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EXISTENCE AND NONEXISTENCE RESULTS FOR HIGHER ORDER DIFFERENTIAL EQUATIONS WITH NON-HOMOGENEOUS INTEGRAL BOUNDARY CONDITIONS

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***Abstract*.** The goal of this paper is to establish the existence and nonexistence of positive solutions to differential equations of order p

$$w^{(p)}(x) + a(x)f(w(x)) = 0, \quad x \in [0, 1],$$

satisfying non-homogeneous integral boundary conditions

$$w^{(i)}(0) = 0, \ i = 0, 1, 2, ..., p-2, \ w^{(r)}(1) - \eta \int_0^1 g(\tau)w^{(r)}(\tau)d\tau = \lambda,$$

where $r \in \{1, 2, \cdots, p-2\}$ but fixed, $p \geq 3$ and $\eta, \lambda \in (0, \infty)$ are parameters, using the Guo–Krasnosel'skii fixed point theorem.

1. Introduction

The theory of differential equations has been used in modeling of physical, biological and medical sciences as well as economics to find optimum investment strategies. In analyzing real world problems, many mathematical models give rise to either initial value problems or boundary value problems. A particular class of problems involving integral boundary conditions arise in thermoelectricity, chemical engineering, plasma physics, and other fields. In these applied settings, only the positive solutions are relevant. To mention a few papers dealing with problems involving integral boundary conditions, see [5, 8, 1, 10, 7] and for the problems involving non-homogeneous boundary conditions, see [2, 11, 13, 9]. The results are now extended to the problem together with non-homogeneous integral boundary conditions.

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Consider the higher order boundary value problem with non-homogeneous integral boundary conditions

$$w^{(p)}(x)+a(x)f(w(x))=0, \quad x\in[0,1], \tag{1.1}$$

$$w^{(i)}(0)=0,\ i=0,1,2,...,p-2,\ w^{(r)}(1)-\eta\int_0^1 g(\tau)w^{(r)}(\tau)d\tau=\lambda, \tag{1.2}$$

where $r\in\{1,2,...,p-2\}$ but fixed, $p\geq 3$ and $\eta,\lambda\in(0,\infty)$ are parameters and establish the existence and nonexistence results. For $\eta=0$ and $\lambda=0$, Eloe and Henderson studied the existence of positive solutions for higher order two-point boundary value problem, see [3]. For $p=3$, $\eta=1$ and $\lambda=0$, Sun and Li addressed the third order problem with integral boundary conditions and proved the existence of positive solutions, see [12]. By a solution of the boundary value problem (1.1) and (1.2), we mean that $w\in C^{(p)}[0,1]$ satisfying (1.1) and (1.2).

In order to prove results, the following assumptions are made throughout the paper:

(C1) $f(x)\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $g(x)\in C([0,1],\mathbb{R}^+)$,

(C2) $a(x)\in C([0,1],\mathbb{R}^+)$ and $a(x)$ is not identically zero on any closed subinterval of $[0,1]$,

(C3) $1-\eta\theta>0$, where $\theta=\int_0^1 g(\tau)\tau^{p-r-1}d\tau$, and

(C4) the nonnegative extended real numbers f_0 and f_∞ exist and are defined by

$$f_0=\lim_{w\to 0^+}\frac{f(w)}{w} \text{ and } f_\infty=\lim_{w\to\infty}\frac{f(w)}{w}.$$

The conditions $f_0=0$ and $f_\infty=\infty$ represent superlinear, where as the other conditions $f_0=\infty$ and $f_\infty=0$ represent sublinear.

The remaining part of the article is structured as follows. The solution to (1.1) and (1.2) is written as an analogous integral equation in terms of kernels, and bounds for kernels are determined in Section 2. In Section 3, we use the Guo–Krasnosel'skii fixed point theorem to demonstrate existence as well as nonexistence results to the problem (1.1) and (1.2). As an application, we present our results through examples.

2. Kernels and their estimates

The solution of the problem (1.1) and (1.2) is expressed as an analogous integral equation involving kernels and several inequalities are established for kernels in this section.

