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REPRESENTATION OF BOUNDED LINEAR OPERATORS ON LAPLACE INTEGRABLE FUNCTIONS

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Abstract. In this paper, a norm is defined on $\mathcal{LP}[a, b]$, the space of all Laplace integrable functions on the closed and bounded interval $[a, b]$, and it is shown that concerning that norm, $\mathcal{LP}[a, b]$ is not complete. And finally, a representation theorem for bounded linear operators on $\mathcal{LP}[a, b]$ is presented.

1. Introduction

In 1909, Riesz [29] proved a theorem which states that all bounded linear functionals $A\colon C[0,1] \to \mathbb{R}$ ($C[0,1]$ is the space of all real-valued continuous functions on $[0,1]$ equipped with the supremum norm $\|\cdot\|_\infty$) can be represented by $A[f(x)] = \int_{[0,1]} f(x)\, d\alpha(x)$, where α is a function of bounded variation on $[0,1]$. This theorem is known as the Riesz representation theorem. After this original work of Riesz, many representation theorems are developed on more general spaces; for example, if $1 \leqslant p < \infty$, μ is a σ-finite positive measure on X and T is a bounded linear functional on $L^p(\mu)$ (the space of all $f\colon X \to \mathbb{R}$ such that $\int_X |f|^p < \infty$), then there is a unique $g \in L^q(\mu)$ ($q = p/(p-1)$) such that $T(f) = \int_X fg\, d\mu$ for all $f \in L^p(\mu)$ [31], furthermore, if X is a locally compact Hausdorff space, and T is a positive linear functional on $C_c(X)$ (the space of all real-valued continuous functions with compact support), then there is a unique Radon measure μ on X such that $T(f) = \int_X f\, d\mu$ for all $f \in C_c(X)$ [6]. Although many representation problems have been successfully proved [5, 8], it is still

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an important topic [2, 10, 21, 22, 23, 24, 30, 35] because it is always convenient to have proper knowledge about the dual of a normed linear space.

Besides the representation problem, extending the set of all real-valued integrable functions is another exciting one [16, 27]. Most of the generalised integrals that have been created so far have discontinuous primitives and are known as *Trigonometric Integrals* [3, 4, 19, 36, 38, 41] since they are used to recover the coefficients of a trigonometric series. The integrals Denjoy, Perron and Henstock [9] have become more popular since, because of continuous primitives, these integrals are easier to deal with and have broader applicability [11, 13, 15, 32, 33, 34, 40, 39, 42] than those with discontinuous primitives. However, these three integrals are equivalent [9]. Pursuing this thought, recently, a generalised integral with continuous primitives called the Laplace integral (the reason behind such naming is the usage of the Laplace derivative [7, 20, 25, 26, 37] to define it) has been constructed [12] by the authors of this paper, which is more general than the Henstock integral.

This paper aims to provide a representation formula for the bounded linear functionals on the Laplace integrable functions. The motivation behind this is [1] and [14], which give representation formulas for bounded linear functionals on the space of all Henstock and T^2 integrable [17] functions. As the proofs of representation problems essentially depend on the topology of the space concerned, we provide a suitable topology, especially a norm topology, on $\mathcal{LP}[a,b]$ (the space of all real-valued Laplace integrable functions on $[a,b]$). Then using that norm and the integration by parts formula for the Laplace integral, we represent bounded linear functionals on $\mathcal{LP}[a,b]$ in terms of Laplace integral.

The rest of this paper is organised as follows. In Section 2, we illustrate all the preliminaries. Section 3 concerns with the incompleteness of $\mathcal{LP}[a,b]$ equipped with the norm given in (3.1) and the representation formula for bounded linear functionals on $\mathcal{LP}[a,b]$. Finally, Section 4 concludes this paper.

2. Preliminaries

Definition 2.1 ([28]). Let f be Perron integrable on a neighbourhood of x. Then f is said to be Laplace differentiable at x if $\exists\, \delta > 0$ such that the following limits

$$\lim_{s\to\infty} s^2 \int_0^\delta e^{-st}[f(x+t) - f(x)]\,dt$$

and

$$\lim_{s\to\infty} (-s^2) \int_0^\delta e^{-st}[f(x-t) - f(x)]\,dt$$

exist and are equal. The common value is then called the Laplace derivative of f at x and denoted by $LD_1 f(x)$.

