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ON NEAR EXACT $G-$BANACH FRAMES

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***Abstract*.** In this paper, near exact $g-$Banach frames are defined and studied. Examples demonstrating the existence of near exact $g-$Banach frames are given. A sufficient condition for a $g-$Banach frame to be a near exact $g-$Banach frame has been obtained. A Krein-Milman-Rutman type stability result for a $g-$Banach frame has been obtained. Finally, we give an application of near exact $g-$Banach frame related to eigenvalue problem.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [3] in 1952 as a tool to study some deep problems in non-harmonic Fourier series. Now a days, frames are regarded as an important and integral tool in signal processing, image processing, data compression, signal detection as well as in the study of Besov spaces and Banach spaces etc. For a nice introduction to frames, one may refer [2].

Feichtinger and Gröchenig [4] extended the concept of frames to Banach spaces and introduced the notion of atomic decomposition for Banach spaces. Later, Gröchenig [5] introduced a more general concept for Banach spaces called Banach frame. He gave the following definition of a Banach frame:

Definition 1.1. Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_{d_1}$ be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_{d_1} \to \mathcal{X}$ be given. Then, the pair $(\{f_n\}, S)$ is called a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_{d_1}$, if

(1) $\{f_n(x)\} \in \mathcal{X}_{d_1}$, for each $x \in \mathcal{X}$.
(2) there exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \le \|\{f_n(x)\}\|_{\mathcal{X}_{d_1}} \le B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{1.1}$$

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(3) S is a bounded linear operator such that $S(\{f_n(x)\}) = x, \quad x \in \mathcal{X}$.

The positive constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : \mathcal{X}_{d_1} \to \mathcal{X}$ is called the reconstruction operator (or the pre-frame operator). The inequality (1.1) is called the frame inequality.

Banach frames and Banach Bessel sequences in Banach spaces were further studied in [6, 7, 8] .

Besides conventional applications, many other applications of frames have been originated which cannot be modeled by one single frame system. In such cases, data assigned to one single frame system becomes too large to be handled numerically. So, it would be beneficial to split larger frame system into a set of smaller systems and process the data locally within each subsystem effectively. Fusion frame system is one of the way to overcome such type of problem. In this direction, fusion frames for Banach spaces were introduced and studied by Kaushik and Kumar [9] as a generalization of Banach frames in Banach spaces. Further, Pathak and Goswami [12] introduced and studied the notion of near exact fusion Banach frames in Banach spaces. For the literature of fusion frames for Banach spaces, one may refer [10, 11].

Abdollahpour et.al [1] generalized the concept of frames for Banach spaces and introduced $g-$Banach frames in Banach spaces. $G-$Banach frames in Banach spaces are generalized frames in Banach spaces which includes Banach frames, frames of subspaces (fusion frames) for Banach spaces as well as many recent generalization of frames in Banach spaces. Infact, $g-$Banach frames are natural generalizations of frames in Banach spaces which provides more choices to reconstruct vectors of the space from reconstruction operator. $G-$Banach Frames and $g-$Banach Bessel sequences in Banach spaces were further studied in [13, 14, 15, 16].

Outline. The paper is organized as follows: In section 2, preliminaries, definition and some key reults which will be used in subsequent part of this paper have been given. In section 3, we defined and studied near exact $g-$Banach frames. Examples have been given towards the existence of near exact $g-$Banach frames. A sufficient condition for a $g-$Banach frame to be a near exact $g-$Banach frame has been obtained and it has been proved that a Banach space possesses a near exact $g-$Banach frame, when it is related in some sense to another Banach space which is having a near exact $g-$Banach frame. A Krein-Milman-Rutman type stability result for a $g-$Banach frame has been obtained. Finally, in section 4, we give a result as an application of near exact $g-$Banach frame related to eigenvalue problem. Proper references are given at the end.

2. Preliminaries

A Banach space of vector valued sequences (or BV-space) is a linear space of sequences with a norm which makes it a Banach space. Let $\mathcal{X}$ be a Banach space and $1 < p < \infty$, then

$$Y = \left\{ \{x_n\} : x_n \in \mathcal{X}, \ \left(\sum_{n \in \mathbb{N}} \|x_n\|^p \right)^{1/p} < \infty \right\}$$

and

$$\ell_\infty = \{\{x_n\} : \sup_{n \in \mathbb{N}} \|x_n\| < \infty, \ x_n \in \mathcal{X}\}$$

are BV spaces of $\mathcal{X}$ (a sequence space generated by the elements of $\mathcal{X}$).

Throughout this paper, $\mathcal{X}$ will denote a Banach space over the scalar field $\mathbb{K}$($\mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}_d$ is an associated Banach space of vector valued sequences indexed by $\mathbb{N}$. $\mathcal{H}$ is a separable Hilbert space, E is a sequence space generated by elements of $\mathcal{H}$ which is also a Banach space with respect to the norms $\|x\|_E = \left(\sum_{n=1}^{\infty} \|x_n\|_{\mathcal{H}}^p\right)^{1/p} < \infty, 1 < p < \infty$ and $\|x\|_E = \sup_{1\leq n<\infty} \|x_n\|_{\mathcal{H}} < \infty,\ x = \{x_n\} \subset E$ and $\mathcal{A}$ is a Banach space of vector valued sequences indexed by $\mathbb{N}$ (associated with E). $B(\mathcal{X}, \mathcal{H})$ is the collection of all bounded linear operators from $\mathcal{X}$ into $\mathcal{H}$, $\overline{span}\{\Lambda_n\}$ is closed linear span of $\{\Lambda_n\}$ in the strong operator topology in $B(\mathcal{X}, \mathcal{H})$.

