Poincare Journal of Analysis & Applications
Vol. 8, No. 2 (2021), 133-143

THE TOPOLOGY OF θ_e-OPEN SETS

ALDISON M. ASDAIN AND MHELMAR A. LABENDIA[†]

Date of Receiving	:	27. 09. 2021
Date of Revision	:	28. 11. 2021
Date of Acceptance	:	30. 12. 2021

Abstract. This paper aims is to introduce a new class of open set defined using e-closure operator, which we call the θ_e-open set. It is worth noting that the family of all θ_e-open sets forms a topology. We then investigate the relationship of this set to the other well-known concepts in topology such as the classical open, θ-open and e-open sets. We also characterize the concepts of θ_e-connected space, some versions of separation axioms with respect to θ_e-open sets, and θ_e-continuous function from an arbitrary topological space into the product space.

1. Introduction and Preliminaries

Open sets are the fundamental building blocks of topology. Many authors have developed different versions of open sets including its weaker and stronger versions. The first initiation was done by Levine [16] in 1963 where he introduce the concepts of semi-open set, semi-closed set and semi-continuity of a function.

A subset O of a topological space X is semi-open [16] if $O \subseteq Cl(Int(O))$. Equivalently, O is semi-open if there exists an open set G in X such that $G \subseteq O \subseteq Cl(G)$. A subset F of X is semi-closed if its complement $X \setminus F$ is semi-open in X. Let A be a subset of a space X. A point $p \in X$ is a semi-closure point of A if for every semi-open set G in X containing x, $G \cap A \neq \varnothing$. We denote by $sCl(A)$ the set of all semi-closure points of A.

In 1968, Veličko [23] introduce the concept θ-continuity between topological spaces and defined the concepts of θ-closure and θ-interior of a set. The work of Veličko was continued by Dickman and Porter [5, 6], Joseph [14], and Long and Herrington [17]. Then several authors have obtained interesting results related to θ-open sets, see [1, 4, 12, 13, 15, 19].

2010 *Mathematics Subject Classification.* 54A10, 54A05.
Key words and phrases. θ_e-open, θ_e-closed, θ_e-connected, θ_e-continuous.
The authors would like to thank the anonymous referee for his/her helpful comments for the improvement of this paper.
[†] *Corresponding author.*
Communicated by. Murad Ozkoc

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The θ-closure and θ-interior of A are, respectively, defined by

$$Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and $Int_\theta(A) = \{x \in X : Cl(U) \subseteq A$ for some open set U containing $x\}$, where $Cl(U)$ is the closure of U in X. A is θ-closed [23] if $Cl_\theta(A) = A$ and θ-open [23] if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser that $\mathcal{T}$.

In 2008, Ekici [8] introduced the notion of e-open set which is a generalization of δ-semi-open sets and δ-preopen sets. He also introduced the notions of $\mathcal{DP}$-sets, $\mathcal{DPE}$-sets, $\mathcal{DP}^*$-sets, $\mathcal{DPE}^*$-sets and obtained some new decomposition of continuity by using the notions of e-continuous functions, $\mathcal{DP}$-continuous functions, $\mathcal{DPE}$-continuous functions, $\mathcal{DP}^*$-continuous functions and $\mathcal{DPE}^*$-continuous functions. Some generalizations of e-open sets were studied in [2, 9, 20].

A set $A \subseteq X$ is said to be regular open [22](resp., regular closed [22]) if $A = Int(Cl(A))$ (resp., $A = Cl(Int(A))$). A point $x \in X$ is said to be a δ-cluster point of A if $A \cap V \neq \varnothing$ for every regular open set V containing x. The set of all δ-cluster points of A is called the δ-closure [23] of A and is denoted by $Cl_\delta(A)$. If $A = Cl_\delta(A)$ then A is called δ-closed and the complement of A, $X \setminus A$ is δ-open. Let A be a subset of a topological space X. Then the δ-interior [23] of A denoted by $Int_\delta(A)$ is the union of all regular open sets contained in A. A subset A of a topological space X is said to be δ-semi-open [21] (resp., e-open [8]) if $A \subseteq Cl(Int_\delta(A))$ (resp., $A \subseteq Cl(Int_\delta(A)) \cup Int(Cl_\delta(A))$). A subset F of X is said to be δ-semi-closed [21] (resp., e-closed [8]) if its complement $X \setminus F$ is δ-semi-open (resp., e-open). The δ-semi-closure [21] (resp., e-closure [8]) of A denoted by $sCl_\delta(A)$ (resp., e-$Cl(A)$) is defined by, $sCl_\delta(A) = \bigcap\{F : F$ is δ-semi-closed and $A \subseteq F\}$ (resp., e-$Cl(A) = \bigcap\{F : F$ is e-closed and $A \subseteq F\}$) and the δ-semi-interior [21] (resp., e-interior [8]) of A, denoted by $sInt_\delta(A)$ (resp., e-$Int(A)$) is defined by the union of all δ-semi-open (resp., e-open) sets contained in A. It is not difficult to see that every δ-semi-open set is an e-open set, hence e-$Cl(A) \subseteq sCl_\delta(A)$.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we define a new type of topology $\mathcal{T}_{\theta_e}$ consisting of θ_e-open sets and investigate its connection to some other well-known variations of open sets.

