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SEMI-CONTINUOUS K-G-FRAMES

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***Abstract*.** In this paper, we introduce the concept of semi-continuous K-g-frames in Hilbert spaces. A characterization of semi-continuous K-g-frames in term of frame operator is given. It is proved that the image of semi-continuous K-g-frame under a bounded linear operator is again a semi-continuous K-g-frame. Also, we give a necessary and sufficient condition for the finite sum of semi-continuous K-g-frames to be semi-continuous K-g-frame. Finally, some stability results for semi-continuous K-g-frame is proved.

1. Introduction

The concept of frames for Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [9], as a part of their research in non-harmonic Fourier series. In particular, they generalized Gabor's method to define frames for Hilbert spaces. Their work on frame theory was somewhat forgotten until 1986, when Daubechies, Grossman, and Meyer [7] brought it back to life. Thereafter, the theory of frames got more attention. Discrete and continuous frames have many application in both pure and applied mathematics. Today, frame theory has been extensively used in many field such as data compression, filter bank theory, coding and many other areas. For more details on frame theory we endorse the book of Christensen [6]. Applications of frames, especially in the last decade, motivated the researcher to find some generalization of frames like continuous frames [1, 14], g-frames [15], K-frames [12] and etc.

Our main purpose in this paper is to study a generalization of frames, named as semi-continuous K-g-frames. We introduce the notion of semi-continuous K-g-frames and set out to generalize some results of [3] to semi-continuous K-g-frames.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces, J is a countable index set, $(\mathcal{X}, \mu)$ is a measure space with positive measure μ, $\{\mathcal{K}_{x,j}\}_{x\in\mathcal{X}.j\in J}$is a sequence of closed subspaces of $\mathcal{K}$, $B(\mathcal{H}, \mathcal{K}_{x,j})$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}_x$. If $\mathcal{K}_{x,j} = \mathcal{H}$ for any $x \in \mathcal{X}, j \in J$, we denote $B(\mathcal{H}, \mathcal{K}_{x,j})$ by $B(\mathcal{H})$.

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Suppose $\mathcal{H}$ be a seperable Hilbert space. Then a countable family of vectors $\{f_k\} \subset \mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there are positive constants C and D such that

$$C\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq D\|f\|^2, \quad f \in \mathcal{H} \tag{1.1}$$

The positive constants C and D are called lower and upper frame bounds, respectively. If the only upper inquality (1.1) holds, then $\{f_k\}$ is called Bessel sequence. Integral representation of discrete frame in Hilbert space was generalized to continuous frame by Kaiser [14] (named as generalized frames) and independently by Ali et al. [1] in 1993: Let $\mathcal{H}$ be a Hilbert space and M a measure space with a positive measure μ. Let $f \in \mathcal{H}$ and $k \to \langle f, f_k\rangle$ is a measurable function on M. Then $\{f_k\}_{k\in M}$ is called a continuous frame for $\mathcal{H}$ if there are positive constants C and D such that

$$C\|f\|^2 \leq \int_M |\langle f, f_k\rangle|^2 d\mu(k) \leq D\|f\|^2, \quad f \in \mathcal{H} \tag{1.2}$$

We refer to [5, 10, 11] for more details on continuous frames. The concept of a discrete frame extended to g-frame by Sun [15]. He defined g-frame as follows:
Let $\{\mathcal{K}_j : j \in J\} \subset \mathcal{K}$ be a sequence of Hilbert spaces. A family $\{\Omega_j \in B(\mathcal{H}, \mathcal{K}_j) : j \in J\}$ is called a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{K}_j : j \in J\}$ if there are two positive constants C,D such that

$$C\|f\|^2 \leq \sum_{j\in\mathcal{J}} ||\Omega_j(f)||^2 \leq D\|f\|^2, \quad f \in \mathcal{H} \tag{1.3}$$

