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APPROXIMATE SOLUTION FOR PROPORTIONAL-DELAY RICCATI DIFFERENTIAL EQUATIONS BY HAAR WAVELET METHOD

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Abstract. In this paper, Haar wavelet based numerical scheme is applied to obtained the approximate solution of Riccati differential equations(RDEs) and proportional-delay Riccati differential equations(PDRDEs)of quadratic nature. The basic idea of this algorithm is that it transform the differential equations into a system of algebraic equations from which approximate numerical solution of the problem is obtained. Several problems are solved and results are compared with Bezier curves method to show the proficiency and simple applicability of the method. We observed that the absolute error is decreasing uniformly with the increase in level of resolution.

1. Introduction

The Riccati differential equations come under the class of nonlinear differential equations. These equations are widely studied for numerous problems of contemporary analysis and its applications, and are not easy to solve explicitly, this make it interesting to investigate the solutions of these equations. Here, we have considered the following Riccati differential equations:

$$y'(t) = q_1(t) + y(t)\left(q_2(t) + q_3(t)y(t)\right), t_0 \le t \le t_f, y(t_0) = y_0, \tag{1.1}$$

where $q_1(t), q_2(t)$ and $q_3(t)(\neq 0)$ are continuous, t_0, t_f and $y(t_0)$ are arbitrary constant and $y(t)$ is unknown function.
Its proportional-delay variant can be written as

$$y'(t) = \psi(t) + by(t) + cy(\alpha t)(d - y(\alpha t)), t_0 \le t \le t_f, y(t_0) = y_0, \tag{1.2}$$

where $c \neq 0,\ b,\ d\ y_0 \in C$, and $\alpha > 0$,$\psi(t)$ is a continuous and $\alpha \neq 1$. When $0 < \alpha < 1$ (1.2) yields a retarded equation, whereas $\alpha > 1$ produces advanced equation. It follows from Picard-Lindelöf theorem that the solutions of (1.1) and (1.2) exists and it is unique.

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For further studies related to Riccati differential equations see [2, 22, 25, 26]. RDEs widely appear in random process, kalman filtering system and network synthesis. It has enormous applications in the field of super symmetric quantum mechanics, quantum chemistry and play a key role in theory of optimal control, stochastic control theory and financial mathematics, diffusion problems, econometric models and dynamic games. Modeling by delay is necessary in many applied physical problems and to accomplish the purpose delay differential equations have been used significantly. Delay differential equations arise inevitably in decision making, mathematical modelling of chemicals, biological, physiological process, economic growth, neural networks and delayed dynamic part, for further applications of RDEs one can see [4, 8, 15, 23]. In recent years, the problem of finding the approximate solution of these differential equations grab attention and have been examined by many mathematicians. Methods already used for numerical solution of Riccati differential equations are variational iteration method(VIM)[9], the modified variational iteration method(MVIM)[14], homotopy perturbation method(HPM)[3], differential transform method(DTM)[22], Bezier curves method[11]. Reproducing kernel Hilbert space method(RKHSM)[17] and Bezier control point method [10] are used to find the approximate solution of delay Riccati differential equation. In [4] Semi-analytical solutions of some nonlinear proportional delay differential equations are discussed. In [1], author studied numerical treatment of stochastic delay differential equation to formulate the one-step scheme to approximate the solution of the problem. [24] studied numerical inclusion of exact periodic solutions for delay Duffing equation and proposed existence of periodic solutions of the forced delay Duffing equations based on the verified numerical computations. Motivated and inspired by the above work, we have used the Haar wavelet method to convert the differential equations into a system of algebraic equations from which approximate numerical solution of the problem is obtained. The main aim of this paper is to use a simple and straight forward numerical technique based on collocation method using Haar wavelet expansion to solve ordinary Riccati differential equations and proportional-delay Riccati differential equations. The paper is organized as: in section 2, notation and preliminaries , basics of wavelets, multiresolution analysis, Haar wavelet and its integrals are mentioned, the description of technique and convergence of Haar wavelet expansion is presented in section 3. Finally, in section 4, the Haar wavelet method is used to solve quadratic Riccati differential equations(QRDEs) and some delay-Riccati differential equations(DRDEs). Solutions are compared with existing methods and observe that the absolute error is decreasing uniformly with the increase in level of resolution.

