int_0^x k_1(x,t)\kappa_1(t,y(t))dt\right)\left(\int_0^x k_2(x,t)\kappa_2(t,y(t))dt\right),\ x \in [0,T), \quad (1.1)$$

where, $y(x)$ is unknown, $f(x)$ and $k_i(x,t)$, $i = 1,2$, are the known functions.

This article is structured in the following way: Properties of Müntz-Legendre polynomials, wavelets, and Müntz-Legendre wavelets are studied in section 2. In section 3, we study the properties of Block pulse functions and we construct a new operational matrix of integration of Müntz-Legendre wavelets. In section 4, a new Müntz-Legendre wavelets operational matrix method for solving nonlinear SQIEs is proposed based on a operational matrix of integration of Müntz-Legendre wavelets. Convergence and error analysis of the proposed method is studied in section 5. In section 6, numerical examples are presented in order to justify the efficiency of the proposed method. Ultimately, the conclusion is drawn in section 7.

2. Müntz-Legendre polynomials, Wavelets, and Müntz-Legendre wavelets

2.1. Müntz-Legendre polynomials. The Müntz-Legendre polynomials [23] $L_m(x)$ are defined on interval $[0,1)$ by:

$$L_m(x) = \sum_{k=0}^{m} \xi_{k,m} x^{\lambda_k}, \qquad \xi_{k,m} = \frac{\prod_{j=0}^{m-1}(\lambda_k + \lambda_j + 1)}{\prod_{j=0, j\neq k}^{m}(\lambda_k - \lambda_j)}, \quad m \in (\mathbb{N}_0). \quad (2.2)$$

The following orthogonality condition is satisfied by these polynomials:

$$\int_0^1 L_n(x)L_m(x)dx = \frac{\delta_{n,m}}{2\lambda_m + 1}, \qquad (m \geq n),$$

where $\delta_{n,m}$ denotes Kronecker delta.

Also, the following recursive formula is satisfied by Müntz-Legendre polynomials:

$$L_m(x) = L_{m-1}(x) - (\lambda_m + \lambda_{m-1} + 1)x^{\lambda_m}\int_x^1 x^{-\lambda_m - 1}L_{m-1}(x)dx, \qquad (x \in [0,1)).$$

In this article, for a real constant γ, we assume that $\lambda_m = m\gamma$.
For instance, for $\gamma = 2$, the first two Müntz-Legendre polynomials are given as follows:

$$L_0(x) = 1,$$
$$L_1(x) = \frac{3x^2}{2} - \frac{1}{2}.$$

2.2. Wavelets and Müntz-Legendre wavelets.

Wavelets. In several fields of science and engineering wavelets have been used very successfully. They form a family of functions generated from the dilation and translation a function called mother wavelet. We have the following family of continuous wavelets, when the translation parameter a and the dilation parameter b vary continuously [23]:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi_{a,b}\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbb{R},\ a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as:

$$\psi_{k,n}(x) = |a_0|^{\frac{k}{2}}\psi(a_0^k x - nb_0),$$

where $\psi_{k,n}(x)$ forms a wavelet basis for $L^2(\mathbb{R})$.

Müntz-Legendre wavelets. Müntz-Legendre wavelets [23] $\psi_{n,m}(x) = \psi(k, n, m, x)$ have four arguments: $n = 1, 2, ..., 2^{k-1}$, k is assumed to be any positive integer, m is the order for Müntz-Legendre polynomials, and x is the normalized time. They are defined on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} (2\lambda_m + 1)^{\frac{1}{2}}\, 2^{\frac{k-1}{2}} L_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where the coefficient $(2\lambda_m + 1)^{\frac{1}{2}}$ is for normality and $m = 0, 1, ..., M - 1$. Here, $L_m(x)$ are the Müntz-Legendre polynomials defined in section 2.1.
For instance, for $\gamma = 2$, $k = 2$, and $M = 2$, we get:

$$\left.\begin{aligned} \psi_{1,0}(x) &= \sqrt{2} \\ \psi_{1,1}(x) &= \sqrt{10}\left(6x^2 - \tfrac{1}{2}\right) \end{aligned}\right\} 0 \leq x < \frac{1}{2},$$

$$\left.\begin{aligned} \psi_{2,0}(x) &= \sqrt{2} \\ \psi_{2,1}(x) &= \sqrt{10}\left(6x^2 - 6x + 1\right) \end{aligned}\right\} \frac{1}{2} \leq x < 1.$$

2.3. Function approximation. Suppose $y(x) \in [0, 1)$ is expanded in terms of the Müntz-Legendre wavelets as:

$$y(x) \simeq y^*(x) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{n,m}\psi_{n,m}(x) = G^T\psi(x). \tag{2.4}$$

Truncating the above infinite series, we get

$$y(x) \simeq y^*(x) = \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} g_{n,m}\psi_{n,m}(x) = G^T\psi(x), \tag{2.5}$$

where G and $\psi(x)$ are $\hat{m}\times 1$ ($\hat{m}=2^{k-1}M$) vectors given by:

$$G=\left[g_{1,0},g_{1,1},..,g_{1,M-1},g_{2,0},..,g_{2,M-1},..,g_{2^{k-1},0},..,g_{2^{k-1},M-1}\right]^T, \tag{2.6}$$

and

$$\psi(x)=\left[\psi_{1,0}(x),\psi_{1,1}(x),..,\psi_{1,M-1}(x),\psi_{2,0}(x),..,\psi_{2,M-1}(x),..,\psi_{2^{k-1},0}(x),..,\psi_{2^{k-1},M-1}(x)\right]^T. \tag{2.7}$$