Definition 2.1. ([1]) Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{X}_d$ be an associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. Let $\{\mathfrak{T}_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{H})$ and $\mathfrak{U} : \mathcal{X}_d \to \mathcal{X}$ be given. Then, the pair $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is called a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$, if

(1) $\{\mathfrak{T}_n(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.

(2) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{\mathfrak{T}_n(x)\}\|_{\mathcal{X}_d} \leq B\|x\|_{\mathcal{X}},\quad x \in \mathcal{X}. \tag{2.2}$$

(3) $\mathfrak{U}$ is a bounded linear operator such that $\mathfrak{U}(\{\mathfrak{T}_n(x)\}) = x,\quad x \in \mathcal{X}$.

The positive constants A and B, respectively, are called the lower and upper frame bounds of the $g-$Banach frame $(\{\mathfrak{T}_n\}, \mathfrak{U})$. The operator $\mathfrak{U} : \mathcal{X}_d \to \mathcal{X}$ is called the reconstruction operator and the inequality (2.2) is called the frame inequality for $g-$Banach frame $(\{\mathfrak{T}_n\}, \mathfrak{U})$.

The following results, proved in [15, 1], stated in the form of lemmas, will be referred to later in this paper.

Lemma 2.2. [15] *Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let $\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H})$ be a sequence of non-zero operators. If $\{\mathfrak{T}_n\}$ is total over $\mathcal{X}$, i.e., $\{x \in \mathcal{X} : \mathfrak{T}_n(x) = 0, n \in \mathbb{N}\} = \{0\}$, then $\mathcal{A} = \{\{\mathfrak{T}_n(x)\} : x \in \mathcal{X}\}$ is a Banach space with the norm given by $\|\{\mathfrak{T}_n(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}$.*

Lemma 2.3. [15] *Let $(\{\mathfrak{T}_n\}, \mathfrak{U})$ $(\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H}),\ \mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ be a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is exact if and only if $\mathfrak{T}_i \notin \overline{span}\{\mathfrak{T}_n\}_{n\neq i}$, for all $i \in \mathbb{N}$.*

Lemma 2.4. [1] *Let $\mathcal{H}$ be a separable Hilbert space. Then, every separable Banach space has a $g-$Banach frame with respect to $\mathcal{H}$ with frame bounds $A = B = 1$.*

3. Main Results

We begin this section with the following definition of near exact $g-$Banach frame:

Definition 3.1. A $g-$Banach frame $(\{\mathfrak{T}_n\}, \mathfrak{U})$ $(\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H}),\ \mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ for $\mathcal{X}$ with respect to $\mathcal{X}_d$ is said to be near exact $g-$Banach frame for $\mathcal{X}$, if it can be transformed into an exact $g-$Banach frame after removing of finite number of its elements.

Towards, the existence of near exact $g-$Banach frames, we give the following examples:

Example 3.2. Let $\mathcal{H}$ be a separable Hilbert space and let

$$E = \ell^\infty(\mathcal{H}) = \left\{ \{x_n\} : x_n \in \mathcal{H};\ \sup_{1 \le n < \infty} \|x_n\|_{\mathcal{H}} < \infty \right\}$$

be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1 \le n < \infty} \|x_n\|_{\mathcal{H}},\ \{x_n\} \subset E.$$

Now, for each $n \in \mathbb{N}$, define $\mathfrak{T}_n : E \to \ell^2(\mathcal{H})$ by

$$\mathfrak{T}_n(x) = \begin{cases} \delta_1^{x_1},\ n = 1, 2, \ldots, k \\ \delta_{n-k}^{x_{n-k}},\ n = k+1, k+2, \ldots, \end{cases} \quad x = \{x_n\} \subset E,\ k \in \mathbb{N},$$

where $\delta_n^x = (0, \ldots, 0 \underbrace{x}_{n^{th}\text{place}}, 0, \ldots)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Then $\{\mathfrak{T}_n\}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{\mathfrak{T}_n(x)\} : x \in E\}$ with the norm given by $\|\{\mathfrak{T}_n(x)\}\|_{\mathcal{A}} = \|x\|_E,\ x \in E$. Define $\mathfrak{U} : \mathcal{A} \to E$ by $\mathfrak{U}(\{\mathfrak{T}_n(x)\}) = x,\ x \in E$. Then $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is a $g-$Banach frame for E with respect to $\mathcal{A}$. Also, since $\mathfrak{T}_i \in \overline{span}\{\mathfrak{T}_n\}_{n \neq i}$ for each $i = 1, 2, ..., k$, then by the Lemma 2.3, $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is not an exact $g-$Banach frame. Again $\{\mathfrak{T}_n\}_{n \neq 1,2,..,k}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $\mathcal{A}_0 = \{\{\mathfrak{T}_n(x)\}_{n \neq 1,2...,k} : x \in E\}$ and a bounded linear operator $\mathfrak{U}_0 : \mathcal{A}_0 \to E$ such that $(\{\mathfrak{T}_n\}_{n \neq 1,2,...,k}, \mathfrak{U}_0)$ is an exact $g-$Banach frame for E.