2. Notation and Preliminaries

Here, we present some preliminary for wavelet and multi-resolution analysis we start this section with the definition of wavelet, wavelet can be described as a real valued function ψ such that : $\int_{-\infty}^{\infty}\psi(t)dt = 0$ and $\int_{-\infty}^{\infty}|\psi(t)|^2dt = 1$. Precisely, wavelets are defined as:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\frac{(t-b)}{a},\ a \neq 0,\ b \in \mathbb{R}, \tag{2.1}$$

the parameters a, b represents the dilation and translation respectively. When the parameters a and b are restricted as $a = a_0^{-j}, b = kb_0a_0^{-j}, a_0 > 0,\ b_0 > 0$, we have the family of discrete wavelets: $\psi_{j,k}(t) = |a_0|^{j/2}\psi(a_0^j t - kb_0)$, where $\psi_{j,k}(t)$ forms a wavelet basis for $L^2(\mathbb{R})$. Note that, if $a_0 = 2$ and $b_0 = 1$, then function $\psi_{j,k}$ will form an orthonormal basis for $L^2(\mathbb{R})$. That is, $\langle\psi_{j,k}, \psi m, n\rangle = \delta_{j,m}\delta_{k,n}$.

Later on, Mallet [19] developed a new idea called multiresolution analysis (MRA), this is one of the most important idea for the construction of orthonormal basis for wavelets. An MRA on $L^2(\mathbb{R})$ is an increasing sequence of close subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying: $(a)\{V_j\} \subset \{V_{j+1}\}$; for all $j \in \mathbb{Z}$, $(b)\bigcup \bar{V_j} = L^2(\mathbb{R})$, $(c) = \bigcap V_j = 0$, $(d)V_{j+1} = V_j \bigoplus W_j$ i.e for each V_j there exists a complement of V_j in V_{j+1}, namely W_j, (e) a function $f(x) \in V_0$ if and only if $D_2jf(x) \in V_j$, (f) there exists a function $\phi \in L^2(\mathbb{R})$ called scaling function such that the collection $\{T_n\phi(x)\}$ is an orthonormal system of translate and $V_0 = \overline{Span}\{T_n\phi(x)\}$. There are several development in the literature for wavelet in recent years, for further studies related to wavelets and MRA one may refer [5, 6, 7, 19, 20, 21]. In this article, we use Haar wavelet which was first introduced by Haar[13] as Haar function. Haar wavelet is square wave having compact support and unitary nature with mean zero. The family of Haar wavelet form simplest orthogonal basis for $L^2[t_0, t_f]$ i.e. any function from $L^2(\mathbb{R})$ can be expressed as infinite Haar series. The family of Haar wavelet for the interval $t \in [t_0, t_f)$ is define as:

The scaling function (or father wavelet) behind the family of Haar wavelets defined on the interval $[t_0, t_f)$ is

$$\mathtt{h}_1(t) = \begin{cases} 1 & t \in [t_0, t_f) \\ 0 & otherwise. \end{cases} \tag{2.2}$$

The Haar wavelet family on the interval $[t_0, t_f)$ is generated by the action of dyadic dilation and integer translation on a fundamental square wave or mother wavelet $\mathtt{h}_2$ which is defined on the interval $[t_0, t_f)$.

$$\mathtt{h}_2(t) = \begin{cases} 1 & t \in [t_0, (t_0 + t_f)0.5) \\ -1 & t \in [(t_0 + t_f)0.5, t_f) \\ 0 & otherwise. \end{cases} \tag{2.3}$$

Each function of the Haar wavelets family defined on the interval $[t_0, t_f)$ except the $\mathtt{h}_1$ function can be expressed as

$$\mathtt{h}_i(t) = \begin{cases} 1 & t \in [t_1(i), t_2(i)) \\ -1 & t \in [t_2(i), t_3(i)) \\ 0 & otherwise, \end{cases} \tag{2.4}$$