2. θ_e-Open Sets and Versions of Open and Closed Functions

This section presents the concepts of θ_e-open sets and θ_e-open and θ_e-closed functions.

Definition 2.1. Let X be a topological space. A subset A of X is said to be θ_e-open (resp., θ_s-open [11]) if for every $x \in A$, there exists an open set U containing x such that $e\text{-}Cl\,(U) \subseteq A$ (resp., $sCl\,(U) \subseteq A$). A subset F of X is called a θ_e-closed (resp., θ_s-closed [11]) if its complement $X \setminus F$ is a θ_e-open (resp., θ_s-open).

Remark 2.2. Let X be a topological space. If $A \subseteq X$ is θ_s-open, then A is θ_e-open but not conversely.

To verify this, suppose that A is θ_s-open. Let $x \in A$. Then for every open set U containing x, $sCl(U) \subseteq A$. Since U is also semi-open [18, p.1009], using [18, Theorem 3.3, p.1010], $e\text{-}Cl(U) \subseteq sCl_\delta(U) = sCl(U) \subseteq A$. Hence, A is θ_e-open.

To show that the converse is not necessarily true, consider $X = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then $\{c\}$ and $\{a, c\}$ are θ_e-open but not θ_s-open.

We also note that $(X, \mathcal{T})$ in Remark 2.2 may be used to verify that the concepts of semi-open set and e-open set are two independent notions since $\{b, c\}$ is semi-open but not e-open and $\{d\}$, $\{a, d\}$, and $\{a, b, d\}$ are e-open but not semi-open.

Remark 2.3. The following diagram holds for a subset of a topological space.

$$\begin{tikzcd}
\theta\text{-open} \arrow[d] & & \text{semi-open} & \delta\text{-semi-open} \arrow[l] \arrow[d] \\
\theta_s\text{-open} \arrow[r] & \theta_e\text{-open} \arrow[r] & \text{open} \arrow[u] \arrow[r] & e\text{-open}
\end{tikzcd}$$

We also remark that the above diagram is also true for their respective closed sets.

The reverse implication for θ_e-open and θ_s-open sets is not true as shown in Remark 2.2 and for open and θ_e-open sets is shown in the following example. Counterexamples for the other reverse implications are found in [8, 11, 18].

Example 2.4. Let $X = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\}$. Then $\{a, c, d\}$ is open but not θ_e-open.

Remark 2.5. Let X be a topological space and $A \subseteq X$. $e\text{-}Cl(A)$ is the smallest closed set containing A, that is, $e\text{-}Cl(A) = \cap\{F : F \text{ is } e\text{-closed and } A \subseteq F\}$.

Theorem 2.6. *Let $\mathcal{T}_{\theta_e}$ be the family of θ_e-open subsets of a topological space X. Then $\mathcal{T}_{\theta_e}$ forms a topology on X.*

Proof. In view of Remark 2.3, $\varnothing, X \in \mathcal{T}_{\theta_e}$.

Now, let $G_1, G_2 \in \mathcal{T}_{\theta_e}$ and let $x \in G_1 \cap G_2$. Then there exist open sets U_1 and U_2 both containing x such that $e\text{-}Cl(U_1) \subseteq G_1$ and $e\text{-}Cl(U_2) \subseteq G_2$. Note that $U_1 \cap U_2$ is an open set containing x and by Remark 2.5,

$$e\text{-}Cl(U_1 \cap U_2) \subseteq e\text{-}Cl(U_1) \cap e\text{-}Cl(U_2) \subseteq G_1 \cap G_2.$$

Thus, $G_1 \cap G_2 \in \mathcal{T}_{\theta_e}$.