The coefficients $g_{n,m}$ in equation (2.5) are computed as follows:

$$g_{n,m}=\int_0^1 y(x)\psi_{n,m}(x)dx. \tag{2.8}$$

Similarly, any arbitrary function $k(x,t)\in[0,1)\times[0,1)$ can be approximated using Müntz-Legendre wavelets as:

$$k(x,t)\simeq\psi^T(x)K\psi(t)=\psi^T(t)K^T\psi(x),$$

where $K=[k_{n,m}]$ is a $\hat{m}\times\hat{m}$ matrix. Using the collocation point $x_j=\frac{j-0.5}{\hat{m}}$, equation (2.7) reduces to $\hat{m}\times\hat{m}$ Müntz-Legendre wavelets coefficient matrix. For instance, for $\gamma=2$, $k=2$ and $M=2$, we get

$$\psi(x)=\begin{bmatrix}\psi_{1,0}(x)\\ \psi_{1,1}(x)\\ \psi_{2,0}(x)\\ \psi_{2,1}(x)\end{bmatrix}=\begin{bmatrix}1.4142 & 1.4142 & 0 & 0\\ -1.2847 & 1.0870 & 0 & 0\\ 0 & 0 & 1.4142 & 1.4142\\ 0 & 0 & -1.2847 & 1.0870\end{bmatrix}.$$

3. Block pulse functions (BPFs) and operational matrix of integration (OMI) of Müntz-Legendre wavelets

3.1. Block pulse functions (BPFs). A set of BPFs [16] $\phi_n(x)$, $n=1,2,...,\hat{m}$ on the interval [0, 1) are defined as follows:

$$\phi_n(x)=\begin{cases}1, & \frac{n-1}{\hat{m}}\le x<\frac{n}{\hat{m}},\\ 0, & otherwise,\end{cases}$$

where $x\in[0,1)$, $n=1,2,...,\hat{m}$ and $h=\frac{1}{\hat{m}}$. The properties of BPFs are as follows:

- The BPFs on the interval $[0,1)$ are disjoint
$$\phi_n(x)\,\phi_m(x)=\delta_{nm}(x),$$
$n,\ m=1,2,...,\hat{m}$ and δ_{nm} is Kronecker delta.
- The BPFs are orthogonal on the interval $[0,1)$.
$$\int_0^1\phi_n(x)\,\phi_m(x)\,dx=h\delta_{nm}(x),\quad n,\ m=1,2,...,\hat{m}.$$
- If $\hat{m}\to\infty$, then the BPFs set is complete; for every $f\in L^2([0,1))$, Parseval's identity holds,
$$\int_0^1 f^2(x)\,dx=\sum_{i=1}^{\infty}f_n^2\left\|\phi_n(x)\right\|^2,$$
where,
$$f_n=\frac{1}{h}\int_0^1 f(x)\phi_n(x)\,dx.$$

Let us consider the first $\hat{m}$ terms of BPFs and we write them as a $\hat{m}$-vector,

$$\phi(x) = \begin{pmatrix}\phi_1(x) & \phi_1(x) & \cdots & \phi_{\hat{m}}(x)\end{pmatrix}^T, \quad x \in [0,1). \tag{3.9}$$

The above representation and disjointness property follows:

$$\phi(x)\,\phi^T(x) = \begin{pmatrix} \phi_1(x) & 0 & \cdots & 0 \\ 0 & \phi_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{\hat{m}}(x) \end{pmatrix}_{\hat{m}\times\hat{m}}.$$

Furthermore, we have $\phi^T(x)\,\phi(x) = 1$ and $\phi(x)\,\phi^T(x)\,F^T = D_F\phi(x)$ where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_1, f_2, ..., f_{\hat{m}})^T$.

The operational matrix of integration of BPF [16] λ is given as:

$$\lambda = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{\hat{m}\times\hat{m}}.$$

3.2. Operational matrix of integration (OMI) of Müntz-Legendre wavelets. Now, we derive the OMI of Müntz-Legendre wavelets P as follows:

Theorem 3.1. *Let $\psi(x)$ and $\phi(x)$ be the $\hat{m}$-dimensional Müntz-Legendre wavelets and BPF vector, respectively. Then, $\psi(x)$ can be expressed using BPF as follows:*

$$\psi(x) = S\phi(x), \tag{3.10}$$

where S is a $\hat{m}\times\hat{m}$ block matrix given by,

$$S_{i,j} = \psi_i\left(\frac{2j-1}{2\hat{m}}\right), \; i,j = 1,2,...,\hat{m}.$$

Proof: *See* [26].

Theorem 3.2. *If $\psi(x)$ is the $\hat{m}$-dimensional Müntz-Legendre wavelet vector defined in equation (2.7), then the integral of this vector is derived as:*

$$\int_0^x \psi(t)dt = S\lambda S^{-1}\psi(x) = P\psi(x). \tag{3.11}$$

where, P is the OMI of Müntz-Legendre wavelet, λ is the OMI for BPF, and S is introduced in (3.10).

Proof: *See* [26].

Remark 3.3. If F is a $\hat{m}$ vector, then

$$\psi(x)\psi^T(x)F = \tilde{F}\psi(x), \tag{3.12}$$

where, $\psi(x)$ is the Müntz-Legendre wavelets coefficient matrix for the collocation point $x_j = \frac{j-0.5}{\hat{m}}$ and $\tilde{F}$ is a $\hat{m}\times\hat{m}$ matrix given by:

$$\tilde{F} = \psi(x)\bar{F}\psi^{-1}(x), \tag{3.13}$$

where $\bar{F} = \mathrm{diag}(\psi^{-1}(x)F)$. Also, for a $\hat{m} \times \hat{m}$ matrix G,

$$\psi^T(x)G\psi(x) = \hat{G}^T\psi(x), \tag{3.14}$$

where $\hat{G}^T = X\psi^{-1}(x)$, in which $X = \mathrm{diag}(\psi^T(x)G\psi(x))$.

4. Müntz-Legendre wavelets operational matrix method to solve non-linear quadratic integral equations

Consider equation (1.1). In equation (1.1) let

$$\left.\begin{aligned} \kappa_1(x, y(x)) &= v_1(x), \\ \kappa_2(x, y(x)) &= v_2(x). \end{aligned}\right\} \tag{4.15}$$

Then from (1.1), we get

$$\left.\begin{aligned} v_1(x) &= \kappa_1\left(x, f(x) + \left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt\right)\right), \\ v_2(x) &= \kappa_2\left(x, f(x) + \left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt\right)\right). \end{aligned}\right\} \tag{4.16}$$