i indicates wavelet number and $t_1(i)= t_0 + (t_f - t_0)k/m, t_2(i) = t_0 + (t_f - t_0)(k + 0.5)/m, t_3(i) = t_0 + (t_f - t_0)(k + 1)/m, m = 2^j, j = 0, 1, 2, 3, ...J,$ and $k = 0, 1, ...m - 1$. The level of resolution is indicated by J while k represents the translation parameter and $i = m + k + 1$ which is true for $i \geq 2$. Haar basis give systematic representation of functions. For further studies related to Haar wavelets see [12, 16, 27, 28, 29]. Any function $f \in L^2[t_0, t_f]$ can be expressed as follows:

$$f(t) = \sum_{i=1}^{2M} a_i\mathtt{h}_i(t) \tag{2.5}$$

The Haar wavelet series coefficients a_i's can be obtained as:

$$a_i = < f(t), \mathtt{h}_i(t) > = \int_{t_0}^{t_f} f(t)\overline{\mathtt{h}_i(t)}dt. \tag{2.6}$$

Collocation points can be calculated as:
$t_\rho = t_0 + \frac{(\rho-0.5)}{2M}(t_f - t_0), \rho = 1, 2, ...2M$ and $M = 2^J$.
We have the following integration of Haar wavelet:

$$I_1\mathtt{h}_i(t) = \begin{cases} t - t_1(i) & t \in [t_1(i), t_2(i)) \\ t_3(i) - t & t \in [t_2(i), t_3(i)) \\ 0 & otherwise. \end{cases} \tag{2.7}$$

3. Description of method.

In order to solve (1.1) and (1.2), by Haar wavelet expansion, let

$$y'(t) \quad = \quad \sum_{i=1}^{2M} a_i \mathtt{h}_i(t). \tag{3.1}$$

on integrating (3.1) with respect to t we compute:

$$y(t) \quad = \quad \sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(t) + y(0). \tag{3.2}$$

Also,

$$y'(\alpha t) \quad = \quad \sum_{i=1}^{2M} a_i \mathtt{h}_i(\alpha t). \tag{3.3}$$

And

$$y(\alpha t) \quad = \quad \sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(\alpha t) + y(0). \tag{3.4}$$

Using (3.1) and (3.2) in (1.1) and (3.3) and (3.4) in (1.2) with collocation points t_ρ, we have the following corresponding system of equations:

$$\begin{aligned} \sum_{i=1}^{2M} a_i \mathtt{h}_i(t_\rho) = q_1(t_\rho) + \left[\sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(t_\rho) + y(0)\right] q_2(t_\rho) \\ + q_3(t_\rho) \left[\sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(t_\rho) + y(0)\right] \left[\sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(t_\rho) + y(0)\right] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \sum_{i=1}^{2M} a_i \mathtt{h}_i(t_\rho) = \psi(t_\rho) + b\left[\sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(t_\rho) + y(0)\right] \\ + c\left[\sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(\alpha t_\rho) + y(0)\right] \left[d - (\sum_{i=1}^{2M} a_i I_1\mathtt{h}_i(\alpha t_\rho) + y(0))\right] \end{aligned} \tag{3.6}$$

Above nonlinear systems of equations can be solved using classical Newton's method, by using these values of Haar coefficient a_i's from (3.5) and (3.6) in (3.1) and (3.2) respectively, we get the approximate solution of (1.1) and (1.2).

3.1. Convergence analysis.

Lemma 3.1. *Assume that $\omega(t)$ be any quadratically integrable function with first derivative bounded on $[0,1)$, then the error norm at Jth level will satisfies the inequality*

$$\|e_j(t)\| \leq \sqrt{\frac{K}{7}}C2^{-(3)2^{J-1}}, \tag{3.7}$$

K and C are real constants.

Proof. One can see [28] for proof. □

4. Numerical problems:

Here, we will present numerical problems and the discussion on Haar wavelet method for solving different type of RDEs and PDRDEs. We have shown that the performance of our technique is sharp and the absolute error is reducing significantly in some problems. Also, experimental rate of convergence is computed, which is characterized as [18]. We begin this section with quadratic Riccati differential equations and subsequently we will consider some DRDEs.