In 2008, Dehghan and Fard [8], proposed continuous g-frames as an extension of g-frames and continuous frames. Hua and Hung [13] in 2016 generalized g-frames to K-g-frames which have certain advantage in practical application. Alizadeh, Faroughi and Rahmani [2] introduced the concept of c-K-g-frames, which are the generalization of K-g-frames. In the recent years, the concept of semi-continuous g-frames in Hilbert spaces using the Fourier transform of the Heisenberg group has been thoroughly investigated by Anirudha [4].
A family$\{\Lambda_{x,j} \in B(\mathcal{H}, \mathcal{K}_{x,j}) : x \in \mathcal{X}, j \in J\}$ is called a semi-continuous g-frame for $\mathcal{H}$ with respect to $(\mathcal{X}, \mu)$, if $\{\Lambda_{x,j} : x \in \mathcal{X}, j \in J\}$ is weakly-measurable, i.e., for any $f \in \mathcal{H}$ and any $j \in J$, the function $x \to \Lambda_{x,j}(f)$ is measurable on $\mathcal{X}$, and if there exist two positive constants C, D such that

$$C\|f\|^2 \leq \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j}(f)||^2 d\mu(x) \leq D\|f\|^2, \quad f \in \mathcal{H} \tag{1.4}$$

If only the right-hand inequality is satisfied, we call$\{\Lambda_{x,j} : x \in \mathcal{X}, j \in J\}$ the semi-continuous g-Bessel sequence for $\mathcal{H}$ with respect to $(\mathcal{X}, \mu)$ with Bessel bound D.

In the present paper, we shall study semi-continuous K-g-frames and prove several intresting results.

This paper is organized as follows: In Section 2, we introduce the semi-continuous version of K-g-frames in Hilbert spaces and characterized semi-continuous K-g-frames in terms of frame operator. Also, we shall investigate the operators preserving semi-continuous K-g-frames. Further we prove a neccesary and sufficient condition for finite sum of semi-continuous K-g-frame to be semi continuous K-g-frame. Finally, in Section 3, we present stablity results of semi-continuous K-g-frames.

2. Semi-continuous K-g-frames

In this section, we introduce semi-continuous K-g-frames for $\mathcal{H}$ along with its associated frame operator. We begin with the following definition:

Definition 2.1. Let K be a bounded linear operator on $\mathcal{H}$. A family $\{\Lambda_{x,j} \in B(\mathcal{H}, \mathcal{K}_{x,j}) : x \in \mathcal{X}, j \in J\}$ is called a *semi-continuous K-g-frame* for $\mathcal{H}$ with respect to $(\mathcal{X}, \mu)$, if $\{\Lambda_{x,j} : x \in \mathcal{X}, j \in J\}$ is weakly-measurable i.e., for any $f \in \mathcal{H}$ and any $j \in J$, the function $x \to \Lambda_{x,j}(f)$ is measurable on $\mathcal{X}$, and if their exist two constants $A, B > 0$ such that

$$A||K^*f||^2 \leq \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j}(f)||^2 d\mu(x) \leq B||f||^2 \text{ for all } \quad f \in \mathcal{H}. \tag{2.1}$$

If the only upper inequality in (2.1) satisfied, then $\{\Lambda_{x,i} : x \in \mathcal{X}, j \in J\}$ is called semi-continuous K-g- Bessel sequence for $\mathcal{H}$ with respect to $(\mathcal{X}, \mu)$ with Bessel bound B.

Next, we define the frame operator for the semi-continuous K-g-frame.

Definition 2.2. Suppose that $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$, with frame bounds A and B. We define $S : H \to H$ by

$$\langle Sf, f\rangle = \int_{\mathcal{X}} \sum_{j\in J} \langle \Lambda^*_{x,j}\Lambda_{x,j}f, f\rangle d\mu(x), f \in \mathcal{H}$$

and we call it the semi-continuous K-g-frame operator.

In the following result, we charaterize semi-continuous K-g-frame for $\mathcal{H}$ in terms of its frame operator.