Example 3.3. Let E be defined as in Example 3.2. Now, for each $n \in \mathbb{N}$, define $\mathfrak{T}_n : E \to \ell^2(\mathcal{H})$ by

$$\mathfrak{T}_{2n-1}(x) = \mathfrak{T}_{2n}(x) = \delta_n^{x_n}, x = \{x_n\} \in E,$$

where $\delta_n^x = (0, \ldots, 0 \underbrace{x}_{n^{th}\text{place}}, 0, \ldots)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Then, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{\mathfrak{T}_n(x)\} : x \in E\}$ and a reconstruction operator $\mathfrak{U} : \mathcal{A} \to E$ such that $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is a $g-$Banach frame for E with respect to $\mathcal{A}$. But, it can be easily verify that $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is not a near exact $g-$Banach frame for E.

In view of Examples 3.2 and 3.3, we give a sufficient condition for a $g-$Banach frame to be near exact.

Theorem 3.4. *Let $(\{\mathfrak{T}_n\}, \mathfrak{U})$ $(\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H}),\ \mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ be a $g-$Banach frame for Banach space $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is near exact if for every infinite sequence $\{\sigma(k)\}_{k=1}^\infty$ of positive integers such that*

$$\overline{span}\{\mathfrak{T}_n\}_{n \neq \sigma(1), \sigma(2), \ldots,} \neq B(\mathcal{X}, \mathcal{H}).$$

Proof. Suppose that $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is not near exact. Then, by omitting a finite number of its elements, that $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is a non-exact $g-$Banach frame. Therefore, by Lemma 2.3, there exists a positive integer $\sigma(1)$ such that

$$\mathfrak{T}_{\sigma(1)} \in \overline{span}\{\mathfrak{T}_n\}_{n \neq \sigma(1)} = B(\mathcal{X}, \mathcal{H}). \tag{3.3}$$

Then,

$$\mathfrak{T}_{\sigma(1)}(x) = lim_{k \to \infty} \sum_{\substack{n=1 \\ n \neq \sigma(1)}}^{k} \alpha_n^{(k)} \mathfrak{T}_n(x),\ x \in \mathcal{X},\ \text{for some scalars } \{\alpha_n\},\ n \in \mathbb{N}.$$

Since $\overline{span}\{\mathfrak{T}_n\}_{n\neq\sigma(1)} = B(\mathcal{X},\mathcal{H})$, there exists a positive integer $n_1 \geq \sigma(1)$ such that

$$\mathrm{dist}\left(\mathfrak{T}_{\sigma(1)},\ \overline{span}\{\mathfrak{T}_n\}^{n_1}_{\substack{n=1\\ n\neq\sigma(1)}}\right) < \frac{1}{2}.$$

By assumption $(\{\mathfrak{T}_i\}_{i=n_1+1}^{\infty}, \mathfrak{U})$ is not exact. Therefore, there exists a positive integer $\sigma(2) \geq n_1 + 1$ such that

$$\mathfrak{T}_{\sigma(2)} \in \overline{span}\{\mathfrak{T}_n\}^{\infty}_{\substack{n=n_1+1\\ n\neq\sigma(1),\sigma(2)}} = \overline{span}\{\mathfrak{T}_n\}^{\infty}_{\substack{n=n_1+1\\ n\neq\sigma(1)}} \tag{3.4}$$

and a positive integer $n_2 \geq \sigma(2)$ such that

$$\mathrm{dist}\left(\mathfrak{T}_{\sigma(2)},\ \overline{span}\{\mathfrak{T}_n\}^{n_2}_{\substack{n=1\\ n\neq\sigma(1),\sigma(2)}}\right) < \frac{1}{4}.$$

Further, since

$$\overline{span}\{\mathfrak{T}_n\}^{n_1}_{\substack{n=1\\ n\neq\sigma(1)}} + \overline{span}\{\mathfrak{T}_n\}^{\infty}_{\substack{n=n_1+1\\ n\neq\sigma(1),\sigma(2)}} \subseteq \overline{span}\{\mathfrak{T}_n\}_{n\neq\sigma(1),\sigma(2)}$$

$$\overline{span}\{\mathfrak{T}_n\}^{n_1}_{\substack{n=1\\ n\neq\sigma(1)}} + \overline{span}\{\mathfrak{T}_n\}^{\infty}_{\substack{n=n_1+1\\ n\neq\sigma(1)}} \subseteq \overline{span}\{\mathfrak{T}_n\}_{n\neq\sigma(1),\sigma(2)} \qquad (\text{using}(3.4))$$

$$\overline{span}\{\mathfrak{T}_n\}_{\substack{n\neq\sigma(1)\\ n\in\mathbb{N}}} \subseteq \overline{span}\{\mathfrak{T}_n\}_{\substack{n\neq\sigma(1),\sigma(2)\\ n\in\mathbb{N}}}.$$

So, we have

$$\overline{span}\{\mathfrak{T}_n\}_{\substack{n\neq\sigma(1),\sigma(2)\\ n\in\mathbb{N}}} = B(\mathcal{X},\mathcal{H}) \qquad (\text{using}(3.3)).$$