$$R_c(J) = \frac{log[E_c(J-1)/E_c(J)]}{log(2)}$$

Problem 1. Consider the following QRDEs taken from [11]

$$y'(t) = 16t^2 - 5 + 8ty(t) + y^2(t),\ 0 \leq t \leq 1,\ y(0) = 1. \tag{4.1}$$

Exact solution of (4.1) is $y(t) = 1 - 4t$. Here we have solved this problem using Haar wavelet expansions and integral of Haar wavelet. The value of Haar coefficients at level $J = 2$ are $a_1 = -4.000000$, $a_2 = 0.000000$, $a_3 = 0.000000$, $a_4 = 0.000000$, $a_5 = 0.000000$, $a_6 = 0.000000$, $a_7 = 0.000000$, $a_8 = 0.000000$. The coefficient a_1 is significantly close to -4 and rest of coefficients are zero correct up to six decimal places. Therefore, from (3.2) solution is $a_1 I_1 h_1(t) + y(0)$ i.e.$-4t + 1$ which is same as exact solution. The error estimate at different level of resolutions are given in Table 1. We observed that good approximation can be achieve with the increase in level of resolution. Comparison between Haar wavelet method and Bezier Curves method [11] is reported in Table 2 which clearly shows that the absolute error is reduced significantly in our case as compare to the Bezier Curves method. Point wise error at $J = 3$ is shown in Table 3. Comparison between exact solution and numerical solution at $J = 3$ is shown in Fig.1 and we observed that both the curves visually coincide.

Problem 2. Solve the equation

$$y'(t) = e^t - e^{3t} + 2e^{2t}y(t) - e^t y^2(t),\ 0 \leq t \leq 1,\ y(0) = 1 \tag{4.2}$$

The exact solution of (4.2) is $y(t) = e^t$[11]. The maximum absolute error at different level of resolution is shown in Table 4. From Table 4 we have observed that error is decreasing with the increase in resolution. Result is compared with Bezier Curves method [11] in Table 5. Comparison between exact solution and numerical solution for

TABLE 1. max. abs. error for Problem 1

value of J	Max. abs. error
2	1.4654E-14
3	1.1546E-14
4	3.1086E-15
5	2.6645E-15
6	3.5527E-15
7	8.8818E-16

TABLE 2. max. abs. error for Problem 1 with J=3

our method	Bezier Curves Method[11]
5.3735E-14	5.3714E-04

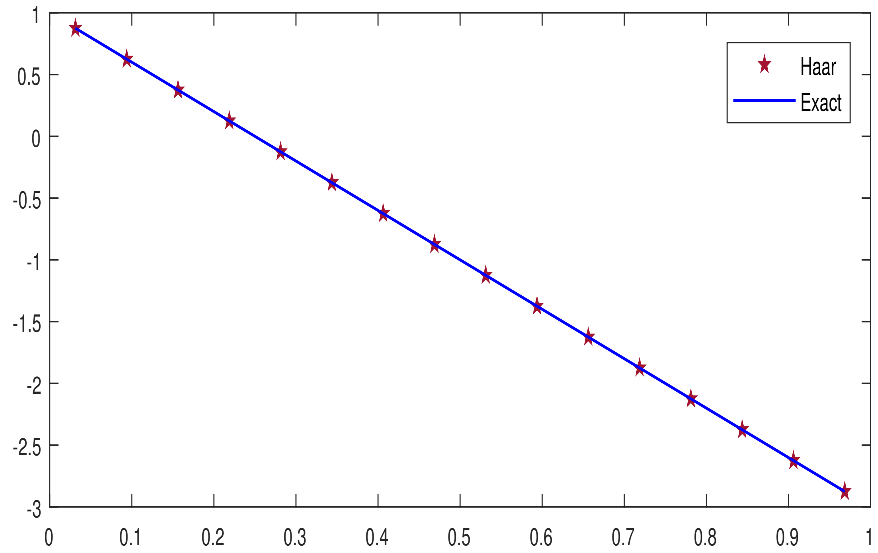

FIGURE 1. Comparison of Exact and Numerical solution at J =3

$J = 4$ is depicted in Fig.2. Computer simulations indicate that by increasing resolution a good approximate solution can be achieved.