Next, let $\{G_\alpha : \alpha \in \mathcal{A}\}$ be a collection of θ_e-open subsets of X. Let $x \in \cup_{\alpha\in\mathcal{A}} G_\alpha$. Then $x \in G_{\alpha_0}$ for some $\alpha_0 \in \mathcal{A}$. Since G_{α_0} is θ_e-open, there exists an open set U_{α_0} containing x such that $e\text{-}Cl(U_{\alpha_0}) \subseteq G_{\alpha_0} \subseteq \cup_{\alpha\in\mathcal{A}} G_\alpha$. Thus, $\cup_{\alpha\in\mathcal{A}} G_\alpha \in \mathcal{T}_{\theta_e}$. Consequently, $\mathcal{T}_{\theta_e}$ is a topology on X. □

Definition 2.7. Let X be a topological space and $A \subseteq X$. Then the θ_e-interior of A is denoted and defined by $Int_{\theta_e}(A)=\cup\{U : U$ is θ_e-open and $U \subseteq A\}$. We note that by Theorem 2.6, $Int_{\theta_e}(A)$ is the largest θ_e-open set contained in A. Moreover, $x \in Int_{\theta_e}(A)$ if and only if there exists a θ_e-open set U containing x such that $U \subseteq A$.

Definition 2.8. Let X be a topological space and $A \subseteq X$. Then the θ_e-closure of A is denoted and defined by $Cl_{\theta_e}(A)=\cap\{F : F$ is θ_e-closed and $A \subseteq F\}$. We note that by Theorem 2.6, $Cl_{\theta_e}(A)$ is the smallest θ_e-closed set containing A. Moreover, $x \in Cl_{\theta_e}(A)$ if and only if for every θ_e-open set U containing x such that $U \cap A \neq \varnothing$.

Remark 2.9. Let X be a topological space and $A, B \subseteq X$. Then

(i) If $A \subseteq B$ then $Cl_{\theta_e}(A) \subseteq Cl_{\theta_e}(B)$.
(ii) A is θ_e-closed if and only if $A = Cl_{\theta_e}(A)$.
(iii) $Cl_{\theta_e}(Cl_{\theta_e}(A)) = Cl_{\theta_e}(A)$.
(iv) $Cl_{\theta_e}(A \cup B) = Cl_{\theta_e}(A) \cup Cl_{\theta_e}(B)$.
(v) If $A \subseteq B$ then $Int_{\theta_e}(A) \subseteq Int_{\theta_e}(B)$.
(vi) A is θ_e-open if and only if $A = Int_{\theta_e}(A)$.
(vii) $Int_{\theta_e}(A) = Int_{\theta_e}(Int_{\theta_e}(A))$.
(viii) $Int_{\theta_e}(A \cap B) = Int_{\theta_e}(A) \cap Int_{\theta_e}(B)$.
(ix) $Cl_{\theta_e}(X \setminus A) = X \setminus Int_{\theta_e}(A)$.
(x) $Int_{\theta_e}(X \setminus A) = X \setminus Cl_{\theta_e}(A)$.
(xi) $x \in Int_{\theta_e}(A)$ if and only if there exists an open set U containing x such that $e\text{-}Cl(U) \subseteq A$.
(xii) $x \in Cl_{\theta_e}(A)$ if and only if for every open set U containing x such that $e\text{-}Cl(U) \cap A \neq \varnothing$.
(xiii) $A \subseteq Cl(A) \subseteq Cl_{\theta_e}(A) \subseteq Cl_\theta(A)$.
(xiv) If A is open in X, then $Cl(A) = C_{\theta_e}(A) = Cl_\theta(A)$.
(xv) $Int_\theta(A) \subseteq Int_{\theta_e}(A) \subseteq Int(A) \subseteq A$.
(xvi) If A is closed in X, then $Int(A) = Int_{\theta_e}(A) = Int_\theta(A)$.

Next, we characterize the concepts of θ_e-open and θ_e-closed functions.

Definition 2.10. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be θ_e-open (resp., θ_e-closed) on X if $f(G)$ is θ_e-open (resp., θ_e-closed) in Y for every open (resp., closed) set G in X.

We note that θ_e-open and θ_e-closed functions are equivalent if f is bijective.

The proofs of the following two results are standard, hence omitted.

Theorem 2.11. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_e-open on X.*
(ii) *$f(Int(A)) \subseteq Int_{\theta_e}(f(A))$ for every $A \subseteq X$.*
(iii) *$f(B)$ is θ_e-open for every basic open set B in X.*
(iv) *For each $x \in X$ and for every open set U in X containing x, there exists an open set V in Y containing $f(x)$ such that $e\text{-}Cl(V) \subseteq f(U)$.*

Theorem 2.12. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_e-closed on X.*
(ii) *$Cl_{\theta_e}(f(G)) \subseteq f(Cl(G))$ for each $G \subseteq X$.*

3. θ_e-Continuous Functions in the Product Space

This section provides a characterization of a θ_e-continuous function from an arbitrary topological space into the product space.