Let us approximate the functions $f(x)$, $k_i(x,t)$, and $v_i(x)$, for $i = 1, 2$, in equation (4.16) using Müntz-Legendre wavelets as follows:

$$\left.\begin{aligned} & v_i(x) \simeq v_i^*(x) = V_i^T\psi(x) = \psi^T(x)V_i, \\ & f(x) \simeq f^*(x) = F^T\psi(x) = \psi^T(x)F, \\ & k_i(x,t) \simeq k_i^*(x,t) = \psi^T(x)K_i\psi(t) = \psi^T(t)K_i^T\psi(x), \ \ i = 1, 2, \end{aligned}\right\} \tag{4.17}$$

where V_1, V_2, F are Müntz-Legendre wavelet coefficient vectors and K_1, K_2 are Müntz-Legendre wavelet coefficient matrices. Substituting equation (4.17) in equation (4.16), we get

$$\begin{cases} \psi^T(x)V_1 = \kappa_1\left(x, \psi^T(x)F + \left(\int_0^x \psi^T(x)K_1\psi(t)\psi^T(t)V_1dt\right)\left(\int_0^x \psi^T(x)K_2\psi(t)\psi^T(t)V_2dt\right)\right), \\ \psi^T(x)V_2 = \kappa_2\left(x, \psi^T(x)F + \left(\int_0^x \psi^T(x)K_1\psi(t)\psi^T(t)V_1dt\right)\left(\int_0^x \psi^T(x)K_2\psi(t)\psi^T(t)V_2dt\right)\right). \end{cases}$$

Using Remark 3.3 and the operational matrix of integration of Müntz-Legendre wavelets, we have

$$\begin{cases} \psi^T(x)V_1 = \kappa_1\left(x, \psi^T(x)F + \left(\psi^T(x)K_1\hat{V}_1P\psi(x)\right)\left(\psi^T(x)K_2\hat{V}_2P\psi(x)\right)\right), \\ \psi^T(x)V_2 = \kappa_2\left(x, \psi^T(x)F + \left(\psi^T(x)K_1\hat{V}_1P\psi(x)\right)\left(\psi^T(x)K_2\hat{V}_2P\psi(x)\right)\right). \end{cases}$$

By assuming $\Upsilon_1 = K_1\hat{V}_1P$, $\Upsilon_2 = K_2\hat{V}_2P$ and using Remark 3.3, we get

$$\begin{cases} \psi^T(x)V_1 = \kappa_1\left(x, \psi^T(x)F + \left(\psi^T(x)\hat{\Upsilon}_1\right)\left(\psi^T(x)\hat{\Upsilon}_2\right)\right), \\ \psi^T(x)V_2 = \kappa_2\left(x, \psi^T(x)F + \left(\psi^T(x)\hat{\Upsilon}_1\right)\left(\psi^T(x)\hat{\Upsilon}_2\right)\right). \end{cases} \tag{4.18}$$

Equation (4.18) can be rewritten as follows:

$$\begin{cases} \psi^T(x)V_1 = \kappa_1\left(x, \psi^T(x)F + \left(\psi^T(x)\hat{\Upsilon}_1\hat{\Upsilon}_2^T\psi(x)\right)\right), \\ \psi^T(x)V_2 = \kappa_2\left(x, \psi^T(x)F + \left(\psi^T(x)\hat{\Upsilon}_1\hat{\Upsilon}_2^T\psi(x)\right)\right). \end{cases}$$

Let $\hat{\Upsilon}_3 = \hat{\Upsilon}_1 \hat{\Upsilon}_2^T$ and using remark 3.3, we get

$$\begin{cases} \psi^T(x)V_1 = \kappa_1\left(x, \psi^T(x)F + \left(\psi^T(x)\hat{\Upsilon}_3\right)\right), \\ \psi^T(x)V_2 = \kappa_2\left(x, \psi^T(x)F + \left(\psi^T(x)\hat{\Upsilon}_3\right)\right). \end{cases} \tag{4.19}$$

Collocating equation (4.19) at $\hat{m}$ nodes defined as:

$$x_i = \frac{2i-1}{2\hat{m}}, \quad i = 1, 2, ..., \hat{m},$$

we get

$$\begin{cases} \psi^T(x_i)V_1 = \kappa_1\left(x_i, \psi^T(x_i)F + \left(\psi^T(x_i)\hat{\Upsilon}_3\right)\right), \\ \psi^T(x_i)V_2 = \kappa_2\left(x_i, \psi^T(x_i)F + \left(\psi^T(x_i)\hat{\Upsilon}_3\right)\right). \end{cases} \tag{4.20}$$

Equation (4.20) is a nonlinear system of equations with $2\hat{m}$ equations and $2\hat{m}$ unknown coefficients $V_1 = \left[v_1^{(1)}\ v_2^{(1)}\ ...v_{\hat{m}}^{(1)}\right]$ and $V_2 = \left[v_1^{(2)}\ v_2^{(2)}\ ...v_{\hat{m}}^{(2)}\right]$, which are to be determined. These equations are solved using well-known Newton's method. Solving these system of equations, the approximate solution of equation (1.1) is obtained as follows:

$$y^*(x) = \psi^T(x)F + \left(\psi^T(x)K_1\widehat{V}_1P\right)\left(\psi^T(x)K_2\widehat{V}_2P\right).$$

5. Convergence and error analysis

Lemma 5.1. *Let $y(x) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0,1)$ and $|y(x)| < \delta$, for every $x \in [0,1)$. Then, the Müntz-Legendre wavelet bases of $y(x)$ on equation (2.4) are bounded as:*

$$|g_{n,m}| < \frac{\delta\,(2\lambda_m+1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}}, \tag{5.21}$$

where, δ is a constant.