Lemma 2.1. *Assume that the condition* $(C3)$ *is fulfilled. If* $\varphi(x)\in C([0,1],\mathbb{R}^+)$ *then the problem*

$$w^{(p)}(x)+\varphi(x)=0, \quad x\in[0,1], \tag{2.1}$$

with (1.2) *has a unique solution and is*

$$\begin{aligned} w(x)={}&\frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)}+\int_0^1 R(x,\vartheta)\varphi(\vartheta)d\vartheta\\ &+\frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1\left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right]\varphi(\vartheta)d\vartheta, \end{aligned} \tag{2.2}$$

$$R(x,\vartheta)=\frac{1}{(p-1)!}\begin{cases}[x^{p-1}(1-\vartheta)^{p-r-1}-(x-\vartheta)^{p-1}], & 0\leq\vartheta\leq x\leq 1,\\ x^{p-1}(1-\vartheta)^{p-r-1}, & 0\leq x\leq\vartheta\leq 1,\end{cases} \tag{2.3}$$

and

$$S(\tau,\vartheta)=\begin{cases}\tau^{p-r-1}(1-\vartheta)^{p-r-1}-(\tau-\vartheta)^{p-r-1}, & 0\le\vartheta\le\tau\le 1,\\ \tau^{p-r-1}(1-\vartheta)^{p-r-1}, & 0\le\tau\le\vartheta\le 1.\end{cases}\tag{2.4}$$

Proof. Let $w(x)$ be the solution of the problem (2.1) and (1.2). Then an equivalent integral equation of (2.1) is

$$w(x)=d_0+d_1x+d_2x^2+\cdots+d_{p-1}x^{p-1}-\frac{1}{(p-1)!}\int_0^x(x-\vartheta)^{p-1}\varphi(\vartheta)d\vartheta.$$

By applying conditions (1.2), one can get

$$\begin{aligned}
&d_i=0,\ \text{for } i=0,1,...,p-2,\quad\text{and}\\
&d_{p-1}=\frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)}+\frac{1}{(p-1)!(1-\eta\theta)}\bigg\{\int_0^1(1-\vartheta)\varphi(\vartheta)d\vartheta\\
&-\eta\int_0^1 g(\tau)\left[\int_0^\tau(\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau\bigg\}.
\end{aligned}$$

Then, the unique solution of (2.1) and (1.2) is

$$\begin{aligned}
w(x)&=\frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)}+\frac{x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\\
&-\frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1 g(\tau)\left[\int_0^\tau(\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau\\
&-\frac{1}{(p-1)!}\int_0^x(x-\vartheta)^{p-1}\varphi(\vartheta)d\vartheta\\
&=\frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)}+\frac{x^{p-1}(1-\eta\theta+\eta\theta)}{(p-1)!(1-\eta\theta)}\int_0^1(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\\
&-\frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1 g(\tau)\left[\int_0^\tau(\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau\\
&-\frac{1}{(p-1)!}\int_0^x(x-\vartheta)^{p-1}\varphi(\vartheta)d\vartheta\\
&=\frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)}+\frac{1}{(p-1)!}\bigg[\int_0^x\left(x^{p-1}(1-\vartheta)^{p-r-1}-(x-\vartheta)^{p-1}\right)\\
&\varphi(\vartheta)d\vartheta+\int_x^1 x^{p-1}(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\bigg]\\
&+\frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1\int_0^1 g(\tau)\tau^{p-r-1}(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\tau d\vartheta\\
&-\frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1 g(\tau)\left[\int_0^\tau(\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau
\end{aligned}$$

$$= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)\varphi(\vartheta)d\vartheta$$
$$+ \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 g(\tau)\Big[\int_0^\tau \big(\tau^{p-r-1}(1-\vartheta)^{p-r-1} - (\tau-\vartheta)^{p-r-1}\big)$$
$$\varphi(\vartheta)d\vartheta + \int_\tau^1 \tau^{p-r-1}(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\Big]d\tau$$
$$= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)\varphi(\vartheta)d\vartheta$$
$$+ \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right]\varphi(\vartheta)d\vartheta.$$

□

We note that $w(x)$ is a solution of the boundary value problem (1.1) and (1.2) if, and only if $w(x)$ is a solution of the integral equation

$$\begin{aligned} w(x) = &\frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\ &+ \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)f(w(\vartheta))d\vartheta, \end{aligned} \tag{2.5}$$

where $R(x,\vartheta)$ and $S(\tau,\vartheta)$ are given in (2.3) and (2.4) respectively.