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be θ_e-continuous if $f^{-1}(G)$ is θ_e-open in X for every open set G in Y.

The proofs of the following two results are standard, hence omitted.

Theorem 3.2. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_e-continuous on X.*
(ii) *$f^{-1}(F)$ is θ_e-closed in X for each closed subset F of Y.*
(iii) *$f^{-1}(B)$ is θ_e-open for each (subbasic) basic open set B in Y.*
(iv) *For every $x \in X$ and every open set V of Y containing $f(x)$, there exists a θ_e-open set U containing x such that $f(U) \subseteq V$.*
(v) *$f(Cl_{\theta_e}(A) \subseteq Cl(f(A))$ for each $A \subseteq X$.*
(vi) *$Cl_{\theta_e}(f^{-1}((B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.*

Theorem 3.3. *Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of subset A of X, where $\mathcal{D}$ is the set $\{0, 1\}$ with discrete topology. Then f_A is θ_e-continuous if and only if A is both θ_e-open and θ_e-closed.*

In the following results, if $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \ldots, A_{\alpha_n}\rangle$, where $K = \{\alpha_1, \ldots, \alpha_n\}$.

If $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ is a finite product, denote $A_1 \times \cdots \times A_n$ by $\langle A_1, \ldots, A_n\rangle$.

Theorem 3.4. *Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is θ_e-continuous on X if and only if $p_\alpha \circ f$ is θ_e-continuous on X for every $\alpha \in \mathcal{A}$.*

Proof. Assume that f is θ_e-continuous on X. Let $\alpha \in \mathcal{A}$ and O_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(O_\alpha)$ is open in Y. Hence, $f^{-1}(p_\alpha^{-1}(O_\alpha)) = (p_\alpha \circ f)^{-1}(O_\alpha)$ is θ_e-open in X. Thus, $p_\alpha \circ f$ is θ_e-continuous for every $\alpha \in \mathcal{A}$.

Conversely, assume that each coordinate function $p_\alpha \circ f$ is θ_e-continuous. Let G_α be open in Y_α. Then $\langle G_\alpha \rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha \rangle)$ is θ_e-open in X. In view of Theorem 3.2, f is θ_e-continuous on X. □

Corollary 3.5. Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is θ_e-continuous on X if and only if each f_α is θ_e-continuous on X for each $\alpha \in \mathcal{A}$.

The next result is immediate since the closure and interior operators are preserved in a finite product, see [7, p.99-100].

Lemma 3.6. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $O = \langle O_1, \ldots, O_n \rangle$ is regular open if and only if each O_i is regular open.*

Theorem 3.7. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $O = \langle O_1, \ldots, O_n \rangle$ is δ-open if and only if each O_i is δ-open.*

Proof. Suppose that $O = \langle O_1, \ldots, O_n \rangle$ is δ-open. Choose $a_i \in O_i = p_i(O)$ for all $i = 1, 2, \ldots, n$. Then there exists $x = \langle a_i \rangle \in O$ such that $p_i(x) = a_i$. Since O is δ-open, there exists a regular open set $U \ni x$ such that $U \subseteq O$. Since $U = \langle U_1, \ldots, U_n \rangle$ is also open [10, Remark 3], by Lemma 3.6, each U_i is regular open containing a_i and $U_i \subseteq O_i$. Thus, each O_i is δ-open.

The converse can be proved similarly. □

Corollary 3.8. Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then

(i) $\langle Cl_\delta(A_1), \ldots, Cl_\delta(A_n) \rangle = Cl_\delta(\langle A_1, \ldots, A_n \rangle)$ and
(ii) $\langle Int_\delta(A_1), \ldots, Int_\delta(A_n) \rangle = Int_\delta(\langle A_1, \ldots, A_n \rangle)$.