Proof. Using Müntz-Legendre wavelets, any arbitrary function $y(x)$ can be approximated as:

$$y^*(x) \simeq \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} g_{n,m}\psi_{n,m}(x) = G^T\psi(x). \tag{5.22}$$

The coefficients $g_{n,m}$ in (5.22) are calculated as follows:

$$g_{n,m} = \int_0^1 y(x)\psi_{n,m}(x)dx. \tag{5.23}$$

Using the definition of $\psi_{n,m}(x)$ i.e., Müntz-Legendre wavelets, we have

$$\psi_{n,m}(x) = 2^{\frac{k-1}{2}}\,(2\lambda_m+1)^{\frac{1}{2}}\,L_m(2^{k-1}x - n + 1), \quad \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}.$$

And therefore, equation (5.23) becomes:

$$g_{n,m} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} y(x)2^{\frac{k-1}{2}}\,(2\lambda_m+1)^{\frac{1}{2}}\,L_m(2^{k-1}x-n+1)dx. \tag{5.24}$$

Let $2^{k-1}x - n + 1 = v$, then equation (5.24) becomes:

$$g_{n,m} = \frac{(2\lambda_m + 1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}} \int_0^1 |y\left(\frac{v+n-1}{2^{k-1}}\right)||L_m(v)|dv. \tag{5.25}$$

Seeing the properties of Müntz-Legendre wavelets, we have

$$\int_0^1 |L_m(v)|dv \leq 1. \tag{5.26}$$

From equations (5.25), (5.26), and the assumption $|y(x)| < \delta$, we get

$$|g_{n,m}| < \frac{\delta\,(2\lambda_m + 1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}}.$$

□

Theorem 5.2. *Let $y(x) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0, 1)$ and $|y(x)| < \delta$ for every $x \in [0, 1)$. By using the Müntz-Legendre wavelet expansion we approximate this function. Let $y^*(x) = \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} g_{nm}\psi_{nm}(x)$ be the Müntz-Legendre wavelet series. Then, the bound of the truncated error $E(x)$ is given as:*

$$\|E(x)\|_2 = \|y(x) - y^*(x)\| \leq \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty} \alpha_m^2\right)^{\frac{1}{2}} + \left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty} \alpha_m^2\right)^{\frac{1}{2}}, \tag{5.27}$$

where,

$$\alpha_m = \frac{\delta\,(2\lambda_m + 1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}}. \tag{5.28}$$

Proof. Any function $y(x) \in L^2[0, 1)$ can be expanded in terms of Müntz-Legendre wavelets as:

$$y(x) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{n,m}\psi_{n,m}(x).$$

If $y^*(x)$ is the expansion truncated by using Müntz-Legendre wavelets, then the error obtained by truncating the above function can be computed as:

$$E(x) = y(x) - y^*(x) = \sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty} g_{n,m}\psi_{n,m}(x) + \sum_{m=M}^{\infty}\sum_{n=1}^{\infty} g_{n,m}\psi_{n,m}(x). \tag{5.29}$$

From equation (5.29), we can write

$$\begin{aligned}
\|E(x)\| &\leq \|\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty} g_{n,m}\psi_{n,m}(x)\| + \|\sum_{m=M}^{\infty}\sum_{n=1}^{\infty} g_{n,m}\psi_{n,m}(x)\| \\
&= \left(\int_0^1 |\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty} g_{n,m}\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 |\sum_{m=M}^{\infty}\sum_{n=1}^{\infty} g_{n,m}\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty} |g_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}} + \left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty} |g_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.30}$$

From Lemma 5.1, using the property:

$$|g_{n,m}| < \frac{\delta\,(2\lambda_m+1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}},$$

equation (5.30) reduces to:

$$\|E(x)\|_2 \le \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}\left|\frac{\delta\,(2\lambda_m+1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}}\right|^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}} + \left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\delta\,(2\lambda_m+1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}}\right|^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}}. \tag{5.31}$$

Let us assume that

$$\alpha_m = \frac{\delta\,(2\lambda_m+1)^{\frac{1}{2}}}{2^{\frac{k-1}{2}}}, \tag{5.32}$$

the definition of $\psi_{n,m}(x)$ i.e., Müntz-Legendre wavelets, we have

$$\psi_{n,m}(x) = 2^{\frac{k-1}{2}}\,(2\lambda_m+1)^{\frac{1}{2}}\,L_m(2^{k-1}x-n+1),\quad \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}.$$

Therefore,

$$\psi^2_{n,m}(x) = 2^{k-1}\,(2\lambda_m+1)\,L^2_m(2^{k-1}x-n+1),\quad \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}. \tag{5.33}$$

Integrating equation (5.33) with respect to x, we get

$$\int_0^1\psi^2_{n,m}(x)dx = 2^{k-1}\,(2\lambda_m+1)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}L^2_m(2^{k-1}x-n+1)dx,\quad \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}. \tag{5.34}$$

Let $2^{k-1}x-n+1=u$, equation (5.34) becomes:

$$\int_0^1\psi^2_{n,m}(x)dx = (2\lambda_m+1)\int_0^1 L^2_m(u)du. \tag{5.35}$$

Using the orthogonality property of Müntz-Legendre wavelets, we have

$$\int_0^1 L^2_m(u)du = \frac{1}{2\lambda_m+1}. \tag{5.36}$$

Substituting equation (5.36) in (5.35), we get

$$\int_0^1\psi^2_{n,m}(x)dx = 1. \tag{5.37}$$

From (5.31), (5.32), and (5.37), we get

$$\|E(x)\|_2 \le \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}\alpha_m^2\right)^{\frac{1}{2}} + \left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}\alpha_m^2\right)^{\frac{1}{2}}.$$

$\square$

Lemma 5.3. *Let $k(x,t) \in L^2(\mathbb{R} \times \mathbb{R})$ be a continuous function on $[0,1) \times [0,1)$ and $|k(x,t)| < \xi$, for each $[x,t) \in [0,1) \times [0,1)$. Then, the Müntz-Legendre wavelet bases of $k(x,t)$ are bounded as:*

$$|k_{n,m}| < \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} \frac{\eta_1 \eta_2}{2^{\frac{k-1}{2}}} \xi, \tag{5.38}$$

where, ξ is any constant and

$$\eta_p = \left(2 \left(\lambda_m\right)_p + 1\right)^{\frac{1}{2}}, \ p = 1, \ 2. \tag{5.39}$$

Proof. Let us approximate $k(x,t)$ as $k^*(x,t) = \psi^T(t) K \psi(x)$. Here $K = [k_{n,m}]$ is a matrix of order $\hat{m} \times \hat{m}$ and

$$|k_{n,m}| \leq \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} \left| \left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle \right|. \tag{5.40}$$

By the definition of inner product:

$$\left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle = \int_0^1 \left[\int_0^1 k(t,x) \psi_{n_1,m_1}\left(x\right) dx \right] \psi_{n_2,m_2}\left(t\right) dt. \tag{5.41}$$