Theorem 2.3. *Let $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$. Then $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$ if and only if there exist a constant $A > 0$, such that $S \geq AKK^*$, where S is the frame operator of $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$.*

Proof. Let $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ be a semi-continuous K-g-frame for H with bound A and B. Then

$$A||K^*f||^2 \leq \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j}(f)||^2 d\mu(x) \leq B||f||^2, \text{ for all } \quad f \in \mathcal{H}.$$

Since S is a frame operator for the frame $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$, we compute

$$\begin{aligned}
\langle Sf, f\rangle &= \int_{\mathcal{X}} \sum_{j\in J} \langle \Lambda^*_{x,j}\Lambda_{x,j}f, f\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \sum_{j\in J} \langle \Lambda_{x,j}f, \Lambda_{x,j}f\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j}f||^2 d\mu(x)
\end{aligned}$$

Now, observe that

$$\begin{aligned}\int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}f||^2d\mu(x) &\geq A||K^*f||^2\\ &= A\langle KK^*f,f\rangle\end{aligned}$$

Therefore S≥AKK*, for all $f\in\mathcal{H}$.

Also, we have

$$\begin{aligned}\int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}f||^2d\mu(x) &\leq B||f||^2\\ &= B\langle f,f\rangle\end{aligned}$$

Thus $\langle Sf,f\rangle\leq B\langle f,f\rangle$ for all $f\in\mathcal{H}$.
Hence, we obtain

$$\langle AKK^*f,f\rangle\leq\langle Sf,f\rangle\leq B\langle f,f\rangle\text{for all }\ f\in\mathcal{H}$$

where S is the semi-continuous K-g-frame operator of $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$. □

Next, we discuss a method of constructing another semi-continuous K-g-frame from a given K-g-frame.

Theorem 2.4. *If $T,K\in B(\mathcal{H})$ and $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$, then $\{\Lambda_{x,j}T^*\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous TK-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$.*

Proof. Since $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$, there exist positive constants A and B such that

$$A||K^*f||^2\leq\int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}(f)||^2d\mu(x)\leq B||f||^2\text{ for all }\quad f\in\mathcal{H}.$$

In addition, we obtain

$$\begin{aligned}A||(TK)^*f||^2 &= A||K^*T^*f||^2\\ &\leq \int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}T^*f||^2d\mu(x)\\ &\leq B||T^*||^2||f||^2,\ \text{for all}\quad f\in\mathcal{H}.\end{aligned}$$

Then

$$A||(TK)^*f||^2\leq\int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}T^*f||^2d\mu(x)\leq B'||f||^2,\ \text{for all}\quad f\in\mathcal{H},$$

where$B'=B||T^*||$. Hence $\{\Lambda_{x,j}T^*\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous TK-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$. □

In the following result, we discuss a necessary condition for a pair of semi-continuous K-g-frames associated with an operator.

Theorem 2.5. *Let $K \in B(\mathcal{H})$ be a dense range, $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ be a semi-continuous K-g-frame for $\mathcal{H}$ and $U \in B(\mathcal{H})$ be a closed range operator. If $\{\Lambda_{x,j}U^*\}_{x\in\mathcal{X},j\in J}$ and $\{\Lambda_{x,j}U\}_{x\in\mathcal{X},j\in J}$ are semi-continuous K-g-frames for $\mathcal{H}$, then the operator U is invertible.*

Proof. Suppose that A_1,B_1 are the frame bounds of semi-continuous K-g-frame $\{\Lambda_{x,j}U^*\}_{x\in\mathcal{X},j\in J}$. Then

$$A_1||K^*f||^2 \leq \int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}U^*f||^2 d\mu(x) \leq B_1||f||^2, \text{ for all } \quad f \in \mathcal{H}.$$

Since $N(U^*) \subseteq N(K^*)$, K^* is an injection. Moreover, since $R(U) = N(U^*)^\perp = \mathcal{H}$, U is surjective. Also if A_2,B_2 are the frame bounds of $\{\Lambda_{x,j}U\}_{x\in\mathcal{X},j\in J}$, then

$$A_2||K^*f||^2 \leq \int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}Uf||^2 d\mu(x) \leq B_2||f||^2, \text{ for all } \quad f \in \mathcal{H}$$

Thus $N(U) \subset N(K^*)$ and so U is injective. Hence U is invertible. □

Next, we state a result which gives a sufficient condition for a semi-continuous K-g-frame.