Lemma 2.2. *If the condition* $(C3)$ *is fulfilled, then the kernels* $R(x,\vartheta)$ *and* $S(x,\vartheta)$ *satisfy the following inequalities:*

(i) $R(x,\vartheta) \ge 0$ *and* $S(x,\vartheta) \ge 0$ *for all* $x,\vartheta \in [0,1]$,
(ii) $R(x,\vartheta) \le R(1,\vartheta)$ *for all* $x,\vartheta \in [0,1]$,
(iii) $R(x,\vartheta) \ge \frac{1}{4^{p-1}}R(1,\vartheta)$ *for all* $x \in I$ *and* $\vartheta \in [0,1]$, *where* $I = [\frac{1}{4},\frac{3}{4}]$.

Proof. The above inequalities are obtained by algebraic computations. □

The Guo–Krasnosel'skii fixed point theorem described below will serve as the foundation for presenting our main findings.

Theorem 2.3. [6, 4] *Let* $\mathfrak{B}$ *be a Banach Space and* ρ *be a cone in* $\mathfrak{B}$. *Suppose* Ω_1 *and* Ω_2 *are any two open subsets of* $\mathfrak{B}$ *such that* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Suppose further the completely continuous operator* $\mathfrak{D} : \rho \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \rho$ *satisfy the following conditions either*

(i) $\|\mathfrak{D}w\| \le \|w\|$, *for* $w \in \rho \cap \partial\Omega_1$ *and* $\|\mathfrak{D}w\| \ge \|w\|$, *for* $w \in \rho \cap \partial\Omega_2$, *or*
(ii) $\|\mathfrak{D}w\| \ge \|w\|$, *for* $w \in \rho \cap \partial\Omega_1$ *and* $\|\mathfrak{D}w\| \le \|w\|$, *for* $w \in \rho \cap \partial\Omega_2$.

Then the operator $\mathfrak{D}$ *has a fixed point in* $\rho \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence and nonexistence results

In this section, we develop criteria for the existence and nonexistence of positive solutions to the problem (1.1) and (1.2).

For this, let $\mathfrak{B} = \{w \mid w \in C[0,1]\}$ be the Banach space with the norm

$$\|w\| = \max_{x\in[0,1]} |w(x)|.$$

Define a cone ρ in $\mathfrak{B}$ as follows:

$$\rho = \Big\{ w \in \mathfrak{B} \mid w(x) \geq 0 \text{ on } x \in [0,1] \text{ and } \min_{x \in I} w(x) \geq \frac{1}{4^{p-1}} \|w\| \Big\}.$$

Let us define an operator $\mathfrak{D} : \rho \to \mathfrak{B}$ as

$$\begin{aligned} \mathfrak{D}w(x) = {} & \frac{\lambda(p-r-1)! x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta) a(\vartheta) f(w(\vartheta)) d\vartheta \\ & + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta) g(\tau) d\tau \right] a(\vartheta) f(w(\vartheta)) d\vartheta. \end{aligned} \tag{3.1}$$

Lemma 3.1. *The operator $\mathfrak{D} : \rho \to \mathfrak{B}$ defined in the equation* (3.1) *is a self map on the cone ρ.*

Proof. From the positivity of kernels $R(x,\vartheta)$, $S(x,\vartheta)$ and for $w \in \rho$, $\mathfrak{D}w(x) \geq 0$ on $x \in [0,1]$. Now, for $w \in \rho$ and by Lemma 2.2, one can obtain

$$\begin{aligned} \mathfrak{D}w(x) = {} & \frac{\lambda(p-r-1)! x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta) a(\vartheta) f(w(\vartheta)) d\vartheta \\ & + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta) g(\tau) d\tau \right] a(\vartheta) f(w(\vartheta)) d\vartheta \\ \leq {} & \frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta) a(\vartheta) f(w(\vartheta)) d\vartheta \\ & + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta) g(\tau) d\tau \right] a(\vartheta) f(w(\vartheta)) d\vartheta \end{aligned}$$

so,

$$\begin{aligned} \|\mathfrak{D}w(x)\| \leq {} & \frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta) a(\vartheta) f(w(\vartheta)) d\vartheta \\ & + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta) g(\tau) d\tau \right] a(\vartheta) f(w(\vartheta)) d\vartheta. \end{aligned} \tag{3.2}$$