Continuing in this way, we get a sequence of indices $\{\sigma(k)\}_{k=1}^{\infty}$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ with $n_{k-1} \leq \sigma(k) \leq n_k$ and $n_0 = 0$ such that

$$\mathrm{dist}\left(\mathfrak{T}_{\sigma(k)},\ \overline{span}\{\mathfrak{T}_n\}^{n_k}_{\substack{n=1\\ n\neq\sigma(1),\sigma(2),\ldots,\sigma(k)}}\right) < \frac{1}{2^k}$$

and

$$\overline{span}\{\mathfrak{T}_n\}_{n\neq\sigma(1),\sigma(2),\ldots,\sigma(k)} = B(\mathcal{X},\mathcal{H}).$$

Thus, we get a sequences of indices $\{\sigma(k)\}_{k=1}^{\infty}$ such that

$$\overline{span}\{\mathfrak{T}_n\}_{n\neq\sigma(1),\sigma(2),\ldots} = B(\mathcal{X},\mathcal{H}).$$

This is a contradiction.
Hence $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is a near exact $g-$Banach frame for $\mathcal{X}$. □

Remark 3.5. The condition in Theorem 3.4 is not necessary (Example 3.2).

Now, in the next result, we give a sufficient condition under which a Banach space $\mathcal{Y}$ possesses a near exact $g-$Banach frame, when $\mathcal{Y}$ is related in some sense to another Banach space $\mathcal{X}$ which is having a near exact $g-$Banach frame.

Theorem 3.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. Let $(\{\mathfrak{T}_n\}, \mathfrak{U})$ $(\{\mathfrak{T}_n\} \subset B(\mathcal{X},\mathcal{H})$, $\mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ be a near exact $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\{\mathfrak{O}_n\} \subset B(\mathcal{Y},\mathcal{H})$ be any sequence of operators. If there exists a linear homeomorphism U from $\mathcal{X}$ onto $\mathcal{Y}$ such that $\mathfrak{O}_n o U = \mathfrak{T}_n,\ n \in \mathbb{N}$, then $\mathcal{Y}$ has a near exact $g-$Banach frame.*

Proof. Since $U : \mathcal{X} \to \mathcal{Y}$ is an onto map. Then, for each $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ such that $U(x) = y$. Suppose that $\mathfrak{O}_n(y) = 0$, for all $n \in \mathbb{N}$. Then $\mathfrak{T}_n(x) = 0$, for all $n \in \mathbb{N}$. This gives $y = 0$. Therefore, by Lemma 2.2, there is an associated Banach space $\mathcal{Y}_d$ and a bounded linear operator T such that $(\{\mathfrak{O}_n\}, T)$ is a $g-$Banach frame for $\mathcal{Y}$ with respect to $\mathcal{Y}_d$.
Now, suppose that $(\{\mathfrak{O}_n\}, T)$ is not a near exact $g-$Banach frame. Then, there exists a reconstruction operator T_0 such that $(\{\mathfrak{O}_n\}_{n\neq\sigma(1),\sigma(2),...}, T_0)$ is an exact $g-$Banach frame. Let A_0 and B_0 be the choice of bounds for $(\{\mathfrak{O}_n\}_{n\neq\sigma(1),\sigma(2),...}, T_0)$. Thus, for each $y \in \mathcal{Y}$, we have

$$A_0\|y\|_{\mathcal{Y}} \leq \|\{\mathfrak{O}_n(y)\}_{n\neq\sigma(1),\sigma(2),...}\|_{\mathcal{Y}_d} \leq B_0\|y\|_{\mathcal{Y}}.$$

This gives

$$\begin{aligned} A_0\|U(x)\|_{\mathcal{Y}} &\leq \|\{\mathfrak{O}_n(U(x))\}_{n\neq\sigma(1),\sigma(2),...}\|_{\mathcal{Y}_d} \\ &= \|\{\mathfrak{T}_n(x)\}_{n\neq\sigma(1),\sigma(2),...}\|_{\mathcal{X}_d},\ x \in \mathcal{X}. \end{aligned}$$

Now, if $\mathfrak{T}_n(x) = 0$ for all $n \neq \sigma(1), \sigma(2), ...$, then $x = 0$. Thus, there exists a reconstruction operator $\mathfrak{U}_0$ such that $(\{\mathfrak{T}_n\}_{n\neq\sigma(1),\sigma(2),...}, \mathfrak{U}_0)$ is a $g-$Banach frame for $\mathcal{X}$. This is a contradiction to the given hypothesis.
Hence $(\{\mathfrak{O}_n\}, T)$ is a near exact $g-$Banach frame for $\mathcal{Y}$ with respect to $\mathcal{Y}_d$. □

Next, we give a necessary and sufficient condition for a sequence of bounded operators to be a near exact $g-$Banach frame.

Theorem 3.7. *Let $(\{\mathfrak{T}_n\}, \mathfrak{U})$ be a near exact $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\{\mathfrak{O}_n\} \subset B(\mathcal{X}, \mathcal{H})$ be a sequence of operators such that $\{\mathfrak{O}_n(x)\} \in \mathcal{X}_d,\ x \in \mathcal{X}$ and $Q : \mathcal{X}_d \to \mathcal{X}_d$ be a bounded linear invertible operator such that $Q(\{\mathfrak{T}_n(x)\}) = \{\mathfrak{O}_n(x)\},\ x \in \mathcal{X}$. Then, there exists a reconstruction operator $T : \mathcal{X}_d \to \mathcal{X}$ such that $(\{\mathfrak{O}_n\}, T)$ is a near exact $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if and only if there exists a positive constant M such that*

$$\|\{\mathfrak{O}_n(x)\}\|_{\mathcal{X}_d} \geq M\|W\{\mathfrak{O}_n(x)\}\|_{\mathcal{X}_d},\ x \in \mathcal{X},$$

where $W : \mathcal{X}_d \to \mathcal{X}_d$ is a bounded linear operator such that $W(\{\mathfrak{O}_n(x)\}) = \{\mathfrak{T}_n(x)\},\ x \in \mathcal{X}$.