Theorem 2.6. *Suppose that $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frames for $\mathcal{H}$ with lower bound A and $U \in B(\mathcal{H})$ with $R(U) \subseteq R(K)$. Then $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous U-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X},j\in J}$.*

Proof. By hypothesis, there exists $\alpha > 0$ such that $UU^* \leq \alpha^2 KK^*$. Hence, for each $f \in \mathcal{H}$ we have

$$A\alpha^{-2}||U^*f||^2 \leq A||K^*f||^2 \leq \int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}(f)||^2 d\mu(x).$$

□

Now, we give a necessary condition for a semi-continuous K-g-frame for $\mathcal{H}$ associated with an operator with closed range.

Theorem 2.7. *Let $K \in B(\mathcal{H})$ is with dense range, $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ and let $U \in B(\mathcal{H})$ be an operator with closed range. If $\{\Lambda_{x,j}U^*\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$, then U is surjective.*

Proof. By hypothesis, there exist a positive constant A such that

$$A||K^*f||^2 \leq \int_{\mathcal{X}}\sum_{j\in J}||\Lambda_{x,j}(f)||^2 d\mu(x), \quad f \in \mathcal{H}.$$

Since K has dense range, so K^* is injective. Therefore, $N(U^*) \subseteq N(K^*)$. Thus U^* is injective. Moreover $R(U) = N(U^*)^\perp = \mathcal{H}$.
Hence U is surjective. □

In the next two results, we give sufficient conditions for the construction of another semi-continuous K-g-frame for $\mathcal{H}$.

Theorem 2.8. *Let $K \in B(\mathcal{H})$ and $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ be a semi-continuous K-g-frame for $\mathcal{H}$ with the frame bounds A,B. If $U \in B(\mathcal{H})$ has closed range with UK=KU, then $\{\Lambda_{x,j}U^*\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frames for R(U).*

Proof. One may note that $\{\Lambda_{x,j}U^*\}_{x\in\chi,j\in J}$ is a semi-continuous g-Bessel sequence with the bound $B||U||^2$. Since U has closed range, it has the pseudo-inverse $U^\dagger$ such that $UU^\dagger = I$. Now, for each $f \in R(U)$, we get $K^*f = (U^\dagger)^*U^*K^*f$. So, we obtain $||K^*f|| \leq ||U^\dagger||.||U^*K^*f||$. Therefore, $||U^\dagger||^{-1}||K^*f|| \leq ||U^*K^*f||$. Also, we have

$$\begin{aligned}\int_\chi \sum_{j\in J} ||\Lambda_{x,j}U^*f||^2 d\mu(x) &\geq A||K^*(U^*f)||^2 \\ &\geq A||U^\dagger||^{-2}||K^*f||^2\end{aligned}$$

□

Theorem 2.9. *Let $\{\Lambda_{x,j}\}_{x\in\chi,j\in J}$ be a semi-comtinuous K-g-frame for $\mathcal{H}$ and $U \in B(\mathcal{H})$ be co-isometry (i.e. $UU^* = I$) with UK=KU. Then $\{\Lambda_{x,j}U^*\}_{x\in\chi,j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$.*

Proof. Let A, B be the frame bounds of $\{\Lambda_{x,j}\}_{x\in\chi,j\in J}$. Then, by Theorem 2.8, $\{\Lambda_{x,j}U^*\}_{x\in\chi,j\in J}$ is a semi-continuous g-Bessel sequence. Since U is co-isometry, for each $f \in \mathcal{H}$, we obtain

$$\int_\mathcal{X} \sum_{j\in J} ||\Lambda_{x,j}U^*f||^2 d\mu(x) \quad \geq \quad A||K^*U^*f||^2 = A||K^*f||^2.$$

□

The following result gives a necessary and sufficient condition for the finite sum of semi-continuous K-g-frames to be a semi-continuous K-g-frame.