Next, if $w \in \rho$, we have from Lemma 2.2 and (3.2) that

$$\begin{aligned} \min_{x \in I} \mathfrak{D}w(x) = {} & \min_{x \in I} \Bigg\{ \frac{\lambda(p-r-1)! x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta) a(\vartheta) f(w(\vartheta)) d\vartheta \\ & + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta) g(\tau) d\tau \right] a(\vartheta) f(w(\vartheta)) d\vartheta \Bigg\} \\ \geq {} & \frac{1}{4^{p-1}} \Bigg[\frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta) a(\vartheta) f(w(\vartheta)) d\vartheta \\ & + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta) g(\tau) d\tau \right] a(\vartheta) f(w(\vartheta)) d\vartheta \Bigg] \\ \geq {} & \frac{1}{4^{p-1}} \|\mathfrak{D}w(x)\|. \end{aligned}$$

Therefore, $\mathfrak{D} : \rho \to \rho$ and the proof is complete. □

Furthermore, $\mathfrak{D}$ is completely continuous operator based on Arzela–Ascoli theorem. Let us take

$$\mathcal{L}_1 = \frac{1}{2}\left\{ \int_0^1 R(1,\vartheta)a(\vartheta)d\vartheta + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)d\vartheta \right\}^{-1} \tag{3.3}$$

and

$$\begin{aligned}\mathcal{L}_2 = \Bigg\{ \frac{1}{4^{2p-2}} \Bigg[&\int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)d\vartheta \\ &+ \frac{\eta}{(p-1)!(1-\eta\theta)} \int_{\vartheta\in I} \left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)d\vartheta \Bigg]\Bigg\}^{-1}.\end{aligned} \tag{3.4}$$

Theorem 3.2. *Suppose the conditions mentioned in* $(C1)$, $(C2)$ *and* $(C3)$ *are fulfilled. If the conditions* $f_0 = 0$ *and* $f_\infty = \infty$ *hold, then for* λ *small enough, the problem* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. By the definition of $f_0 = 0$, there exist $\mathcal{L}_1 > 0$ and $\mathcal{H}_1 > 0$ such that

$$f(w) \le \mathcal{L}_1 w, \ 0 < w \le \mathcal{H}_1. \tag{3.5}$$

Let λ be satisfy

$$0 < \lambda \le \frac{(p-1)!(1-\eta\theta)\mathcal{H}_1}{(p-r-1)!2}. \tag{3.6}$$

Choose $w \in \rho$ and $\|w\| = \mathcal{H}_1$. Then from Lemma 2.2, and by inequalities (3.5) and (3.6), we have

$$\begin{aligned}\mathfrak{D}w(x) &= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\ &\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\ &\le \frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\ &\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\ &\le \frac{\mathcal{H}_1}{2} + \mathcal{L}_1\Bigg\{ \int_0^1 R(1,\vartheta)a(\vartheta)w(\vartheta)d\vartheta \\ &\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)w(\vartheta)d\vartheta \Bigg\} \\ &\le \frac{\mathcal{H}_1}{2} + \mathcal{L}_1\Bigg\{ \int_0^1 R(1,\vartheta)a(\vartheta)d\vartheta \\ &\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)d\vartheta \Bigg\}\|w\| \\ &= \frac{\mathcal{H}_1}{2} + \frac{\mathcal{H}_1}{2} = \mathcal{H}_1,\end{aligned}$$

so that

$$\mathfrak{D}w(x) \leq \mathcal{H}_1.$$

Therefore, $\|\mathfrak{D}w(x)\| \leq \mathcal{H}_1 = \|w\|$. If we set

$$\Omega_1 = \{w \in \mathfrak{B} \mid \|w\| < \mathcal{H}_1\},$$

then

$$\|\mathfrak{D}w(x)\| \leq \|w\|, \text{ for } w \in \rho \cap \partial\Omega_1. \tag{3.7}$$

Furthermore, by definition of $f_\infty = \infty$, there exist $\mathcal{L}_2 > 0$ and $\bar{\mathcal{H}}_2 > 0$ such that

$$f(w) \geq \mathcal{L}_2 w, \; w \geq \bar{\mathcal{H}}_2. \tag{3.8}$$

Let $\mathcal{H}_2 = \max\left\{2\mathcal{H}_1, 4^{p-1}\bar{\mathcal{H}}_2\right\}$. Choose $w \in \rho$ and $\|w\| = \mathcal{H}_2$. Then

$$\min_{x\in I} w(x) \geq \frac{1}{4^{p-1}}\|w\| \geq \bar{\mathcal{H}}_2. \tag{3.9}$$