By the definition of Müntz-Legendre wavelets, equation (5.41) reduces to:

$$\begin{aligned} &\left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle \\ &= 2^{k-1} \eta_1 \eta_2 \int_{\frac{n_2-1}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} \left[\int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n_1}{2^{k-1}}} k(t,x) L_{m_1}\left(2^{k-1}x - n_1 + 1\right) dx \right] L_{m_2}\left(2^{k-1}t - n_2 + 1\right) dt. \end{aligned} \tag{5.42}$$

Let $2^{k-1}x - n_1 + 1 = v$ and $2^{k-1}t - n_2 + 1 = u$. Then equation (5.37) becomes:

$$\begin{aligned} &\left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle \\ &= \frac{\eta_1 \eta_2}{2^{k-1}} \int_0^1 \left[\int_0^1 \left[k\left(\frac{v + n_1 - 1}{2^{k-1}}, \frac{u + n_2 - 1}{2^{k-1}} \right) \right] L_{m_1}\left(v\right) dv \right] L_{m_2}\left(u\right) du. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle \\ &\leq \frac{\eta_1 \eta_2}{2^{k-1}} \int_0^1 \int_0^1 \left| k\left(\frac{v + n_1 - 1}{2^{k-1}}, \frac{u + n_2 - 1}{2^{k-1}} \right) \right| \left| L_{m_1}\left(v\right) \right| \left| L_{m_2}\left(u\right) \right| dv du. \end{aligned} \tag{5.43}$$

In the hypothesis it is assumed that $|k(x,t)| < \xi$ and hence, equation (5.43) becomes:

$$\left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle < \frac{\eta_1 \eta_2}{2^{k-1}} \xi \int_0^1 \int_0^1 \left| L_{m_1}\left(v\right) \right| \left| L_{m_2}\left(u\right) \right| dv du. \tag{5.44}$$

From equation (5.26) and (5.44), we get

$$\left\langle \left\langle k\left(x,t\right), \psi_{n_1,m_1}\left(x\right) \right\rangle, \psi_{n_2,m_2}\left(t\right) \right\rangle < \frac{\eta_1 \eta_2}{2^{k-1}} \xi. \tag{5.45}$$

And hence from (5.40), we get

$$|k_{n,m}| < \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} \frac{\eta_1 \eta_2}{2^{\frac{k-1}{2}}} \xi.$$

□

Theorem 5.4. *Let $k(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ be a continuous function on $[0,1)\times[0,1)$ and $|k(x,t)| < \xi$ for all $[x,t] \in [0,1)\times[0,1)$. By using the Müntz-Legendre wavelet expansion we approximate this function.*
Let $k^*(x,t) = \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)$ be the Müntz-Legendre wavelet series. Then, the bound of the truncated error $E(x,t)$ can be given as:

$$\begin{aligned}\|E(x,t)\|_2 &= \|k(x,t)-k^*(x,t)\|_2\\ &< \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}}\\ &+ \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}},\end{aligned} \tag{5.46}$$

where,

$$\rho_{n,m} = \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\xi. \tag{5.47}$$

where η_1 and η_2 are given in equation (5.39).

Proof. Any function $k(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ can be expanded in terms of Müntz-Legendre wavelets as:

$$k(x,t) = \sum_{m_1=0}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t).$$

If this expansion is truncated by using Müntz-Legendre wavelets, then then the error obtained by truncating the above function can be computed as:

$$\begin{aligned}E(x,t) &= k(x,t)-k^*(x,t)\\ &= \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=2^{k-1}+1}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)\\ &+ \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)\\ &+ \sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t)\\ &+ \sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty} k_{n,m}\psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t).\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|E(x,t)\|_2 \le & \left\| \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=2^{k-1}+1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right\|_2 \\
& + \left\| \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=M}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right\|_2 \\
& + \left\| \sum_{m_1=0}^{M-1} \sum_{n_1=2^{k-1}+1}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right\|_2 \\
& + \left\| \sum_{m_1=M}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_2=M}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right\|_2 \\
\le & \left(\int_0^1 \int_0^1 \left| \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=2^{k-1}+1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
& + \left(\int_0^1 \int_0^1 \left| \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=M}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
& + \left(\int_0^1 \int_0^1 \left| \sum_{m_1=0}^{M-1} \sum_{n_1=2^{k-1}+1}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
& + \left(\int_0^1 \int_0^1 \left| \sum_{m_1=M}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_2=M}^{\infty} \sum_{n_2=1}^{\infty} k_{n,m} \psi_{n_1,m_1}(x)\, \psi_{n_2,m_2}(t) \right|^2 dxdt \right)^{\frac{1}{2}} \\
\le & \left(\sum_{m_1=0}^{M-1} \sum_{n_1=2^{k-1}+1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{\infty} |k_{n,m}|^2 \int_0^1 \psi_{n_1,m_1}^2(x)\, dx \int_0^1 \psi_{n_2,m_2}^2(t)\, dt \right)^{\frac{1}{2}} \\
& + \left(\sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=M}^{\infty} \sum_{n_2=1}^{\infty} |k_{n,m}|^2 \int_0^1 \psi_{n_1,m_1}^2(x)\, dx \int_0^1 \psi_{n_2,m_2}^2(t)\, dt \right)^{\frac{1}{2}} \\
& + \left(\sum_{m_1=0}^{M-1} \sum_{n_1=2^{k-1}+1}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_2=1}^{\infty} |k_{n,m}|^2 \int_0^1 \psi_{n_1,m_1}^2(x)\, dx \int_0^1 \psi_{n_2,m_2}^2(t)\, dt \right)^{\frac{1}{2}} \\
& + \left(\sum_{m_1=M}^{\infty} \sum_{n_1=1}^{\infty} \sum_{m_2=M}^{\infty} \sum_{n_2=1}^{\infty} |k_{n,m}|^2 \int_0^1 \psi_{n_1,m_1}^2(x)\, dx \int_0^1 \psi_{n_2,m_2}^2(t)\, dt \right)^{\frac{1}{2}}.
\end{aligned}
\tag{5.48}
$$

From equation (5.37), we have

$$\int_0^1 \psi^2_{n_1,m_1}(x)dx = 1,$$

and

$$\int_0^1 \psi^2_{n_2,m_2}(x)dx = 1.$$

And hence equation (5.48) reduces to:

$$\begin{aligned}\|E(x,t)\|_2 \le &\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.49}$$