Definitions of $\underline{LD}_1^+ f(x)$, $\underline{LD}_1^- f(x)$, $\overline{LD}_1^+ f(x)$, $\overline{LD}_1^- f(x)$ are obvious. For the definition of higher order Laplace derivative, $LD_n f$ $(n \geqslant 2)$, see [20, 18].

Definition 2.2 ([12]). Let $f\colon [a,b] \to \mathbb{R}$ and let $U\colon [a,b] \to \mathbb{R}$. Then U is said to be a major function of f if U is continuous, $\underline{LD}_1 U \geqslant f$ on $[a,b]$ and $\underline{LD}_1 U > -\infty$ on $[a,b]$. Moreover, a function $V\colon [a,b] \to \mathbb{R}$ is said to be a minor function of f if $-V$ is a major function of f.

Lemma 2.3 ([12]). *If U and V are a major and a minor function of f respectively, then $U - V$ is non-decreasing.*

Definition 2.4 ([12]). Let $f\colon [a,b] \to \mathbb{R}$. By Lemma 2.3, we have

$$\sup\left\{V_a^b \mid V \text{ is a minor function of } f\right\} \leqslant \inf\left\{U_a^b \mid U \text{ is a major function of } f\right\}.$$

If the equality holds with a finite value, f is said to be Laplace integrable on $[a,b]$ and the common value, denoted by $\int_a^b f$, is said to be the Laplace integral of f on $[a,b]$. We denote the set of all Laplace integrable functions on $[a,b]$ by $\mathcal{LP}[a,b]$, and we say $F \in C[a,b]$ (the space of all continuous functions over $[a,b]$) is a primitive of $f \in \mathcal{LP}[a,b]$ if $F(x) - F(a) = \int_a^x f$ for $x \in [a,b]$.

If we consider the function $f : [0,1] \to \mathbb{R}$ defined in Equation (4.12) of [12], then we have $\phi = LD_1 f \in \mathcal{LP}[0,1] \setminus \mathcal{HK}[0,1]$ (Theorem 4.2 of [12]). Therefore, $\mathcal{HK}[a,b]$ is a proper subset of $\mathcal{LP}[a,b]$ since by definition, Henstock integrability implies Laplace integrability.

Theorem 2.5 ([12]).

A. *Let $f \in \mathcal{LP}[a,b]$ and $F(x) = \int_a^x f$. If U and V are major and minor functions of f respectively, then each of the functions $U - F$ and $F - V$ are non-decreasing.*

B. a. *If $f \in \mathcal{LP}[a,b]$, the f is Laplace integrable on every sub-interval of $[a,b]$.*

   b. *Let $c \in (a,b)$. If f is Laplace integrable on both intervals $[a,c]$ and $[c,b]$, then f is Laplace integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.*

C. *If $f \in \mathcal{LP}[a,b]$, then f is finite valued a.e on $[a,b]$.*

D. *Let $f \in \mathcal{LP}[a,b]$ and let $g = f$ a.e. on $[a,b]$. Then $g \in \mathcal{LP}[a,b]$ and $\int_a^b g = \int_a^b f$.*

E. *Let $f, h \in \mathcal{LP}[a,b]$. Then:*

   a. *$kf + lh \in \mathcal{LP}[a,b]$ and $\int_a^b (kf + lh) = k\int_a^b f + l\int_a^b h$ for each $k,\, l \in \mathbb{R}$.*

   b. *If $f \leqslant h$ a.e. on $[a,b]$, then $\int_a^b f \leqslant \int_a^b h$.*

Theorem 2.6 ([12]). *Let $f \in \mathcal{LP}[a,b]$ and let $F(x) = \int_a^x f$ for all $x \in [a,b]$. Then F is continuous on $[a,b]$.*

Theorem 2.7 (Fundamental theorem of calculus I [12]). *If $F\colon [a,b] \to \mathbb{R}$ is continuous and $LD_1 F$ exists finitely nearlyeverywhere on $[a,b]$, then $LD_1 F \in \mathcal{LP}[a,b]$ and $\int_a^x LD_1 F = F(x) - F(a)$ for all $x \in [a,b]$.*

Theorem 2.8 ([12]). *Let $LD_1 F = f$ on $[a,b]$. If $f \in \mathcal{LP}[a,b]$, then $\int_a^x f = F(x) - F(a)$ for all $x \in [a,b]$.*