Theorem 3.9. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, \ldots, n$. If $O = \langle O_1, \ldots, O_n \rangle$ is e-open, then each O_i is e-open.*

Proof. Suppose that $O = \langle O_1, \ldots, O_n \rangle$ is e-open. Then by Corollary 3.8,

$$\begin{aligned} O &\subseteq Int(Cl_\delta(O)) \cup Cl(Int_\delta(O)) \\ &= \langle Int(Cl_\delta(O_1)), \ldots, Int(Cl_\delta(O_n)) \rangle \cup \langle Cl(Int_\delta(O_1)), \ldots, Cl(Int_\delta(O_n)) \rangle \\ &\subseteq \langle Int(Cl_\delta(O_1)) \cup Cl(Int_\delta(O_1)), \ldots, Int(Cl_\delta(O_n)) \cup Cl(Int_\delta(O_n)) \rangle . \end{aligned}$$

Hence, for every $i = 1, 2, \ldots, n$, $O_i \subseteq Int(Cl_\delta(O_i)) \cup Cl(Int_\delta(O_i))$. Thus, each O_i is e-open. □

Theorem 3.10. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $e\text{-}Cl(\langle A_1, \ldots, A_n \rangle) \subseteq \langle e\text{-}Cl(A_1), \ldots, e\text{-}Cl(A_n) \rangle$.*

Proof. In view of [3, Lemma 2.4] and using Corollary 3.8, we have

$$\begin{aligned}
& e\text{-}Cl(\langle A_1, \ldots, A_n \rangle) \\
= \; & \langle A_1, \ldots, A_n \rangle \cup [Int(Cl_\delta(\langle A_1, \ldots, A_n \rangle)) \cap Cl(Int_\delta(\langle A_1, \ldots, A_n \rangle))] \\
= \; & \langle A_1, \ldots, A_n \rangle \cup \langle Int(Cl_\delta(A_1)) \cap Cl(Int_\delta(A_1)), \ldots, \\
& \quad Int(Cl_\delta(A_n)) \cap Cl(Int_\delta(A_n)) \rangle \\
\subseteq \; & \langle A_1 \cup [Int(Cl_\delta(A_1)) \cap Cl(Int_\delta(A_1))], \ldots, \\
& \quad A_n \cup [Int(Cl_\delta(A_n)) \cap Cl(Int_\delta(A_n))] \rangle \\
= \; & \langle e\text{-}Cl(A_1), \ldots, e\text{-}Cl(A_n) \rangle \, ,
\end{aligned}$$

thereby completing the proof. □

Theorem 3.11. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then*

(i) $Cl_{\theta_e}(\langle A_1, \ldots, A_n \rangle) \subseteq \langle Cl_{\theta_e}(A_1), \ldots, Cl_{\theta_e}(A_n) \rangle$ and
(ii) $\langle Int_{\theta_e}(A_1), \ldots, Int_{\theta_e}(A_n) \rangle \subseteq Int_{\theta_e}(\langle A_1, \ldots, A_n \rangle)$.

Proof. (i) Let $x = \langle a_i \rangle \in Cl_{\theta_e}(\langle A_1, \ldots, A_n \rangle)$. Suppose that there exists an open set $O_j \ni a_j$ such that $e\text{-}Cl(O_j) \cap A_j = \varnothing$. Then $\langle O_1, \ldots, O_j, \ldots, O_n \rangle$ is open containing x and

$$\begin{aligned}
& e\text{-}Cl(\langle O_1, \ldots, O_j, \ldots, O_n \rangle) \cap \langle A_1, \ldots, A_j, \ldots, A_n \rangle \\
\subseteq \; & \langle e\text{-}Cl(O_1) \cap A_1, \ldots, e\text{-}Cl(O_j) \cap A_j, \ldots, e\text{-}Cl(O_n) \cap A_n \rangle \\
= \; & \varnothing,
\end{aligned}$$

a contradiction. Thus, $x \in \langle Cl_{\theta_e}(A_1), \ldots, Cl_{\theta_e}(A_n) \rangle$.

(ii) Let $x = \langle a_i \rangle \in \langle Int_{\theta_e}(A_1), \ldots, Int_{\theta_e}(A_n) \rangle$. Then $a_i \in Int_{\theta_e}(A_i)$ for all $i = 1, 2, \ldots, n$. This means that there exists $O_i \ni a_i$ such that $e\text{-}Cl(O_i) \subseteq A_i$. Then $\langle O_1, \ldots, O_n \rangle$ is open containing x and by Theorem 3.10,

$$e\text{-}Cl(\langle O_1, \ldots, O_n \rangle) \subseteq \langle e\text{-}Cl(O_1), \ldots, e\text{-}Cl(O_n) \rangle \subseteq \langle A_1, \ldots, A_n \rangle \, .$$

Hence, $x \in Int_{\theta_e}(\langle A_1, \ldots, A_n \rangle)$. □

Following the same argument as in Theorem 3.11 (ii), we have the following result.