Proof. Let $A_{\mathfrak{T}}$ and $B_{\mathfrak{T}}$ be the frame bounds for $g-$Banach frame $(\{\mathfrak{T}_n\}, \mathfrak{U})$. Then, for each $x \in \mathcal{X}$, we have

$$\begin{aligned} MA_{\mathfrak{T}}\|x\|_{\mathcal{X}} &\leq M\|\{\mathfrak{T}_n(x)\}\|_{\mathcal{X}_d} \\ &= M\|W(\{\mathfrak{O}_n(x)\})\|_{\mathcal{X}_d} \\ &\leq \|\{\mathfrak{O}_n(x)\}\|_{\mathcal{X}_d} \\ &\leq \|Q\|\|\{\mathfrak{T}_n(x)\}\|_{\mathcal{X}_d} \\ &\leq B_{\mathfrak{T}}\|Q\|\|x\|_{\mathcal{X}}. \end{aligned}$$

Therefore, we obtain

$$MA_{\mathfrak{T}}\|x\|_{\mathcal{X}} \leq \|\{\mathfrak{O}_n(x)\}\|_{\mathcal{X}_d} \leq B_{\mathfrak{T}}\|Q\|\|x\|_{\mathcal{X}},\ x \in \mathcal{X}.$$

Let $T = \mathfrak{U}W$. Then $T : \mathcal{X}_d \to \mathcal{X}$ is a bounded linear operator such that

$$T(\{\mathfrak{O}_n(x)\}) = x,\ x \in \mathcal{X}.$$

Thus $(\{\mathfrak{O}_n\}, T)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$.
Now, suppose that $(\{\mathfrak{O}_n\}, T)$ is not a near exact $g-$Banach frame. Then, there exists a reconstruction operator T_0 such that $(\{\mathfrak{O}_n\}_{n\neq\sigma(1)\sigma(2),\ldots}, T_0)$ is a $g-$Banach frame for $\mathcal{X}$. Let A_0 and B_0 be the choice of bounds for $(\{\mathfrak{O}_n\}_{n\neq\sigma(1)\sigma(2),\ldots}, T_0)$. Then, for each $x \in \mathcal{X}$, we have

$$\begin{aligned} A_0\|x\|_{\mathcal{X}} &\leq \|\{\mathfrak{O}_n(x)\}_{n\neq\sigma(1)\sigma(2),\ldots}\|_{\mathcal{X}_d} \\ &\leq \|Q\|\|\{\mathfrak{T}_n(x)\}_{n\neq\sigma(1)\sigma(2),\ldots}\|_{\mathcal{X}_d}. \end{aligned}$$

This gives

$$A_0\|Q\|^{-1}\|x\|_{\mathcal{X}} \leq \|\{\mathfrak{T}_n(x)\}_{n\neq\sigma(1)\sigma(2),\ldots}\|_{\mathcal{X}_d},\ x \in \mathcal{X}.$$

Thus, by Lemma 2.2, there exists a reconstruction operator $\mathfrak{U}_0$ such that $(\{\mathfrak{T}_n(x)\}_{n\neq\sigma(1)\sigma(2),\ldots}, \mathfrak{U}_0)$ is a $g-$Banach frame for $\mathcal{X}$. This is a contradiction.
Hence $(\{\mathfrak{O}_n\}, T)$ is a near exact $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$.

Conversely, Let $(\{\mathfrak{O}_n\}, T)$ be a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with frame bounds $A_{\mathfrak{O}}$ and $B_{\mathfrak{O}}$. Let $U : \mathcal{X} \to \mathcal{X}_d$ be a mapping, defined by $U(x) = \{\mathfrak{T}_n(x)\},\ x \in \mathcal{X}$ and $W = UT$. Then $W : \mathcal{X}_d \to \mathcal{X}_d$ is a bounded linear operator such that

$$W(\{\mathfrak{O}_n(x)\}) = \{\mathfrak{T}_n(x)\},\ x \in \mathcal{X}.$$

Thus, there exists a $M > 0$ which depends on bounds of $g-$Banach frames $(\{\mathfrak{T}_n\}, \mathfrak{U})$ and $(\{\mathfrak{O}_n\}, T)$ such that

$$\|\{\mathfrak{O}_n(x)\}\|_{\mathcal{X}_d} \geq M\|W(\{\mathfrak{O}_n(x)\})\|_{\mathcal{X}_d},\ x \in \mathcal{X}.$$

□

The following result provides us a Krein-Milman-Rutman type stability result.