Theorem 2.10. *Let $K \in B(\mathcal{H})$ be a co-isometry and let $\{\Lambda^i_{x,j}\}$, $i = 1, 2, \ldots, k$ be semi-continuous K-g-frames for $\mathcal{H}$. Let $\alpha_i, i = 1, 2, \ldots, k$ be any scalars. Then $\{\sum_{i=1}^k \alpha_i\Lambda^i_{x,j}\}$ is a semi-continuous K-g-frame for $\mathcal{H}$ if and only if there exists $\beta > 0$ and some $p \in \{1, 2, \ldots, k\}$ such that*

$$\beta \int_\mathcal{X} \sum_{j\in J} \|\Lambda^p_{x,j}f\|^2 d\mu(x) \leq \int_\mathcal{X} \sum_{j\in J} \left|\left| \left(\sum_{i=1}^k \alpha_i\Lambda^i_{x,j}\right) f\right|\right|^2 d\mu(x), \quad f \in \mathcal{H}.$$

Proof. For each $1 \leq p \leq k$, let A_p and B_p be bounds of semi continuous K-g frame $\{\Lambda^p_{x,j}\}$ and let $\beta > 0$ be a constant satisfying

$$\beta \int_\mathcal{X} \sum_{j\in J} \|\Lambda^p_{x,j}f\|^2 d\mu(x) \leq \int_\mathcal{X} \sum_{j\in J} \left|\left| \left(\sum_{i=1}^k \alpha_i\Lambda^i_{x,j}\right) f\right|\right|^2 d\mu(x), \quad f \in \mathcal{H}.$$

Then

$$\beta A_p\|K^*f\|^2 \leq \beta \int_\mathcal{X} \sum_{j\in J} \|\Lambda^p_{x,j}f\|^2 d\mu(x) \leq \int_\mathcal{X} \sum_{j\in J} \left|\left| \left(\sum_{i=1}^k \alpha_i\Lambda^i_{x,j}\right) f\right|\right|^2 d\mu(x), \quad f \in \mathcal{H}.$$

and

$$\int_{\mathcal{X}} \sum_{j\in J} \left|\left| \left(\sum_{i=1}^{k} \alpha_i \Lambda^i_{x,j} \right) f \right|\right|^2 d\mu(x) \le k \sum_{i=1}^{k} |\alpha_i|^2 \int_{\mathcal{X}} \sum_{j\in J} \left|\left| \left(\Lambda^i_{x,j}\right) f \right|\right|^2 d\mu(x)$$

$$\le k(\max|\alpha|^2) \left(\sum_{i=1}^{k} B_i \right) \|f\|^2, \quad f \in \mathcal{H}.$$

Hence $\{\sum_{i=1}^{k} \alpha_i \Lambda^i_{x,j}\}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with bounds βA_p and $k(\max|\alpha|^2)\left(\sum_{i=1}^{k} B_i\right)$.

Conversely, let $\{\sum_{i=1}^{k} \alpha_i \Lambda^i_{x,j}\}$ be a semi-continuous K-g-frame for $\mathcal{H}$ with bounds A and B and let for any $p \in \{1, 2, \ldots, k\}$, $\{\Lambda^p_{x,j}\}$ be a semi-continuous K-g-frame for $\mathcal{H}$ with bounds A_p, B_p. Then, for any $p \in \{1, 2, \ldots, k\}$, we have

$$A_p \|K^* f\|^2 \le \int_{\mathcal{X}} \sum_{j\in J} \|\Lambda^p_{x,j} f\|^2 d\mu(x) \le B_p \|f\|^2, \quad f \in \mathcal{H}$$

and

$$A\|K^* f\|^2 \le \int_{\mathcal{X}} \sum_{j\in J} \left|\left| \left(\sum_{i=1}^{k} \alpha_i \Lambda^i_{x,j} \right) f \right|\right|^2 d\mu(x) \le B\|f\|^2. \;\; f \in \mathcal{H}$$