Then from Lemma 2.2, and by inequalities (3.8) and (3.9), we can conclude that

$$\begin{aligned}
\mathfrak{D}w(x) &= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\
&\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\
&\geq \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\
&\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\
&\geq \min_{x\in I}\left\{ \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \right. \\
&\quad \left. + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \right\} \\
&\geq \frac{1}{4^{p-1}}\left\{ \int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \right. \\
&\quad \left. + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_{\vartheta\in I} \left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \right\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\mathcal{L}_2}{4^{p-1}}\left\{\int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)w(\vartheta)d\vartheta\right.\\
&\quad\left.+\frac{\eta}{(p-1)!(1-\eta\theta)}\int_{\vartheta\in I}\left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)w(\vartheta)d\vartheta\right\}\\
&\geq \frac{\mathcal{L}_2}{4^{2p-2}}\left\{\int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)d\vartheta\right.\\
&\quad\left.+\frac{\eta}{(p-1)!(1-\eta\theta)}\int_{\vartheta\in I}\left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)d\vartheta\right\}\|w\|\\
&=\|w\|,
\end{aligned}
$$

so that

$$\mathfrak{D}w(x) \geq \|w\|.$$

Hence, $\|\mathfrak{D}w\| \geq \|w\|$. Now, if we take

$$\Omega_2 = \{w \in \mathfrak{B} \mid \|w\| < \mathcal{H}_2\},$$

then

$$\|\mathfrak{D}w\| \geq \|w\|, \text{ for } w \in \rho \cap \partial\Omega_2. \tag{3.10}$$

When Theorem 2.3 applied to the equations (3.7) and (3.10), the operator $\mathfrak{D}$ has at least one fixed point $w \in \rho \cap (\Omega_2 \backslash \overline{\Omega_1})$ and that w is the positive solution of (1.1) and (1.2). □

Theorem 3.3. *Suppose the conditions mentioned in* ($C1$), ($C2$) *and* ($C3$) *are fulfilled. If the conditions $f_0 = 0$ and $f_\infty = \infty$ holds, then for λ large enough, the problem* (1.1) *and* (1.2) *has no positive solution.*

Proof. We assume that for any positive integer q, there exist $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_q < \cdots$ with $\lim\limits_{q\to\infty} \lambda_q = \infty$ such that the problem

$$
\begin{aligned}
&w^{(p)}(x) + a(x)f(w(x)) = 0, \quad x \in [0,1],\\
&w^{(i)}(0) = 0,\ i = 0,1,2,...,p-2,\ w^{(r)}(1) - \eta\int_0^1 g(\tau)w^{(r)}(\tau)d\tau \quad = \lambda_q,
\end{aligned}
$$

has a positive solution and is $w_q(x)$. Then by the equation (2.5), we have

$$
\begin{aligned}
w_q(1) &= \frac{\lambda_q(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta\\
&\quad+\frac{\eta}{(p-1)!(1-\eta\theta)}\int_0^1\left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)f(w(\vartheta))d\vartheta\\
&\geq \frac{\lambda_q(p-r-1)!}{(p-1)!(1-\eta\theta)} \to \infty \text{ as } q \to \infty.
\end{aligned}
$$

Thus, $\|w_q\| \to \infty$ as $q \to \infty$.

Since the definition of $f_\infty = \infty$, there exist $\mathcal{L}_2 > 0$ and $\overline{\mathcal{H}} > 0$ such that

$$f(w) \geq 2\mathcal{L}_2 w,\ w \geq \overline{\mathcal{H}}.$$

Let q be the large enough that $\|w_q\| \geq \overline{\mathcal{H}}$. Then

$$\begin{aligned}
\|w_q\| &\geq w_q(1) \\
&= \frac{\lambda_q(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta)a(\vartheta)f(w_q(\vartheta))d\vartheta \\
&\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w_q(\vartheta))d\vartheta \\
&\geq \int_0^1 R(1,\vartheta)a(\vartheta)f(w_q(\vartheta))d\vartheta \\
&\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w_q(\vartheta))d\vartheta \\
&\geq 2\mathcal{L}_2 \Bigg\{ \int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)w_q(\vartheta)d\vartheta \\
&\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_{\vartheta\in I} \left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)\tau\right] a(\vartheta)w_q(\vartheta)d\vartheta \Bigg\} \\
&\geq \frac{2\mathcal{L}_2}{4^{2p-2}} \Bigg\{ \int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)d\vartheta \\
&\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_{\vartheta\in I} \left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)d\vartheta \Bigg\} \|w_q\| \\
&= 2\|w_q\|,
\end{aligned}$$