Using Lemma 5.3, we have

$$|k_{n,m}| < \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\xi. \tag{5.50}$$

Let

$$\rho_{n,m} = \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\xi. \tag{5.51}$$

From (5.50) and (5.51), we have

$$|k_{n,m}| < \rho_{n,m}. \tag{5.52}$$

From equations (5.49) and (5.52), we get

$$\begin{aligned}\|E(x,t)\|_2 < &\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}\rho^2_{n,m}\right)^{\frac{1}{2}} + \left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho^2_{n,m}\right)^{\frac{1}{2}}\\ &+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}\rho^2_{n,m}\right)^{\frac{1}{2}} + \left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho^2_{n,m}\right)^{\frac{1}{2}}.\end{aligned}$$

□

Theorem 5.5. *Let $y(x)$ and $y^*(x)$ be the exact and approximate solution of equation (1.1), respectively. Also, let us assume that*

(1) $\|y(x)\|_2 \le \beta$,

(2) $\|k_i(x,t)\|_2 \le M_i$, $i = 1$ and 2,

(3) $H^2\mathfrak{R}(M_1+\varrho_1)(M_2+\varrho_2)\left(\mathfrak{R}(1+\beta)+((\Omega_m)_1)\right) < 1$,

where, $H = \|x\|$. Also, let us assume that the nonlinear terms in equation (1.1) satisfy Lipschitz condition i.e.,

$$\|\kappa_1(x,y(x))-\kappa_1(x,y^*(x))\| + \|\kappa_2(x,y(x))-\kappa_2(x,y^*(x))\| \le \mathfrak{R}\|y(x)-y^*(x)\|, \tag{5.53}$$

and the linear growth condition i.e.,

$$\|\kappa_1(x,y(x))\| + \|\kappa_2(x,y(x))\| \leq \Re\left(1+\|y(x)\|\right), \tag{5.54}$$

then,

$$\|y(x)-y^*(x)\| \leq \frac{\alpha + \left(\left(M_1H^2\varrho_2\Re + M_2H^2\varrho_1\Re\right)(1+\beta) + M_1M_2H^2\left(\left(\Omega_m\right)_1 + \left(\Omega_m\right)_2\right)\right)\Re(1+\beta)}{1-H^2\Re(M_1+\varrho_1)(M_2+\varrho_2)\left(\Re(1+\beta)+\left(\left(\Omega_m\right)_1\right)\right)},$$

where $\left(\left(\Omega_m\right)_i, \varrho_i\right.$ for $i=1,2$ are obtained from Theorem 5.2 and Theorem 5.4 respectively.

Proof. Consider equation (1.1).
Let

$$\left.\begin{aligned}\kappa_1(x,y(x)) &= v_1(x),\\ \kappa_2(x,y(x)) &= v_2(x).\end{aligned}\right\} \tag{5.55}$$

Then from (1.1), we get

$$\left.\begin{aligned}v_1(x) &= \kappa_1\left(x, f(x) + \left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt\right)\right),\\ v_2(x) &= \kappa_2\left(x, f(x) + \left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt\right)\right).\end{aligned}\right\} \tag{5.56}$$

Also we define

$$\left.\begin{aligned}\hat{u}_1(x) &= \kappa_1(x,y^*(x)),\\ \hat{u}_2(x) &= \kappa_2(x,y^*(x)).\end{aligned}\right\} \tag{5.57}$$

Using (5.55) equation (1.1) can be rewritten as

$$y(x) = f(x) + \left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt\right). \tag{5.58}$$

Let $y(x)$ and $y^*(x)$ be the exact and approximate solution of equation (1.1) respectively. Let $f^*(x)$, $k_1^*(x,t)$, $k_2^*(x,t)$, $v_1^*(x)$, and $v_2^*(x)$ be the truncated approximations of $f(x)$, $k_1(x,t)$, $k_2(x,t)$, $v_1(x)$, and $v_2(x)$ respectively using Müntz-Legendre wavelets. Then the approximate solution of equation (1.1) is as follows:

$$y^*(x) = f^*(x) + \left[\left(\int_0^x k_1^*(x,t)v_1^*(t)dt\right)\left(\int_0^x k_2^*(x,t)v_2^*(t)dt\right)\right]. \tag{5.59}$$

where

$$\left.\begin{aligned}v_1^*(x) &= \kappa_1^*(x,y^*(x)),\\ v_2^*(x) &= \kappa_2^*(x,y^*(x)).\end{aligned}\right\}$$

Subtracting (5.58) and (5.59), we get

$$\begin{aligned}y(x)-y^*(x) = f(x) - f^*(x) &+ \left[\left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt\right)\right]\\ &- \left[\left(\int_0^x k_1^*(x,t)v_1^*(t)dt\right)\left(\int_0^x k_2^*(x,t)v_2^*(t)dt\right)\right].\end{aligned} \tag{5.60}$$

Adding and subtracting $[(k_1(x,t)v_1(t)dt)\,(k_2^*(x,t)v_2^*(t)dt)]$ to equation (5.60) and rewriting, we get

$$\begin{aligned} y(x)-y^*(x) &= f(x)-f^*(x) \\ &\quad + \left(\int_0^x k_1(x,t)v_1(t)dt\right)\left(\int_0^x k_2(x,t)v_2(t)dt - \int_0^x k_2^*(x,t)v_2^*(t)dt\right) \\ &\quad + \left(\int_0^x k_2^*(x,t)v_2^*(t)dt\right)\left(\int_0^x k_1(x,t)v_1(t)dt - \int_0^x k_1^*(x,t)v_1^*(t)dt\right). \end{aligned} \tag{5.61}$$

Therefore, from equation (5.61), we get

$$\begin{aligned} \|y(x)-y^*(x)\| &\le \|f(x)-f^*(x)\| \\ &\quad + \|k_1(x,t)\|\|v_1(t)\|\|t\|^2\|k_2(x,t)v_2(t)-k_2^*(x,t)v_2^*(t)\| \\ &\quad + \|k_2(x,t)\|\|v_2^*(t)\|\|t\|^2\|k_1(x,t)v_1(t)-k_1^*(x,t)v_1^*(t)\|. \end{aligned} \tag{5.62}$$