Hence, we obtain

$$\beta \int_{\mathcal{X}} \sum_{j\in J} \|\Lambda^p_{x,j} f\|^2 d\mu(x) \le \int_{\mathcal{X}} \sum_{j\in J} \left|\left| \left(\sum_{i=1}^{k} \alpha_i \Lambda^i_{x,j} \right) f \right|\right|^2 d\mu(x),$$

where $\beta = \dfrac{A}{B_p}$. □

3. Stability of semi-continuous K-g-frames

In this section, we discuss the stability of semi-continuous K-g-frames and give a sufficient condition for the stability of a semi-continuous K-g-frame for $\mathcal{H}$.

Theorem 3.1. *Let $\{\Lambda_{x,j}\}_{x\in\mathcal{X}, j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with bounds A and B and $\{\Gamma_{x,j}\}_{x\in\mathcal{X}, j\in J}$ be a family of operators, such that $\{\Gamma_{x,j}\}_{x\in\mathcal{X}, j\in J}$ is weakly-measurable for all $f \in \mathcal{H}$. If there exist constants $\lambda_1, \lambda_2, \gamma \ge 0$, such that max$\{\lambda_2, \frac{\gamma}{A} + \lambda_1\} < 1$ and for each $f, g \in \mathcal{H}$*

$$\int_{\mathcal{X}} \sum_{j\in J} |\langle (\Lambda^*_{x,j}\Lambda_{x,j} - \Gamma^*_{x,j}\Gamma_{x,j}) f, g \rangle| d\mu(x) < \lambda_1 \int_{\mathcal{X}} \sum_{j\in J} |\langle \Lambda^*_{x,j}\Lambda_{x,j} f, g\rangle| d\mu(x)$$

$$+ \lambda_2 \int_{\mathcal{X}} \sum_{j\in J} |\langle \Gamma^*_{x,j}\Gamma_{x,j} f, g\rangle| d\mu(x) + \gamma ||K^* f||^2$$

then $\{\Gamma_{x,j}\}_{x\in\mathcal{X}, j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{x,j}\}_{x\in\mathcal{X}, j\in J}$ with bounds

$$\frac{(1-\lambda_1)A - \gamma}{1+\lambda_2}, \frac{(1+\lambda_1)B + \gamma||K||^2}{1-\lambda_2}$$

Proof. For each $f, g \in \mathcal{H}$, we compute

$$\begin{aligned}
|\int_{\mathcal{X}} \sum_{j\in J} \langle \Gamma^*_{x,j}\Gamma_{x,j} f, g\rangle d\mu(x)| &\leq \int_{\mathcal{X}} \sum_{j\in J} |\langle(\Lambda^*_{x,j}\Lambda_{x,j} - \Gamma^*_{x,j}\Gamma_{x,j})f, g\rangle| d\mu(x) \\
&\quad + \int_{\mathcal{X}} \sum_{j\in J} |\langle \Lambda^*_{x,j}\Lambda_{x,j} f, g\rangle| d\mu(x) \\
&\leq \lambda_1 \int_{\mathcal{X}} \sum_{j\in J} |\langle \Lambda^*_{x,j}\Lambda_{x,j} f, g\rangle| d\mu(x) + \lambda_2 \int_{\mathcal{X}} \sum_{j\in J} |\langle \Gamma^*_{x,j}\Gamma_{x,j} f, g\rangle| d\mu(x) \\
&\quad + \gamma ||K^* f||^2 + \int_{\mathcal{X}} \sum_{j\in J} |\langle \Lambda^*_{x,j}\Lambda_{x,j} f, g\rangle| d\mu(x)
\end{aligned}$$