Theorem 3.8. *Let $\mathcal{X}$ be a separable Banach space. Let $(\{\mathfrak{T}_n\}, \mathfrak{U})$ be a non-near exact $g-$Banach for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then, for any sequence of positive number $\{\eta_n\}$, there exists a $g-$Banach frame $(\{\mathfrak{O}_n\}, T)$ for $\mathcal{X}$ such that*

$$\|\mathfrak{T}_n(x) - \mathfrak{O}_n(x)\|_{\mathcal{X}_d} \leq \eta_n,\ \textit{for all } n \in \mathbb{N},\ x \in \mathcal{X}.$$

Proof. Since $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is a non-near exact $g-$Banach frame for $\mathcal{X}$, then one can find an infinite sequence of indices $\{\sigma(n)\}$ such that

$$\overline{span}\{\mathfrak{T}_i\}_{i\neq\sigma(1),\sigma(2),\ldots} = \overline{span}\{\mathfrak{T}_n\}$$

Since $\mathcal{X}$ is separable, then by the Lemma 2.4, there exists a total sequence of bounded linear operators $\{\widetilde{\mathfrak{T}}_n\} \subset B(\mathcal{X}, \mathcal{H})$ such that $\{\widetilde{\mathfrak{T}}_n(x)\} \in \mathcal{X}_d,\ x \in \mathcal{X}$ and $\|\{\widetilde{\mathfrak{T}}_n(x)\}\|_{\mathcal{X}_d} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Now, define

$$\begin{cases} \mathfrak{O}_{\sigma(n)} = \mathfrak{T}_{\sigma(n)} + \eta_{\sigma(n)}\widetilde{\mathfrak{T}}_n,\ n \in \mathbb{N} \\ \mathfrak{O}_{\alpha(n)} = \mathfrak{T}_{\alpha(n)},\ n \in \mathbb{N} \end{cases}$$

where $\{\alpha_n\} = \mathbb{N}\backslash\{\sigma(n)\}$.
Then $\{\mathfrak{O}_n(x)\} \subset \mathcal{X}_d$ is such that $\|\mathfrak{T}_n(x) - \mathfrak{O}_n(x)\|_{\mathcal{X}_d} < \eta_n$, for all $n \in \mathbb{N}$. Let $\mathfrak{O}_n(x) = 0$, for all $n \in \mathbb{N}$. Then $\mathfrak{T}_n(x) = 0$, for all $n \in \mathbb{N}$. Therefore $\eta_\sigma(n)\widetilde{\mathfrak{T}}_n(x) = -\mathfrak{T}_{\sigma(n)}(x) = 0$. Since $\eta_{\sigma(n)} \neq 0,\ n \in \mathbb{N}$, then $\widetilde{\mathfrak{T}}_n(x) = 0$, for all $n \in \mathbb{N}$. Since $\{\widetilde{\mathfrak{T}}_n\}$ is total over $\mathcal{X}$, $x = 0$. Thus $\{\mathfrak{O}_n\}$ is total over $\mathcal{X}$. Therefore, by Lemma 2.2, there exists an associated Banach

space $\mathcal{X}'_d = \{\{\mathfrak{O}_n(x)\} : x \in \mathcal{X}\}$ with the norm given by $\|\{\mathfrak{O}_n(x)\}\|_{\mathcal{X}'_d} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$ and a reconstruction operator $T : \mathcal{X}'_d \to \mathcal{X}$ given by $T(\{\mathfrak{O}_n(x)\}) = x,\ x \in \mathcal{X}$ such that $(\{\mathfrak{O}_n\}, T)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}'_d$. □

4. Application

In this section, we give the following result as an application of near exact $g-$Banach frame related to eigen value problem.

Theorem 4.1. *Let* $(\{\mathfrak{T}_n\}, \mathfrak{U})$ $(\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H}),\ \mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ *be a near exact* $g-$*Banach frame for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$. *Let* $\{\mathfrak{F}_k\}_{k=1}^m \subset \mathcal{X}^*$ *be a linearly independent set and for each* k $(1 \le k \le m)$, *there exists an* $\xi_k \in \mathcal{X}$ *such that*

$$\mathfrak{T}_i(\xi_k) = \lim_{l\to\infty} \sum_{\substack{n=1\\ n\neq i}}^{m_l} \alpha_n^{(l)} \mathfrak{T}_n(\xi_k),\ \textit{for all } i \in \mathbb{N}, \tag{4.5}$$

where $\{m_l\}$ *is an increasing sequence of positive integers.*
If $\left(\left\{\mathfrak{T}_i + \dfrac{1}{p}\displaystyle\sum_{k=1}^{m}\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_n^{(l)}\mathfrak{T}_n(\xi_k)\right)\mathfrak{F}_k\right\}, \mathfrak{U}_0\right)$, *where* p *is any real number, is* $g-$*Banach frame for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$, *then* $-p$ *is not an eigenvalue of the matrix*

$$\begin{pmatrix} \mathfrak{F}_1(\xi_1) & \mathfrak{F}_1(\xi_2) & \dots & \mathfrak{F}_1(\xi_m) \\ \mathfrak{F}_2(\xi_1) & \mathfrak{F}_2(\xi_2) & \dots & \mathfrak{F}_2(\xi_m) \\ \dots & \dots & \dots & \dots \\ \mathfrak{F}_m(\xi_1) & \mathfrak{F}_m(\xi_2) & \dots & \mathfrak{F}_m(\xi_m) \end{pmatrix}.$$