Let $H=\|x\|$. Using the linear growth model i.e. equation (5.53) and Theorem 5.2, for $i=1,2$, we have

$$\|u(x)-v_i^*(x)\| \le \|v_i(x)-\hat{u}_i(x)\| + \|\hat{u}_i(x)-v_i^*(x)\| \le \mathfrak{R}\|y(x)-y^*(x)\| + (\Omega_m)_i\,. \tag{5.63}$$

Using equations (5.54) and (5.63), we have

$$\|v_i^*(x)\| \le \|v_i(x)-v_i^*(x)\| + \|v_i(x)\| \le \mathfrak{R}\|y(x)-y^*(x)\| + (\Omega_m)_i + \mathfrak{R}\,(1+\beta)\,. \tag{5.64}$$

Using assumption (2) and Theorem 5.4, we have

$$\|k_2^*(x,t)\| \le \|k_2(x,t)-k_2^*(x,t)\| + \|k_2(x,t)\| \le \varrho_2 + M_2. \tag{5.65}$$

Using Theorem 5.4, equation (5.63), and (5.64), we have

$$\begin{aligned} \|k_i(x,t)v_i(t)-k_i^*(x,t)v_i^*(t)\| &\le \|k_i(x,t)\|\|v_i(x)-v_i^*(x)\| \\ &\quad + \|k_i(x,t)-k_i^*(x,t)\|\left(\|v_i(x)-v_i^*(x)\|+\|v_i(x)\|\right) \\ &\le M_i\left(\mathfrak{R}\|y(x)-y^*(x)\| + (\Omega_m)_i\right) + \varrho_i \\ &\quad + \mathfrak{R}\|y(x)-y^*(x)\| + (\Omega_m)_i + \mathfrak{R}\,(1+\beta)\,. \end{aligned} \tag{5.66}$$

From equations (5.62) to (5.66), Theorem 5.2, and Theorem 5.4 we have

$$\|y(x)-y^*(x)\| \le \frac{\alpha + \left(\left(M_1H^2\varrho_2\mathfrak{R} + M_2H^2\varrho_1\mathfrak{R}\right)(1+\beta) + M_1M_2H^2\left((\Omega_m)_1+(\Omega_m)_2\right)\right)\mathfrak{R}\,(1+\beta)}{1-H^2\mathfrak{R}(M_1+\varrho_1)(M_2+\varrho_2)\left(\mathfrak{R}\,(1+\beta)+\left((\Omega_m)_1\right)\right)}.$$

□

6. Numerical Experiments

Test problem 6.1. Let us consider the nonlinear quadratic integral equation [28]:

$$y(x) = \left(x^2 - \frac{x^{15}}{1350}\right) + \left(\int_0^x ty^2(t)dt\right)\left(\int_0^x \frac{t^2}{25}y^3(t)dt\right), \tag{6.67}$$

which has the exact solution $y(x)=x^2$. Table 1 compares the absolute values of test problem 6.1 obtained from method illustrated in section 5 at some selected points for different values of $\hat{m}$ i.e., for $\hat{m}=4,8,$ and 16, and table 2 shows the maximum absolute errors (E_m) of test problem 6.1. Figure 1 shows the graph of exact and approximate solution of test problem 6.1 for $\hat{m}=16$, and figure 2 shows the absolute errors of test problem 6.1 for different values of $\hat{m}$ i.e., for $\hat{m}=4,8,$ and 16.

TABLE 1. Absolute errors of test problem 6.1 for different values of $\hat{m}$.

x_i	$\hat{m}$=4	$\hat{m}$=8	$\hat{m}$=16
0	3.4951e-05	6.2369e-05	7.2494e-06
0.1	9.5927e-06	2.8297e-15	8.0225e-16
0.2	9.0627e-11	1.9689e-12	8.6866e-13
0.3	2.1146e-10	1.7647e-11	2.8203e-12
0.4	6.4521e-08	2.1411e-09	9.1223e-10
0.5	3.2140e-07	6.7665e-08	3.2359e-09
0.6	3.2140e-07	6.7665e-08	3.2359e-09
0.7	3.0435e-05	5.7040e-06	4.0603e-07
0.8	7.0158e-05	2.9866e-05	3.0161e-06
0.9	7.3228e-05	2.0680e-06	1.5806e-07

TABLE 2. Comparison of maximum absolute errors of test problem 6.1.

$\hat{m}$	E_m
4	7.3228e-05
8	6.2369e-05
16	7.2494e-06

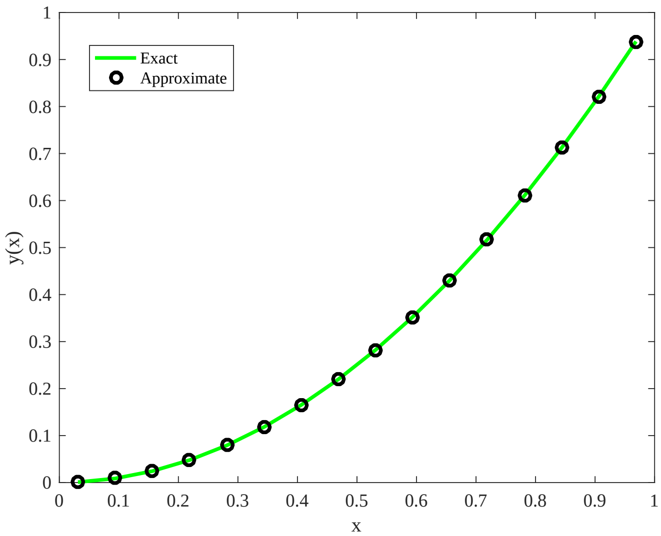

FIGURE 1. Exact and approximate solution of test problem 6.1 for $\hat{m} = 16$.