This gives

$$(1-\lambda_2)\int_{\mathcal{X}} \sum_{j\in J} |\langle \Gamma^*_{x,j}\Gamma_{x,j} f, g\rangle| d\mu(x) \leq (1+\lambda_1)\int_{\mathcal{X}} \sum_{j\in J} |\langle \Lambda^*_{x,j}\Lambda_{x,j} f, g\rangle| d\mu(x) + \gamma ||K^* f||^2.$$

Then, for all $f \in \mathcal{H}$, we have

$$\begin{aligned}
\int_{\mathcal{X}} \sum_{j\in J} ||\Gamma_{x,j} f||^2 d\mu(x) &\leq \frac{1+\lambda_1}{1-\lambda_2} B||f||^2 + \frac{\gamma}{1-\lambda_2}||K||^2||f||^2 \\
&\leq \frac{(1+\lambda_1)B + \gamma ||K||^2}{1-\lambda_2}||f||^2.
\end{aligned}$$

Therefore, $\{\Gamma_{x,j}\}_{x\in\mathcal{X}, j\in J}$ is a semi-continuous K-g-Bessel family for $\mathcal{H}$.
Further, note that for each $f in \mathcal{H}$, we have

$$\begin{aligned}
\int_{\mathcal{X}} \sum_{j\in J} ||\Gamma_{x,j} f||^2 d\mu(x) &= \int_{\mathcal{X}} \sum_{j\in J} |\langle \Gamma^*_{x,j}\Gamma_{x,j} f - \Lambda^*_{x,j}\Lambda_{x,j} f + \Lambda^*_{x,j}\Lambda_{x,j} f, f\rangle| d\mu(x) \\
&\geq \int_{\mathcal{X}} \sum_{j\in J} |\langle \Lambda^*_{x,j}\Lambda_{x,j} f, f\rangle| d\mu(x) - \int_{\mathcal{X}} \sum_{j\in J} |\langle \Gamma^*_{x,j}\Gamma_{x,j} f - \Lambda^*_{x,j}\Lambda_{x,j} f, f\rangle| d\mu(x) \\
&\geq \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j} f||^2 d\mu(x) - \lambda_1 \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j} f||^2 d\mu(x) \\
&\quad - \lambda_2 \int_{\mathcal{X}} \sum_{j\in J} ||\Gamma_{x,j} f||^2 d\mu(x) - \gamma ||K^* f||^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\mathcal{X}} \sum_{j\in J} ||\Gamma_{x,j} f||^2 d\mu(x) &\geq \frac{1-\lambda_1}{1+\lambda_2} \int_{\mathcal{X}} \sum_{j\in J} ||\Lambda_{x,j} f||^2 d\mu(x) - \frac{\gamma}{1+\lambda_2}||K^* f||^2 \\
&\geq \frac{1-\lambda_1}{1+\lambda_2} A||K^* f||^2 - \frac{\gamma}{1+\lambda_2}||K^* f||^2 \\
&= \frac{(1-\lambda_1)A - \gamma}{1+\lambda_2}||K^* f||^2.
\end{aligned}$$

□

In the following result, we obtain a sufficient condition for the stability of a semi-continuous K-g-frame.

Theorem 3.2. *Let* $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ *be a semi-continuous* K*-g-frame for* $\mathcal{H}$ *with bounds* A *and* B*. Suppose that* $\{\Gamma_{x,j}\}_{x\in\mathcal{X},j\in J} \in B(\mathcal{H}, K_{x,j})$ *for any* $x \in \mathcal{X}, j \in J$*. Let* $\lambda_1, \lambda_2, \mu \geq 0$ *be constants such that* $\max(\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2) < 1$ *and that*

$$\left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,j} - \Gamma_{x,j})f\|^2 d\mu(x)\right)^{1/2} \leq \lambda_1 \left(\int_{\mathcal{X}} \sum_{j\in J} \|\Lambda_{x,j} f\|^2 d\mu(x)\right)^{1/2} + \lambda_2 \left(\int_{\mathcal{X}} \sum_{j\in J} \|\Gamma_{x,j} f\|^2 d\mu(x)\right)^{1/2} + \mu\|K^* f\|,$$

for all $f \in \mathcal{H}$*. Then* $\{\Gamma_{x,j}\}$ *is a semi-continuous* K*-g-frames for* $\mathcal{H}$*.*