Proof. We prove the result for $m = 3$. Suppose $-p$ is an eigenvalue of the matrix

$$\begin{pmatrix} \mathfrak{F}_1(\xi_1) & \mathfrak{F}_1(\xi_2) & \mathfrak{F}_1(\xi_3) \\ \mathfrak{F}_2(\xi_1) & \mathfrak{F}_2(\xi_2) & \mathfrak{F}_2(\xi_3) \\ \mathfrak{F}_3(\xi_1) & \mathfrak{F}_3(\xi_2) & \mathfrak{F}_3(\xi_3) \end{pmatrix}.$$

Then

$$\begin{vmatrix} \mathfrak{F}_1(\xi_1) + p & \mathfrak{F}_1(\xi_2) & \mathfrak{F}_1(\xi_3) \\ \mathfrak{F}_2(\xi_1) & \mathfrak{F}_2(\xi_2) + p & \mathfrak{F}_2(\xi_3) \\ \mathfrak{F}_3(\xi_1) & \mathfrak{F}_3(\xi_2) & \mathfrak{F}_3(\xi_3) + p \end{vmatrix} = 0.$$

So, there exists scalars $\alpha,\ \beta,\ \gamma$ not all zero, such that

$$\begin{aligned} \alpha\mathfrak{F}_1(\xi_1) + \beta\mathfrak{F}_1(\xi_2) + \gamma\mathfrak{F}_1(\xi_3) &= -\alpha p \\ \alpha\mathfrak{F}_2(\xi_1) + \beta\mathfrak{F}_2(\xi_2) + \gamma\mathfrak{F}_2(\xi_3) &= -\beta p \\ \alpha\mathfrak{F}_3(\xi_1) + \beta\mathfrak{F}_3(\xi_2) + \gamma\mathfrak{F}_3(\xi_3) &= -\gamma p. \end{aligned}$$

Now, let $z = -\alpha p\xi_1 - \beta p\xi_2 - \gamma p\xi_3$. Then z is a non-zero element in $\mathcal{X}$ and

$$\mathfrak{T}_i(z) = -\alpha p\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_i^{(l)}\mathfrak{T}_n(\xi_1)\right) - \beta p\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_i^{(l)}\mathfrak{T}_n(\xi_2)\right) - \gamma p\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_i^{(l)}\mathfrak{T}_n(\xi_3)\right),$$

for all $i \in \mathbb{N}$.

Now, $\mathfrak{F}_k(z) = -\alpha p\mathfrak{F}_k(\xi_1) - \beta p\mathfrak{F}_k(\xi_2) - \gamma p\mathfrak{F}_k(\xi_3),\ k = 1, 2, 3$. Therefore $\mathfrak{F}_1(z) = p^2\alpha$, $\mathfrak{F}_2(z) = p^2\beta$ and $\mathfrak{F}_3(z) = p^2\gamma$. Thus, we get

$$\left(\mathfrak{T}_i + \frac{1}{p}\sum_{k=1}^{3}\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_n^{(l)}\mathfrak{T}_n(\xi_k)\right)\mathfrak{F}_k\right)(z) = 0,\ \text{for all } i \in \mathbb{N}$$

Since $\left(\left\{\mathfrak{T}_i + \frac{1}{p}\sum_{k=1}^{3}\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_n^{(l)}\mathfrak{T}_n(\xi_k)\right)\mathfrak{F}_k\right\}, \mathfrak{U}_0\right)$ is a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$, it follows that $z = 0$. This is a contradiction. $\square$

Corollary 4.2. Let $(\{\mathfrak{T}_n\}, \mathfrak{U})$ $(\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H}), \mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ be a near exact $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\{\mathfrak{F}_k\}_{k=1}^m \subset \mathcal{X}^*$ be a linearly independent set and for each k $(1 \leq k \leq m)$, there exists an $\xi_k \in \mathcal{X}$ as defined in (4.5). If $\left(\left\{\mathfrak{T}_i + \sum_{k=1}^{m}\left(\lim_{l\to\infty}\sum_{\substack{n=1\\ n\neq i}}^{m_l}\alpha_n^{(l)}\mathfrak{T}_n(\xi_k)\right)\mathfrak{F}_k\right\}, \mathfrak{U}_0\right)$ is $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$, then -1 is not an eigenvalue of the matrix as given in Theorem 4.1.

Proof. Take $p = 1$ in Theorem 4.1.

As an applicability of Theorem 4.1, we give the following example:

Example 4.3. Let $\mathcal{H}$ be a separable Hilbert space and let

$$E = \ell^\infty(\mathcal{H}) = \left\{\{x_n\} : x_n \in \mathcal{H};\ \sup_{1\leq n<\infty}\|x_n\|_{\mathcal{H}} < \infty\right\}$$

be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1\leq n<\infty}\|x_n\|_{\mathcal{H}},\ \{x_n\} \in E.$$

Now, for each $n \in \mathbb{N}$, define $\mathfrak{T}_n : E \to \ell^2(\mathcal{H})$ by

$$\begin{cases}\mathfrak{T}_1(x) = \delta_1^{x_1}\\ \mathfrak{T}_n(x) = \delta_{n-1}^{x_{n-1}},\ n \geq 2,\ n \in \mathbb{N},\ x = \{x_n\} \in E,\end{cases}$$

where $\delta_n^x = (0, \ldots, 0\ \underbrace{x}_{n^{th}\text{place}}\ , 0, \ldots)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Then $\{\mathfrak{T}_n\}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{\mathfrak{T}_n(x)\} : x \in \mathcal{X}\}$ and a reconstruction operator $\mathfrak{U} : \mathcal{A} \to E$ such that $(\{\mathfrak{T}_n\}, \mathfrak{U})$ is a near exact $g-$Banach frame for E with respect to $\mathcal{A}$.