Problem 3. Consider the RDE

$$y'(t) = y(t) - 50ty^2 + f(t),\ 0 \le t \le 1,\ y(0) = 1. \tag{4.3}$$

The function $f(t)$ is chosen such that (4.3) has exact solution $y(t) = \frac{1}{1+25t^2}$. We have calculated maximum absolute error and experimental rate of convergence at J=4,5,6,7,8,9. The maximum absolute error is decreasing from order 10^{-3} for $J = 4$ to order 10^{-6} for $J = 9$. Further, we observed that the maximum absolute error decreases by increasing the levels of resolution and the numerical rate of convergence approaches to 2, thus confirming the theoretical results (studied by authors in [18]). Finally, the Comparison between approximate and exact solution is depicted in Fig.4

TABLE 3. Comparison Between Exact Solution and Haar Solution for Problem 1 at J=3

$t_\rho = (\rho - 0.5)/2^{J+1}$ $t_\rho = (1/32)$	Haar solution	Exact	Error
1	0.8750	0.8750	0.0222E-014
3	0.6250	0.6250	0.0666E-014
5	0.3750	0.3750	0.0666E-014
7	0.1250	0.1250	0.0333E-014
9	-0.1250	-0.1250	0.0222E-014
11	-0.3750	-0.3750	0.0888E-014
13	-0.6250	-0.6250	0.1554E-014
15	-0.8750	-0.8750	0.1554E-014
17	-1.1250	-1.1250	0.1332E-014
19	-1.3750	-1.3750	0.1332E-014
21	-1.6250	-1.6250	0.1332E-014
23	-1.8750	-1.8750	0.0444E-014
25	-2.1250	-2.1250	0.0888E-014
27	-2.3750	-2.3750	0.2665E-014
29	-2.6250	-2.6250	0.4885E-014
31	-2.8750	-2.8750	0.8882E-014

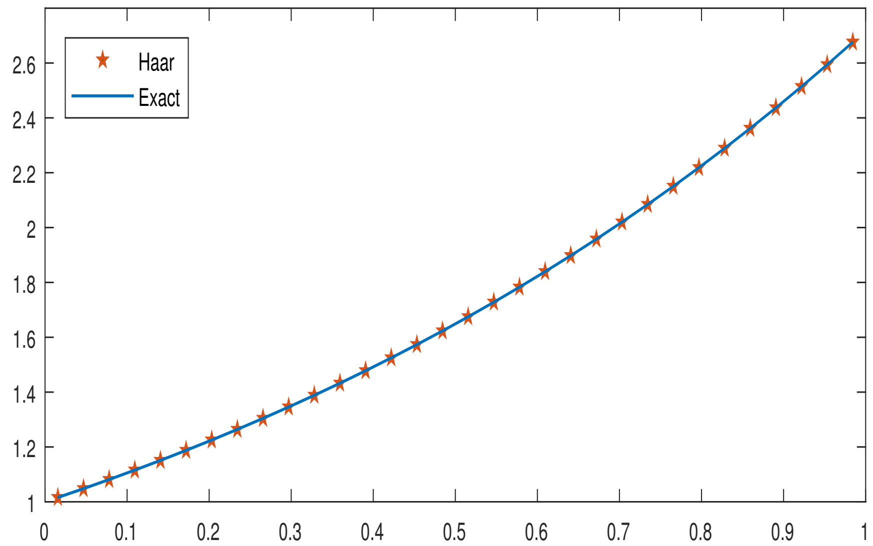

FIGURE 2. Comparison between Exact and Numerical solution at J=4

Problem 4. Solve the proportional-delay Riccati differential equation

$$y'(t) = 1 - 2y^2(\frac{t}{2}), \; 0 \leq t \leq 2\pi, \; y(0) = 0. \tag{4.4}$$

TABLE 4. max. abs. error for Problem 2

Value of J	Max. abs. error
4	2.5841E-04
5	6.5049E-05
6	1.6315E-05
7	4.0855E-06
8	1.0238E-06
9	2.5889E-07

TABLE 5. max. abs. error for Problem-2 with J=5

our method	Bezier Curves Method[11]
6.5049E-05	8.0175E-04

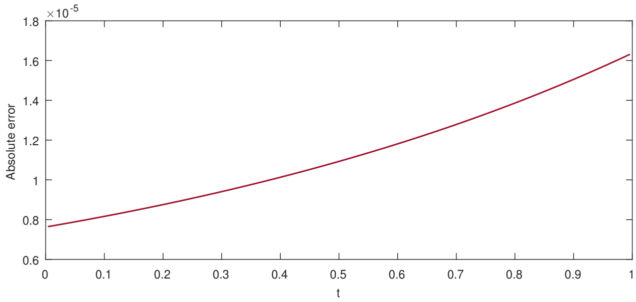

FIGURE 3. Absolute Error(Problem 2) at J=6.