Proof. We have

$$A\|K^* f\|^2 \leq \int_{\mathcal{X}} \sum_{j\in J} \|\Lambda_{x,j} f\|^2 d\mu(x) \leq B\|f\|^2, \quad f \in \mathcal{H}$$

Using hypothesis, we obtain

$$\left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,j} - \Gamma_{x,j})f\|^2 d\mu(x)\right)^{1/2} \leq (\lambda_1\sqrt{B} + \|K^*\|\mu)||f|| + \lambda_2 \left(\int_{\mathcal{X}} \sum_{j\in J} \|\Gamma_{x,j} f\|^2 d\mu(x)\right)^{1/2}.$$

Hence we compute

$$\int_{\mathcal{X}} \sum_{j\in J} \|(\Gamma_{x,j})f\|^2 d\mu(x) \leq B\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu\|K^*\|}{\sqrt{B}}}{1-\lambda_2}\right)^2.$$

Again

$$\left(\int_{\mathcal{X}} \sum_{j\in J} \|\Gamma_{x,j} f\|^2 d\mu(x)\right)^{1/2} \geq \left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,j})f\|^2 d\mu(x)\right)^{1/2} + \left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,i} - \Gamma_{x,j})f\|^2 d\mu(x)\right)^{1/2}$$

$$\geq \left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,j})f\|^2 d\mu(x)\right)^{1/2} - \lambda_1 \left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,i})f\|^2 d\mu(x)\right)^{1/2}$$
$$- \lambda_2 \left(\int_{\mathcal{X}} \sum_{j\in J} \|\Gamma_{x,j}f\|^2 d\mu(x)\right)^{1/2} - \mu\|K^*f\|.$$

This gives

$$(1+\lambda_2)\left(\int_{\mathcal{X}} \sum_{j\in J} \|\Gamma_{x,j}f\|^2 d\mu(x)\right)^{\frac{1}{2}} \geq (1-\lambda_1)\left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,i})f\|^2 d\mu(x)\right)^{\frac{1}{2}} - \mu\|K^*f\|.$$

Hence,

$$\left(\int_{\mathcal{X}} \sum_{j\in J} \|(\Gamma_{x,i})f\|^2 d\mu(x)\right) \geq A\left(1 - \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}}}{1+\lambda_2}\right)^2 \|K^*f\|^2.$$

□

Finally, we give the following stability result.

Corollary 3.3. Let $\{\Lambda_{x,j}\}_{x\in\mathcal{X},j\in J}$ is a semi-continuous K-g-frame for $\mathcal{H}$ with bounds A and B and $\{\Gamma_{x,j}\}_{x\in\mathcal{X},j\in J} \in B(\mathcal{H}, K_{x,j})$ for any $x \in \mathcal{X}, j \in J$. Let there exists a constant $0 < M < A$ such that

$$\int_{\mathcal{X}} \sum_{j\in J} \|(\Lambda_{x,j} - \Gamma_{x,j})f\|^2 d\mu(x) \leq M\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H}.$$

Then $\{\Gamma_{x,j}\}$ is a semi-continuous K-g-frames for $\mathcal{H}$ with bounds $A\left(1-\sqrt{\frac{M}{A}}\right)^2$ and $B\left(1+\sqrt{\frac{M}{B}}\|K\|\right)^2$.

Proof. Let $\lambda_1 = \lambda_2 = 0$ and $\mu = \sqrt{M}$. As $M < A$, by Theorem 3.2 $\{\Gamma_{x,j}\}$ is a semi continuous K-g-frames for $\mathcal{H}$ with frame bounds $A\left(1-\sqrt{\frac{M}{A}}\right)^2$ and $B\left(1+\sqrt{\frac{M}{B}}\|K\|\right)^2$. □

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