which leads to contradiction, and hence the theorem. □

Theorem 3.4. *Suppose the conditions mentioned in* $(C1)$, $(C2)$ *and* $(C3)$ *are fulfilled. If the conditions* $f_0 = \infty$ *and* $f_\infty = 0$ *hold, then for any* $\lambda \in (0,\infty)$, *the problem* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. By definition of $f_0 = \infty$, there exist $\mathcal{L}_3 > 0$ and $\mathcal{J}_1 > 0$ such that

$$f(w) \geq \mathcal{L}_3 w, \; 0 < w \leq \mathcal{J}_1,$$

where $\mathcal{L}_3 \geq \mathcal{L}_2$ and $\mathcal{L}_2$ is given in (3.4). Then for $w \in \rho$ and $\|w\| = \mathcal{J}_1$, we have

$$\begin{aligned}
\mathfrak{D}w(x) &= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\
&\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\
&\geq \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\
&\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta
\end{aligned}$$

$$\geq \min_{x\in I}\Bigg\{\int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta$$
$$+\frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)}\int_0^1\left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)f(w(\vartheta))d\vartheta\Bigg\}$$
$$\geq \frac{1}{4^{p-1}}\Bigg\{\int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta$$
$$+\frac{\eta}{(p-1)!(1-\eta\theta)}\int_{\vartheta\in I}\left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)f(w(\vartheta))d\vartheta\Bigg\}$$
$$\geq \frac{\mathcal{L}_2}{4^{p-1}}\Bigg\{\int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)w(\vartheta)d\vartheta$$
$$+\frac{\eta}{(p-1)!(1-\eta\theta)}\int_{\vartheta\in I}\left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)w(\vartheta)d\vartheta\Bigg\}$$
$$\geq \frac{\mathcal{L}_2}{4^{2p-2}}\Bigg\{\int_{\vartheta\in I} R(1,\vartheta)a(\vartheta)d\vartheta$$
$$+\frac{\eta}{(p-1)!(1-\eta\theta)}\int_{\vartheta\in I}\left[\int_{\tau\in I} S(\tau,\vartheta)g(\tau)d\tau\right]a(\vartheta)d\vartheta\Bigg\}\|w\|$$
$$=\|w\|,$$

Therefore, $\|\mathfrak{D}w\|\geq\|w\|$. Now, if we set

$$\Omega_3=\{w\in\mathfrak{B}\mid\|w\|<\mathcal{J}_1\},$$

then

$$\|\mathfrak{D}w\|\geq\|w\|,\ \text{for } w\in\rho\cap\partial\Omega_3. \tag{3.11}$$

Further, by definition of $f^{\infty}=0$, there exist $\mathcal{L}_4>0$ and $\bar{\mathcal{J}}_2>0$ such that

$$f(w)\leq\mathcal{L}_4 w,\ w\geq\bar{\mathcal{J}}_2,$$

where $\mathcal{L}_4\leq\mathcal{L}_1$ and $\mathcal{L}_1$ is given in (3.3).

We establish the result by considering the following.

Case (i): Assume that the function f is bounded. Then there exists $\mathcal{M}>0$ such that

$$f(w)\leq\mathcal{M},\ \text{for } 0\leq w<\infty.$$

In this case, we choose

$$\mathcal{J}_2\geq\max\left\{2\mathcal{J}_1,\ \frac{\mathcal{M}}{\mathcal{L}_1},\ \frac{2\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)}\right\},$$

and then for $w \in \rho$ and $\|w\| = \mathcal{J}_2$, we have

$$\begin{aligned}
\mathfrak{D}w(x) &= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\
&\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\
&\le \frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)} + \mathcal{M}\int_0^1 R(1,\vartheta)a(\vartheta) \\
&\quad + \frac{\eta\mathcal{M}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)d\vartheta \\
&\le \frac{\mathcal{J}_2}{2} + \frac{\mathcal{J}_2}{2} = \mathcal{J}_2.
\end{aligned}$$