Theorem 3.12. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, 2, \ldots, n$. If each O_i is θ_e-open in Y_i, then $O = \langle O_1, \ldots, O_n \rangle$ is θ_e-open in Y.*

Theorem 3.13. *Let $X = \Pi\{X_i : 1 \leq i \leq n\}$ and $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be finite product spaces and for each $i = 1, 2, \ldots, n$, let $f_i : X_i \to Y_i$ be a function. If each f_i is θ_e-continuous on X_i, then the function $f : X \to Y$ defined by $f(\langle x_i \rangle) = \langle f_i(x_i) \rangle$ is θ_e-continuous on X.*

Proof. Let $\langle V_1, \ldots, V_n \rangle$ be a basic open set in Y. Then

$$f^{-1}(\langle V_1, \ldots, V_n \rangle) = \left\langle f_1^{-1}(V_1), \ldots, f_n^{-1}(V_n) \right\rangle .$$

Since each f_i is θ_e-continuous, $f_i^{-1}(V_i)$ is θ_e-open in X_i. Let $x = \langle x_i \rangle \in f^{-1}(\langle V_1, \ldots, V_n \rangle)$. Then $x_i \in f_i^{-1}(V_i)$ for all $i = 1, 2, \ldots, n$. This means that there exists an open set $O_i \ni x_i$ such that $e\text{-}Cl(O_i) \subseteq f_i^{-1}(V_i)$. Then $\langle O_1, \ldots, O_n \rangle$ is open in X containing x and by Theorem 3.10,

$$\begin{aligned} e\text{-}Cl(\langle O_1, \ldots, O_n \rangle) &\subseteq \langle e\text{-}Cl(O_1), \ldots, e\text{-}Cl(O_n) \rangle \\ &\subseteq \left\langle f_1^{-1}(V_1), \ldots, f_n^{-1}(V_n) \right\rangle \\ &= f^{-1}(\langle V_1, \ldots, V_n \rangle). \end{aligned}$$

This implies that $f^{-1}(\langle V_1, \ldots, V_n \rangle)$ is θ_e-open in X. Thus, f is θ_e-continuous on X. $\square$

4. θ_e-Connected Space and Versions of Separation Axioms

This section characterizes the concepts of θ_e-connected space and some versions of separation axioms.

Definition 4.1. A topological space X is said to be a θ_e-connected (resp., θ-connected [24], connected) if it is not the union of two nonempty disjoint θ_e-open (resp., θ-open, open) sets. Otherwise, X is θ_e-disconnected (resp., θ-disconnected [24], disconnected). A subset B of X is θ_e-connected (resp., θ-connected [24], connected) if it is θ_e-connected (resp., θ-connected, connected) as a subspace of X.

Theorem 4.2. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_e-connected.*
(ii) *The only subsets of X that are both θ_e-open and θ_e-closed are $\varnothing$ and X.*
(iii) *No θ_e-continuous function $f : X \to \mathcal{D}$ is surjective.*

By Remark 2.3 and the equivalence of connected and θ-connected spaces [12], the following result holds.

Theorem 4.3. *A topological space X is θ_e-connected if and only if X is connected.*

Remark 4.4. The following diagram holds for a subset of a topological space.

$$\theta\text{-connected} \longleftrightarrow \theta_e\text{-connected} \longleftrightarrow \text{connected}$$

Definition 4.5. A topological space X is said to be

(i) θ_e-Hausdorff if given any pair of distinct points p, q in X there exist disjoint θ_e-open sets U and V such that $p \in U$ and $q \in V$;
(ii) θ_e-regular if for each closed set F and each point $x \notin F$, there exist disjoint θ_e-open sets U and V such that $x \in U$ and $F \subseteq V$;
(iii) θ_e-normal if for every pair of disjoint closed sets E and F of X, there exist disjoint θ_e-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

In view of Remark 2.3, every θ_e-Hausdorff (resp., θ_e-regular, θ_e-normal) space is Hausdorff (resp., regular, normal).

Theorem 4.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is θ_e-Hausdorff.
(ii) Let $x \in X$. For $y \neq x$, there exists a θ_e-open set U containing x such that $y \notin Cl_{\theta_e}(U)$.

(iii) For each $x \in X$, $C = \cap\{Cl_{\theta_e}(U) : U$ is θ_e-open containing $x\}=\{x\}$.

Proof. (i)⇒(ii): Note that for every distinct points $x, y \in X$, there exist disjoint θ_e-open sets U and V such that $x \in U$ and $y \in V$. Using Remark 2.9 (xii), $y \notin Cl_{\theta_e}(U)$.