Theorem 2.9 (Fundamental theorem of calculus II [12]). *Let $f \in \mathcal{LP}[a,b]$ and let $F(x) = \int_a^x f$, $x \in [a,b]$. Then $LD_1 F = f$ a.e. on $[a,b]$.*

Definition 2.10 ([12]). Let f be Laplace integrable on a neighbourhood of x. If $\exists\,\delta > 0$ such that the following limits

$$\lim_{s\to\infty} s\int_0^\delta e^{-st} f(x+t)\,dt \quad \text{and} \quad \lim_{s\to\infty} s\int_0^\delta e^{-st} f(x-t)\,dt$$

exist and are equal, then the common value is denoted by $LD_0 f(x)$. Furthermore, f is said to be Laplace continuous at x if $LD_0 f(x) = f(x)$.

It is easy to prove that definition of Laplace continuity does not depend on δ, and the continuity of f at x implies $LD_0 f(x) = f(x)$.

Theorem 2.11 ([12]). *Let $f \in \mathcal{LP}[a,b]$ and let $F(x) = \int_a^x f$.*
A. *If f is Laplace continuous at $x \in [a,b]$, then $LD_1F(x) = f(x)$.*
B. *If f is continuous at $x \in [a,b]$, then $F'(x) = f(x)$*

Theorem 2.12 (Hake's theorem for Laplace integral [12]). *Let $f\colon [a,b] \to \mathbb{R}$. Then $f \in \mathcal{LP}[a,b]$ iff for every $c \in (a,b)$, $f|_{[a,c]} \in \mathcal{LP}[a,c]$ and there exists $L \in \mathbb{R}$ such that $\lim\limits_{c\to b-} \int_a^c f = L$. In this case, we write $\int_a^b f = L$.*

Theorem 2.13 (Integration by parts [12]). *Let $f \in \mathcal{LP}[a,b]$ and let $g \in \mathcal{BV}[a,b]$ (the space of all bounded variation functions defined over $[a,b]$). Then $fG \in \mathcal{LP}[a,b]$ and $\int_a^b fG = F(b)G(b) - (\mathsf{P})\int_a^b Fg$, where $F(x) = \int_a^x f$, $G(x) = \int_a^x g$ and $(\mathsf{P})\int_a^b$ is the definite Perron integral.*

3. Main Results

Lemma 3.1. *Let $F(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$, and let $F = 0$ a.e. on $[a,b]$. Then $f = 0$ a.e. on $[a,b]$.*

Proof. Let B_1, B_2 be two subsets of $[a,b]$ with measure $b-a$ such that $F = 0$ on B_1 and $LD_1F = f$ on B_2 (see Theorem 2.9). Then
$$B_3 = B_1 \cap B_2 \setminus \{b\}$$
is of measure $b-a$. Now it is enough to prove that $f = 0$ on B_3. Let $x \in B_3$, and let $\delta > 0$ be such that $x + \delta \in [a,b]$, then we have
$$\begin{aligned} f(x) &= \lim_{s\to\infty} s^2 \int_x^{x+\delta} e^{-s(y-x)}[F(y) - F(x)]\,dy \\ &= \lim_{s\to\infty} s^2 \int_x^{x+\delta} e^{-s(y-x)} F(y)\,dy = \lim_{s\to\infty} s^2 \int\limits_{B_3\cap[x,x+\delta]} e^{-s(y-x)}F(y)\,dy = 0. \qquad \square \end{aligned}$$

Let $\mathcal{LP}_*[a,b]$ be the set of all equivalence classes of Laplace integrable functions, where the equivalence relation is "$f \sim g$ if and only if $f = g$ a.e. on $[a,b]$". Then $\mathcal{LP}_*[a,b]$ forms a vector space. From now on, we shall use the same symbol $\mathcal{LP}[a,b]$ to denote $\mathcal{LP}_*[a,b]$ and the space of all Laplace integrable functions on $[a,b]$, and hope it will not confuse. Now define
$$\|f\|_{\mathcal{LP}} = \|F\|_D, \tag{3.1}$$
where $F(x) = \int_a^x f(t)\,dt$ for $f \in \mathcal{LP}[a,b]$, and $\|\cdot\|_D$ is the Alexiewicz' norm [1]. Then $(\mathcal{LP}[a,b], \|\cdot\|_{\mathcal{LP}})$ forms a normed linear space.