The exact solution of (4.4) is $y(t) = sin(t)$[4]. In this problem $\alpha = 1/2$, which is the case of retarded equation. We have solve this problem using Haar wavelet expansion and integral of Haar wavelet. The maximum absolute error for different resolution is shown in Table 6. Comparison between analytic and approximate solution at $J = 5$ is shown in Fig.5. Maximum absolute error is $3.0790E - 06$ at $J = 10$. We observed that by increasing the value of J a good approximation can be achieved.

TABLE 6. max. abs. error for Problem-4.

Value of J	Max. abs. error
5	3.1062E-03
6	7.8308E-04
7	1.9648E-04
8	4.9204E-05
9	1.2307E-05
10	3.0790E-06

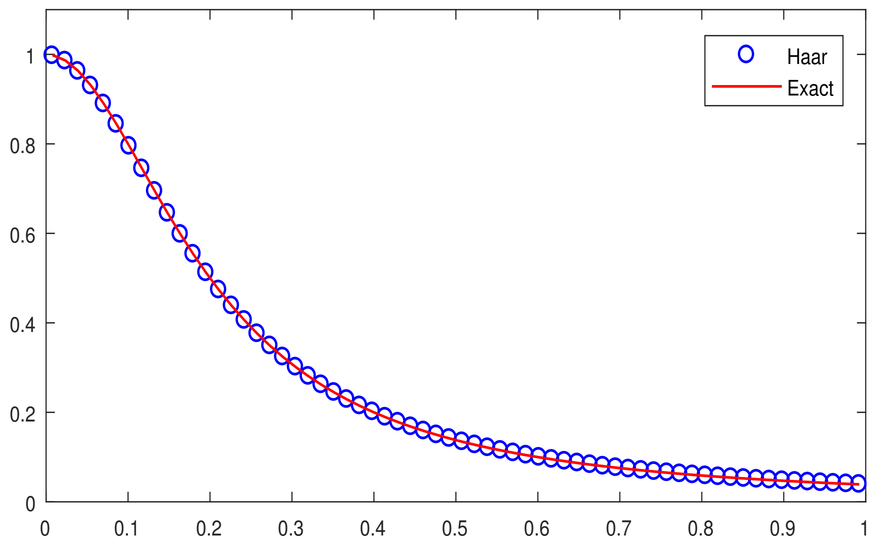

FIGURE 4. Comparison Between Exact and Numerical Solution at J=5

Problem 5. Solve the delay equation

$$y'(t) = \frac{1}{4}y(t) + y(\frac{t}{2})(1 - y(\frac{t}{2})),\ 0 \leq t \leq 1,\ y(0) = 1. \tag{4.5}$$

The equation (4.5) is an example of proportional-delay variant of Riccati differential equation and it possesses the periodic solution

$$y(t) = \frac{1}{2} + \frac{1}{2}cos(\frac{\sqrt{2}t}{4}) + \frac{\sqrt{2}}{2}sin(\frac{\sqrt{2}t}{4}).$$

First, we will transform (4.5) into the following system of algebraic equations,

$$\sum_{i=1}^{2M} a_i \mathtt{h}_i(t_\rho) - \frac{1}{4}\left[\sum_{i=1}^{2M} a_i I_1 \mathtt{h}_i(t_\rho) + 1\right] - \left[\sum_{i=1}^{2M} a_i I_1 \mathtt{h}_i(\frac{t_\rho}{2}) + 1\right]\left[1 - (\sum_{i=1}^{2M} a_i I_1 \mathtt{h}_i(\frac{t_\rho}{2}) + 1)\right] = 0. \tag{4.6}$$

On solving the system of algebraic equation in (4.6), we obtained the values of Haar coefficients $a_i^{,}s$. Using the values of a_i along with given initial condition $y(0) = 1$ in (3.2) approximate numerical solution for (4.5) is obtained. Maximum absolute error for different values of J is presented in Table 7 and the maximum absolute error is $7.4500E - 09$ for $J = 8$. In Table 9 pointwise error between exact solution and Haar solution at $J = 3$ is reported. Finally, comparison between exact solution and numerical solution at $J = 3$ is depicted graphically in Fig.7. We observed that a better approximation can be achieved with the increase in value of J.