Let $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in E^*$ be given by

$$\mathfrak{F}_1 = (1, 0, 0, 0, ...),\ \mathfrak{F}_2 = (0, 1, 0, 0, ...),\ \mathfrak{F}_3 = (0, 0, 1, 0, ...)$$

and $\xi_1, \xi_2, \xi_3 \in E$ be given by

$$\xi_1 = (-1, 0, 0, 0, ...),\ \xi_2 = (0, -1, 0, 0, ...),\ \xi_3 = (0, 0, -1, 0, ...).$$

Then -1 is an eigenvalue of the matrix

$$\begin{pmatrix}\mathfrak{F}_1(\xi_1) & \mathfrak{F}_1(\xi_2) & \mathfrak{F}_1(\xi_3)\\ \mathfrak{F}_2(\xi_1) & \mathfrak{F}_2(\xi_2) & \mathfrak{F}_2(\xi_3)\\ \mathfrak{F}_3(\xi_1) & \mathfrak{F}_3(\xi_2) & \mathfrak{F}_3(\xi_3)\end{pmatrix}.$$

Therefore, by Theorem 4.1, there exists no reconstruction operator $\mathfrak{U}_0 : \mathcal{A} \to E$ such that $\left(\left\{ \mathfrak{T}_i + \sum_{k=1}^{3} \left(\lim_{l\to\infty} \sum_{\substack{n=1\\ n\neq i}}^{m_l} \alpha_n^{(l)} \mathfrak{T}_n(\xi_k) \right) \mathfrak{F}_k \right\}, \mathfrak{U}_0 \right)$ is a $g-$Banach frame for E with respect to $\mathcal{A}$, where $\mathfrak{T}_i(\xi_k) = \lim_{l\to\infty} \sum_{\substack{n=1\\ n\neq i}}^{m_l} \alpha_n^{(l)} \mathfrak{T}_n(\xi_k)$, for all $i \in \mathbb{N}$ and $k = 1, 2, 3$.

References

[1] M.R. Abdollahpour, M.H. Faroughi and A. Rahimi: *PG*−Frames in Banach Spaces, Methods of Functional Analysis and Topology, 13(3)(2007), 201-210.
[2] O. Christensen: An introduction to Frames and Riesz Bases, Birkhäuser, Boston (2003).
[3] R.J. Duffin and A.C. Schaeffer: A class of non-harmonic Fourier series, Trans. Amer. Math. Soc., 72(1952), 341–366.
[4] H. G. Feichtinger and K. H. Gröchenig, A unified approach to atomic decompositions via integrable group representations, Function spaces and Applications, *Lecture Notes in Mathematics*, 1302 (1988), 52-73.
[5] K.H. Gröchenig, Describing functions: Atomic decompositions versus frames, Monatsh. fur Mathematik, 112(1991), 1-41.
[6] P.K. Jain, S. K. Kaushik and L. K. Vashisht, On Banach frames, Indian J. Pure Appl. Math., 37(5)(2006), 265-272.
[7] P.K. Jain, S. K. Kaushik and L. K. Vashisht, On perturbation of Banach frames, Int. J. Wavelets, Multiresolution and Information Processing, 4(3)(2006), 559-565.
[8] P.K. Jain, S. K. Kaushik and L. K. Vashisht, On stability of Banach frames, Bull. Korean Math. Soc., 44(1) (2007), 73-81.
[9] S.K. Kaushik and Varinder Kumar, Frames of subspaces for Banach spaces, Int. J. Wavelets, Multiresolution and Information Processing, 8(2)(2010), 243-252.
[10] S.K. Kaushik and Varinder Kumar, A note on fusion Banach frames, Arch. Math., 46(3)(2010), 203-209.
[11] S.K. Kaushik and Varinder Kumar, On fusion frames in Banach spaces, Georgian Math. J., 18(2011), 121-130.
[12] H.K. Pathak and Mayur Puri Goswami, Some characterization of fusion Banach frames, Int. J. Wavelets, Multiresolution and Information Processing, 13(3)(2015), 1550015, 11pp.
[13] Ghanshyam Singh Rathore and Tripti Mittal; On *G*−Banach Bessel sequences in Banach spaces, Global Journal of Pure and Applied Mathematics, 13(9) (2017), 5129-5142.
[14] Ghanshyam Singh Rathore and Tripti Mittal; On Weighted *G*−Banach frames in Banach spaces, Poincare Journal of Analysis and Applications, 2018(2(II)), Special Issue (IWWFA-III, Delhi), 13-26.
[15] Ghanshyam Singh Rathore and Tripti Mittal; On *G*−Banach frames, Jordan Journal of Mathematical and Statistics, Jordan Journal of Mathematics and Statistics (JJMS), 12(4) (2019), 498 - 519 .
[16] Ghanshyam Singh Rathore and Tripti Mittal; On Perturbation of Weighted *G*−Banach frames in Banach spaces, Journal of the Indian Math. Soc., 87(1-2) (2020), 114-130.

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