Theorem 3.2. $\mathcal{LP}[0,1]$ *is not complete.*

Proof. Consider the sequence $\{f_n\}$ given in Theorem 3.4 of [14]. Then $\{f_n''\}$ is a Cauchy sequence in $T^2[0,1]$ but does not converge, and $f_n' \in C[0,1]$ for all $n \in \mathbb{N}$ since $\omega' \in C[0,1]$ and $\omega'(0) = 0 = \omega'(1)$. Furthermore, f_n'' exists nearlyeverywhere in $[0,1]$ for all $n \in \mathbb{N}$, so by (2.7) we get f_n'' is Laplace integrable for all $n \in \mathbb{N}$. Thus

$\{f_n''\}$ is a Cauchy sequence in $\mathcal{LP}[0,1]$ not converging in $\mathcal{LP}[0,1]$ which completes the proof. $\square$

Theorem 3.3 (**Representation Theorem**). *For each bounded linear operator*

$$T\colon \mathcal{LP}[a,b] \to \mathbb{R},$$

there exists a unique $h \in \mathcal{BV}[a,b]$ such that h is continuous at the right on $[a,b)$, $h(b)=0$ and

$$T(f) = \int_a^b f(t)h_1(t)\,dt \qquad \textit{for all } f \in \mathcal{LP}[a,b], \tag{3.2}$$

where $h_1(t) = \int_t^b h(u)\,du$ for $t \in [a,b]$ and $\|T\| = \underset{a\le t\le b}{Var}\, h(t)$.
Conversely, let $h \in \mathcal{BV}[a,b]$ be such that it is continuous at the right on $[a,b)$ and $h(b)=0$, let $h_1(t) = \int_t^b h(u)\,du$, and let

$$T(f) = \int_a^b f(t)h_1(t)\,dt \qquad \textit{for all } f \in \mathcal{LP}[a,b].$$

Then $T \in \mathcal{LP}^*[a,b]$ with $\|T\| = \underset{a\le t\le b}{Var}\, h(t)$.

Proof. Let $F(x) = \int_a^x f(t)\,dt$ for $f \in \mathcal{LP}[a,b]$. Then $F \in C[a,b]$. Define

$$S = \left\{ F \in C[a,b] \mid F(x) = \int_a^x f(t)\,dt \text{ for some } f \in \mathcal{LP}[a,b] \right\}.$$

Then S is a linear subspace of $\mathcal{HK}[a,b]$. Define $T^*\colon S \to \mathbb{R}$ by

$$T^*(F) = T(f).$$

Then T^* is bounded linear functional on S with $\|T^*\| = \|T\|$. Thus by Hahn-Banach extension theorem there is a bounded linear functional

$$T_E^*\colon \mathcal{HK}[a,b] \to \mathbb{R}$$

such that $\|T_E^*\| = \|T^*\|$ and $T_E^* = T^*$ on S. Now by [1], there is a unique function $h \in \mathcal{BV}[a,b]$ which is continuous at the right on $[a,b)$ and $h(b)=0$ such that

$$T_E^*(F) = (D)\int_a^b F(t)h(t)\,dt, \qquad \text{for } F \in \mathcal{HK}[a,b]$$

with $\|T_E^*\| = \underset{a\le t\le b}{Var}\, h(t)$. Thus for $F(x) = \int_a^x f(t)\,dt$, $f \in \mathcal{LP}[a,b]$, we have

$$\begin{aligned}
T(f) = T^*(F) &= (D)\int_a^b F(t)h(t)\,dt \\
&= F(b)\int_a^b h(t)\,dt - \int_a^b f(t)\left(\int_a^t h(u)\,du\right) dt \\
&= F(b)\int_a^b h(t)\,dt - \int_a^b f(t)\left(\int_a^b h(u)\,du - \int_t^b h(u)\,du\right) dt \\
&= F(b)\int_a^b h(t)\,dt - \int_a^b f(t)\,dt\int_a^b h(u)\,du + \int_a^b f(t)\int_t^b h(u)\,du\,dt \\
&= \int_a^b f(t)h_1(t)\,dt,
\end{aligned}$$

where $h_1(t) = \int_t^b h(u)\,du$ and $\|T\| = \|T_E^*\| = \underset{a\le t\le b}{Var}\, h(t)$.
Let $k \in \mathcal{BV}[a,b]$ be another function such that it is right continuous on $[a,b)$, $k(b) = 0$, and

$$T(f) = \int_a^b f(t)k_1(t)\,dt \qquad \text{for } F \in \mathcal{HK}[a,b],$$

where $k_1(t) = \int_t^b k(u)\,du$, $t \in [a,b]$. Now taking $f = \chi_{[a,s]}$ and using the continuity of h_1 and k_1 we have

$$\begin{aligned}
\int_a^s [h_1(t) - k_1(t)]\,dt = 0 &\implies h_1(t) = k_1(t) \text{ on } [a,s] \\
&\implies h_1 = k_1 \text{ on } [a,b], \quad [s \text{ is arbitrary}] \\
&\implies h = k \text{ a.e. on } [a,b] \;\; [\text{by Lemma 3.1}].
\end{aligned}$$