(ii) ⇒ (iii): It is not difficult to see that $x \in C$. By assumption, for every $y \neq x$, there exists a θ_e-open set U containing x such that $y \notin Cl_{\theta_e}(U)$. Thus, $y \notin C$. Since y is arbitrary, $C = \{x\}$.

(iii)⇒(i): Let $x, y \in X$ such that $x \neq y$. By assumption, there exists a θ_e-open set U containing x such that $y \notin Cl_{\theta_e}(U)$. In view of Remark 2.9 (xii), there exists a θ_e-open set V containing y such that $U \cap V = \varnothing$. Hence, X is a θ_e-Hausdorff. □

Theorem 4.7. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is θ_e-regular.

(ii) For each $x \in X$ and an open set U containing x, there exists θ_e-open set V such that $x \in V \subseteq Cl_{\theta_e}(V) \subseteq U$.

(iii) For each $x \in X$ and closed set F with $x \notin F$, there exists a θ_e-open set V containing x such that $F \cap Cl_{\theta_e}(V) = \varnothing$.

Proof. (i)⇒(ii): Let $x \in X$ and U be an open set containing x. Using the assumption, there exist disjoint θ_e-open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. By Remark 2.9 (i), (ii), $Cl_{\theta_e}(V) \subseteq Cl_{\theta_e}(X \setminus W) = X \setminus W$. Moreover, $Cl_{\theta_e}(V) \cap (X \setminus U) \subseteq Cl_{\theta_e}(V) \cap W = \varnothing$. Hence, $Cl_{\theta_e}(V) \subseteq U$. Therefore, $x \in V \subseteq Cl_{\theta_e}(V) \subseteq U$.

(ii)⇒(iii): Let $x \in X$ and F be a closed set in X with $x \notin F$. By assumption, there exists a θ_e-open set V containing x such that $V \subseteq Cl_{\theta_e}(V) \subseteq X \setminus F$. This means that $Cl_{\theta_e}(V) \cap F = \varnothing$.

(iii)⇒(i): Let $x \in X$ and F be a closed set with $x \notin F$. By assumption, there exists a θ_e-open set V containing x such that $Cl_{\theta_e}(V) \cap F = \varnothing$. Note that $X \setminus Cl_{\theta_e}(V)$ is a θ_e-open set and $F \subseteq X \setminus Cl_{\theta_e}(V)$. Furthermore, $V \cap X \setminus Cl_{\theta_e}(V) = \varnothing$. Hence, X is a θ_e-regular. □

Theorem 4.8. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is θ_e-normal.

(ii) For each closed set A and an open set $U \supseteq A$, there exists a θ_e-open set V containing A such that $Cl_{\theta_e}(V) \subseteq U$.

(iii) For each pair of disjoint closed sets A and B, there exists a θ_e-open set V containing A such that $Cl_{\theta_e}(V) \cap B = \varnothing$.

Proof. (i)⇒(ii): Let A be a closed set and U be an open set in X containing A. Then A and $X \setminus U$ are disjoint closed sets in X. By assumption, there exist disjoint θ_e-open sets V and W such that $A \subseteq V$ and $X \setminus U \subseteq W$. Using Remark 2.9 (i), (ii), $Cl_{\theta_e}(V) \subseteq Cl_{\theta_e}(X \setminus W) = X \setminus W$. Hence, $Cl_{\theta_e}(V) \subseteq X \setminus W \subseteq U$.

(ii)⇒(iii): Let A and B be a disjoint closed sets in X. Then $A \subseteq X \setminus B$ and $X \setminus B$ is open. By assumption, there exists a θ_e-open set V containing A such that $Cl_{\theta_e}(V) \subseteq X \setminus B$. This means that $Cl_{\theta_e}(V) \cap B = \varnothing$.

(iii)⇒(i): Let A and B be disjoint closed sets in X. By assumption, there exists a θ_e-open set V containing A such that $Cl_{\theta_e}(V) \cap B = \varnothing$. Then $B \subseteq X \setminus Cl_{\theta_e}(V)$. Since V and $X \setminus Cl_{\theta_e}(V)$ are disjoint θ_e-open sets, X is θ_e-normal. □

A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 4.9. *Let X be a T_1-space. Then*

(i) If X is θ_e-regular, then X is θ_e-Hausdorff.
(ii) If X is θ_e-normal, then X is θ_e-regular.