So, $\|\mathfrak{D}w\| \le \|w\|$.
Case (ii): Assume that the function f is unbounded. Choose

$$\mathcal{J}_2 \ge \max\left\{2\mathcal{J}_1,\ \bar{\mathcal{J}}_2,\ \frac{2\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)}\right\},$$

such that

$$f(w) \le f(\mathcal{J}_2),\ \text{for } 0 \le w \le \mathcal{J}_2.$$

Then for $w \in \rho$ and $\|w\| = \mathcal{J}_2$, we have

$$\begin{aligned}
\mathfrak{D}w(x) &= \frac{\lambda(p-r-1)!x^{p-1}}{(p-1)!(1-\eta\theta)} + \int_0^1 R(x,\vartheta)a(\vartheta)f(w(\vartheta))d\vartheta \\
&\quad + \frac{\eta x^{p-1}}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(w(\vartheta))d\vartheta \\
&\le \frac{\lambda(p-r-1)!}{(p-1)!(1-\eta\theta)} + \int_0^1 R(1,\vartheta)a(\vartheta)f(\mathcal{J}_2)d\vartheta \\
&\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)f(\mathcal{J}_2)d\vartheta \\
&\le \frac{\mathcal{J}_2}{2} + \mathcal{L}_1\mathcal{J}_2\Bigg\{R(1,\vartheta)a(\vartheta)d\vartheta \\
&\quad + \frac{\eta}{(p-1)!(1-\eta\theta)} \int_0^1 \left[\int_0^1 S(\tau,\vartheta)g(\tau)d\tau\right] a(\vartheta)d\vartheta\Bigg\} \\
&= \frac{\mathcal{J}_2}{2} + \frac{\mathcal{J}_2}{2} = \mathcal{J}_2.
\end{aligned}$$

So, $\|\mathfrak{D}w\| \le \|w\|$. Therefore, in either case by setting

$$\Omega_4 = \{w \in \rho \mid \|w\| < \mathcal{J}_2\},$$

then

$$\|\mathfrak{D}w\| \le \|w\|,\ \text{for } w \in \rho \cap \partial\Omega_4. \tag{3.12}$$

When Theorem 2.3 applied to the equations (3.11) and (3.12), the operator $\mathfrak{D}$ has at least one fixed point $w \in \rho \cap (\Omega_4 \setminus \bar{\Omega}_3)$ and that w is the positive solution of (1.1) and (1.2). □

4. Examples

Consider the following examples to demonstrate our results.

Example 4.1. Let $p = 4$ and $r = 2$. Consider the problem

$$\left.\begin{aligned} &w^{(4)}(x) + a(x)f(w(x)) = 0, \quad x \in [0,1],\\ &w^{(i)}(0) = 0,\ i = 0,1,2,\ w^{(2)}(1) - \eta\int_0^1 g(\tau)w^{(2)}(\tau)d\tau = \lambda, \end{aligned}\right\} \tag{4.1}$$

where $a(x) = 1$, $f(w(x)) = w^2(x)$ and $g(x) = x$. Then all the assumptions of Theorem 3.2 are fulfilled and hence, the problem (4.1) has at least one positive solution by taking λ such that $0 < \lambda \le \frac{5}{2}\mathcal{H}_1$, where $\mathcal{H}_1 > 0$. According to Theorem 3.3, the problem (4.1) has no positive solution for λ large enough.

Example 4.2. Let $p = 5$ and $r = 2$. Consider the problem

$$\left.\begin{aligned} &w^{(5)}(x) + a(x)f(w(x)) = 0, \quad x \in [0,1],\\ &w^{(i)}(0) = 0,\ i = 0,1,2,3,\ w^{(2)}(1) - \eta\int_0^1 g(\tau)w^{(2)}(\tau)d\tau = \lambda, \end{aligned}\right\} \tag{4.2}$$

where $a(x) = 1$, $f(w(x)) = sin^2 w(x)$ and $g(x) = x^2$. Then all of Theorem 3.4 assumptions are fulfilled and hence, the problem (4.2) has at least one positive solution for any $\lambda \in (0, \infty)$.

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