If $t_0 \in [a,b]$ be arbitrary, by the continuity of h and k, we have

$$\begin{aligned}
&\lim_{s\to\infty} s\int_0^\delta e^{-st}(h-k)(t_0+t)\,dt = (h-k)(t_0) \\
&\implies 0 = (h-k)(t_0) \implies h(t_0) = k(t_0) \\
&\implies h = k \text{ on } [a,b], \quad [t_0 \text{ is arbitrary}].
\end{aligned}$$

Conversely, let $h \in \mathcal{BV}[a,b]$ be such that it is continuous at the right on $[a,b)$ and $h(b) = 0$, let $h_1(t) = \int_t^b h(u)\,du$, and let

$$T(f) = \int_a^b f(t)h_1(t)\,dt \qquad \text{for all } f \in \mathcal{LP}[a,b].$$

Then T is linear operator on $\mathcal{LP}[a,b]$. We only need to show that T is bounded. For $\psi(t) = \int_a^t F(u)\,du$, we have

$$\begin{aligned}
\int_a^b f(t)h_1(t)\,dt &= h_1(b)\int_a^b f(t)\,dt + \int_a^b F(t)h(t)\,dt = \int_a^b F(t)h(t)\,dt \\
&= h(b)\int_a^b F(t)\,dt - \int_a^b (\int_a^t F(u)\,du) dh = -\int_a^b \psi(t)\,dh
\end{aligned}$$

which implies $|T(f)| \leq \|\psi\|_\infty(\underset{a\leq t\leq b}{Var}\, h(t)) = \|f\|_{\mathcal{LP}}(\underset{a\leq t\leq b}{Var}\, h(t))$. Thus

$$\|T\| \leqslant \underset{a\leq t\leq b}{Var}\, h(t).$$

The first part of this theorem implies that there is a unique $h^* \in \mathcal{BV}[a,b]$ such that h^* is right continuous on $[a,b)$, $h^*(b) = 0$ and

$$T(f) = \int_a^b f(t)h_1^*(t)\,dt$$

with $\|T\| = \underset{a\leq t\leq b}{Var}\, h^*(t)$, where $h_1^*(t) = \int_t^b h^*(u)\,du$. Since h^* is unique, we have $\|T\| = \underset{a\leq t\leq b}{Var}\, h(t)$. □

Remark 3.4. If we consider $\|f\|_{\mathcal{LP}} = \|F\|_\infty$, then $\mathcal{HK}[a,b]$ becomes a normed linear subspace of $\mathcal{LP}[a,b]$, and any $T \in \mathcal{HK}^*[a,b]$ can be represented as $T(f) = \int fg$ [1], where $g \in \mathcal{BV}[a,b]$ is unique. Thus the extension operator $T^*\colon \mathcal{LP}[a,b] \to \mathbb{R}$ of T can be expressed as $T^*(f) = \int fg$ for $f \in \mathcal{LP}[a,b]$, and this creates the problem since we have no certain answer so far about the integrability of fg when g is only of bounded variation [12].

4. Conclusion

In this paper, we have shown that $\mathcal{LP}[a,b]$ is an incomplete normed linear space concerning the norm in (3.1). Then a representation theorem has been given for linear functionals $T\colon \mathcal{LP}[a,b] \to \mathbb{R}$.

In the representation problem of continuous linear functionals on $\mathcal{HK}[a,b]$, Alexiewicz considered the supremum norm of the first primitive $F(x) = \int_{[a,x]} f\,dx$, where $x \in [a,b]$ and $f \in \mathcal{HK}[a,b]$. However, this norm was not taken in this paper. Instead, the supremum norm of the second primitive $F_1(x) = \int_a^x \int_a^t f(u)\,du$ is considered. In the future, attempts will be taken to represent bounded linear functionals on $\mathcal{LP}[a,b]$ equipped with the Alexiewicz's norm.

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