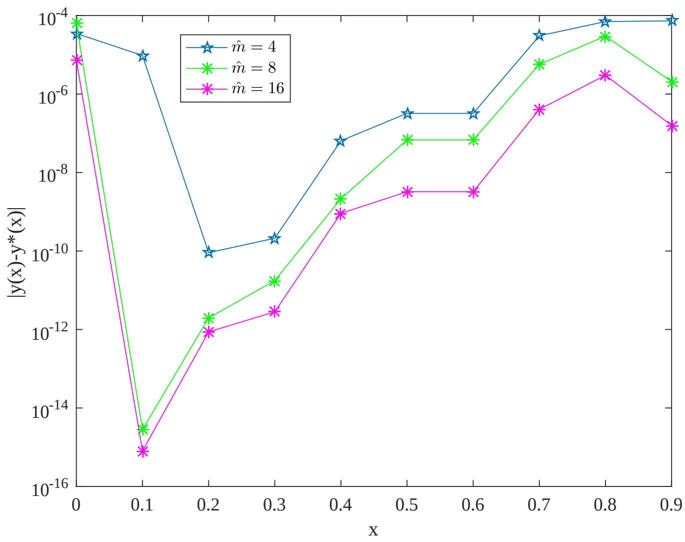

FIGURE 2. Comparison of absolute values of test problem 6.1 for different values of $\hat{m}$ i.e., for $\hat{m} = 4, 8$, and 16.

Test problem 6.2. Let us consider the nonlinear quadratic integral equation [28]

$$y(x) = \left(x - (e^x - 1) \left(\frac{x^3}{30} + \frac{x^5}{50} \right) \right) + \left(\int_0^x \left(\frac{t^2+1}{10} \right) y^2(t) dt \right) \left(\int_0^x e^{y(t)} dt \right), \quad (6.68)$$

which has the exact solution $y(x) = x$. Table 3 compares the absolute values of test problem 6.2 obtained from method illustrated in section 5 at some selected points for different values of $\hat{m}$ i.e., for $\hat{m} = 4, 8$, and 16, and table 4 shows the maximum absolute errors (E_m) of test problem 6.2. Figure 3 shows the graph of exact and approximate solution of test problem 6.2 for $\hat{m} = 16$, and figure 4 shows the absolute errors of test problem 6.2 for different values of $\hat{m}$ i.e., for $\hat{m} = 4, 8$, and 16.

7. Conclusion

In this paper, we have introduced a new computational method for solving nonlinear quadratic integral equations. We have used the Müntz-Legendre wavelets and its operational matrix of integration for solving nonlinear quadratic integral equations. Nonlinear quadratic integral equations are reduced to a system of nonlinear algebraic equations by using Müntz-Legendre wavelets and its operational matrix of integration. These nonlinear system of equations are solved by using Newton's method. Error analysis of the proposed method is presented. Numerical experiments show that the proposed method is accurate and efficient. Moreover, the results show that the maximum absolute error decreases with the increasing values of $\hat{m}$.

TABLE 3. Absolute errors of test problem 6.2 for different values of $\hat{m}$.

x_i	$\hat{m}$=4	$\hat{m}$=8	$\hat{m}$=16
0	1.5122e-02	1.6532e-03	1.6897e-04
0.1	2.4542e-03	1.4249e-05	4.6243e-06
0.2	2.6631e-04	8.1146e-05	6.7983e-06
0.3	6.0971e-04	3.6015e-05	3.5504e-06
0.4	1.6527e-03	1.3140e-04	1.1674e-05
0.5	4.7946e-03	3.5189e-04	3.2103e-05
0.6	4.7946e-03	3.5189e-04	3.2103e-05
0.7	1.9782e-02	1.5501e-03	1.5237e-04
0.8	3.4528e-02	2.9536e-03	2.9360e-04
0.9	3.6931e-02	5.5008e-03	5.2952e-04

TABLE 4. Comparison of maximum absolute errors of test problem 6.2.

$\hat{m}$	E_m
4	3.6931e-02
8	5.5008e-03
16	5.2952e-04

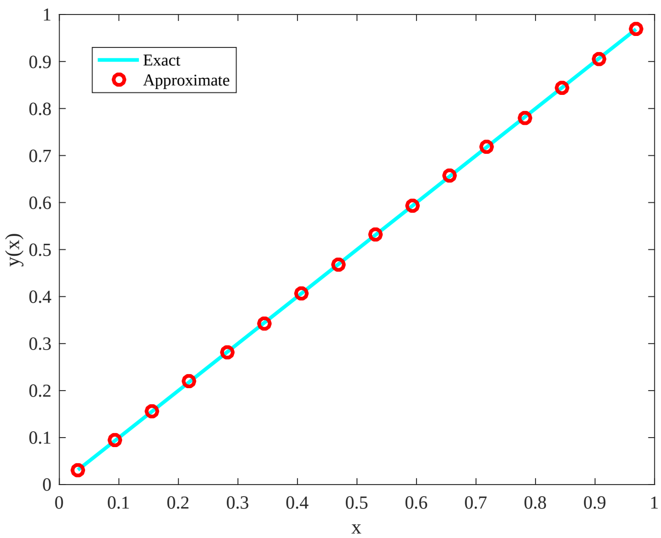

FIGURE 3. Exact and approximate solution of test problem 6.2 for $\hat{m} = 16$.

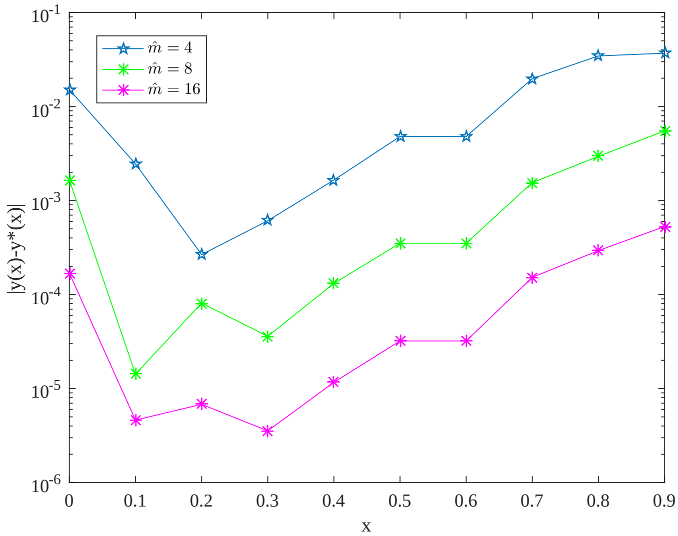

FIGURE 4. Comparison of absolute values of test problem 6.2 for different values of $\hat{m}$ i.e., for $\hat{m} = 4, 8,$ and 16.

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