Proof. (i): Suppose that X is θ_e-regular. Since X is a T_1-space, for each $x, y \in X$ with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$. This implies that $x \notin X \setminus U$ and $y \notin X \setminus V$. Since X is θ_e-regular, there exist disjoint θ_e-open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Note that $y \in X \setminus U$. Hence, $y \in B$. Thus, X is θ_e-Hausdorff.

We can prove (ii) by following the same argument used in (i). □

5. Conclusion and Recommendations

The paper has introduced the concept of θ_e-open set and described its connection to the other well-known concepts such as the classical open, θ-open and e-open sets. The paper has also defined and characterized the concepts of θ_e-open and θ_e-closed functions, θ_e-continuous function, θ_e-connected space, and some versions of separation axioms. A worthwhile direction for further investigation is to establish versions of Urysohn's Lemma and Tietze Extension Theorem with respect to θ_e-open sets.

References

[1] T. A. Al-Hawary, On supper continuity of topological spaces, MATEMATIKA, 21 (2005), 43-49.
[2] B. S. Ayhan and M. Özkoç, Almost e^*-continuous functions and their characterizations, J. Nonlinear Sci. Appl., 9 (2016), 6408-6423.
[3] B. S. Ayhan and M. Özkoç, On generalized e-closed sets, J. Adv. Stud. Topol., 5 (2014), 14-21.
[4] M. Caldas, R. Latif, and S. Jafari, Sobriety via θ-open sets, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S), 56 (2010), 163-167.
[5] R. F. Dickman Jr. and J. R. Porter, θ-closed subsets of Hausdorff spaces, Pacific J. Math., 59 (1975), 407-415.
[6] R. F. Dickman Jr. and J. R. Porter, θ-perfect and θ-absolutely closed functions, Illinois J. Math., 21 (1977), 42-60.
[7] J. Dugundji, *Topology*, Prentice-Hall of India Private Ltd., New Delhi, 1975.
[8] E. Ekici, On e-open sets, $\mathcal{DP}^*$-sets and $\mathcal{DPE}^*$-sets and decompositions of continuity, The Arabian J. for Sci. and Eng., 33 (2008), 269-282.
[9] E. Ekici, On e^*-open sets and $(\mathcal{D}, \mathcal{S})^*$-sets, Math. Morav., 13 (2009), 29-36.
[10] A. M. Farhan and X. S. Yang, New type of strongly continuous functions in topological spaces via δ-β-open sets, Eur. J. Pure Appl. Math., 8 (2015), 185-200.
[11] J. Hassan and M. Labendia, θ_s-open sets and θ_s-continuity in the product space, J. Math. Computer Sci., 25 (2022), 182-190.
[12] S. Jafari, Some properties of quasi-θ-continuous function, Far East J. Math. Sci., 6 (1998), 689-696.
[13] D. Jankovic, θ-regular spaces, Internat. J. Math. and Math. Sci., 8 (1986), 615-619.
[14] J. E. Joseph, θ-closure and θ-subclosed graphs, Math. Chronicle, 8 (1979), 99-117.
[15] M. Kovar, On θ-regular spaces, Internat. J. Math. and Math Sci., 17 (1994), 687-692.

[16] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
[17] P. E. Long and L. L. Herrington, The τ_θ-topology and faintly continuous functions, Kyungpook Math. J., 22 (1982), 7-14.
[18] T. Noiri, Remarks on δ-semi-open sets and δ-preopen sets, Demonstr. Math., 36 (2003), 1007-1020.
[19] T. Noiri and S. Jafari, Properties of (θ, s)-continuous functions, Topology Appl., 123 (2002), 167-179.
[20] M. Özkoç and B. S. Ayhan, On characterizations of weakly e^*-continuity, Divulg. Mat., 21 (2020), 9-20.
[21] J. H. Park, B. Y. Lee, and M. J. Son, On δ-semiopen sets in topological space, J. Indian Acad. Math., 19 (1997), 59-67.
[22] M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41 (1937), 375-381.
[23] N. V. Veličko, H-closed topological spaces, Amer. Math. Soc. Transl., 78 (1968), 103-118.
[24] N. V. Veličko, On the theory of H-closed topological spaces, Sib. Math. J., 8 (1967), 569-575.

(Aldison M. Asdain) Department of Mathematics and Statistics, Mindanao State University- Iligan Institute of Technology, 9200 Iligan City, Philippines
E-mail address: `aldison.asdain@g.msuiit.edu.ph`

(Mhelmar A. Labendia) Department of Mathematics and Statistics, Mindanao State University- Iligan Institute of Technology, 9200 Iligan City, Philippines
E-mail address, Corresponding author: `mhelmar.labendia@g.msuiit.edu.ph`