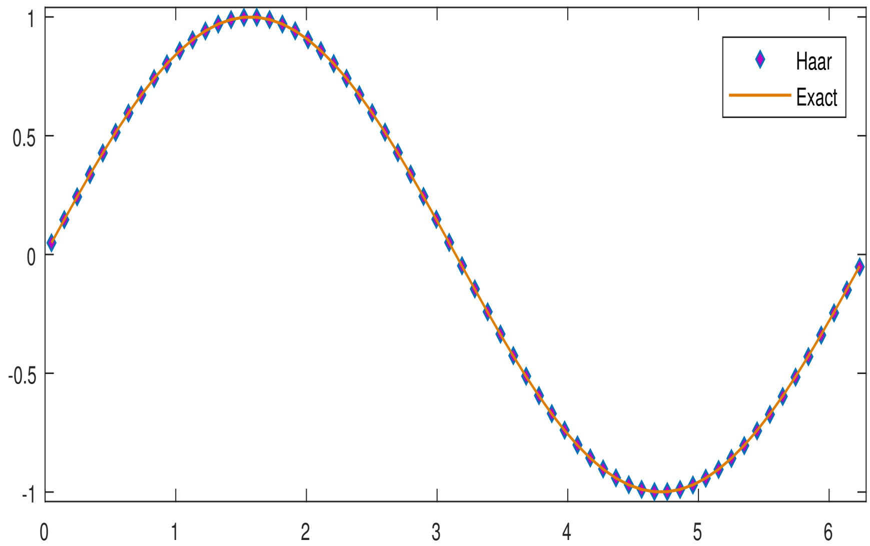

FIGURE 5. Comparison Between Exact and Numerical Solution at J=5.

TABLE 7. max. abs. error for Problem-5.

Value of J	Max. abs. error
3	3.0352E-05
4	7.6091E-06
5	1.9048E-06
6	4.7652E-07
7	1.1917E-07
8	7.4500E-09

TABLE 8. max. abs. error for Problem 6.

Value of J	Max. abs. error
4	5.8484E-04
5	1.5242E-04
6	3.8914E-05
7	9.8315E-06
8	2.4709E-06
9	6.1935E-07

Problem 6. Solve the delay equation

$$y'(t) = -\frac{1}{8}y(t) + y(\frac{t}{2})(1 - y(\frac{t}{2})),\ 0 \leq t \leq 4\pi,\ y(0) = \frac{1}{4}. \tag{4.7}$$

Note that (4.7) is first order RDE with proportional delay. The exact solution for this problem is

$$y(t) = \frac{1}{2} - \frac{1}{4}cos(\frac{\sqrt{5}t}{8}) + \frac{\sqrt{5}}{4}sin(\frac{\sqrt{5}t}{8}).$$

On applying the numerical technique mentioned in section 3 maximum absolute error between exact solution and numerical solution at different values of J is recorded in Table 8. From Table 8 we have observed that Maximum absolute error decreases with the increase in resolution. The maximum absolute error is decreasing from order 10^{-4} at $J = 4$ to 10^{-7} at $J = 9$. The rate of convergence approaches to 2, which follows

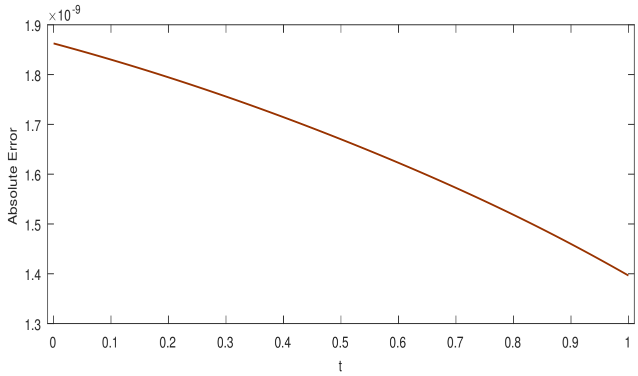

FIGURE 6. Absolute Error(Problem 5) at J=10

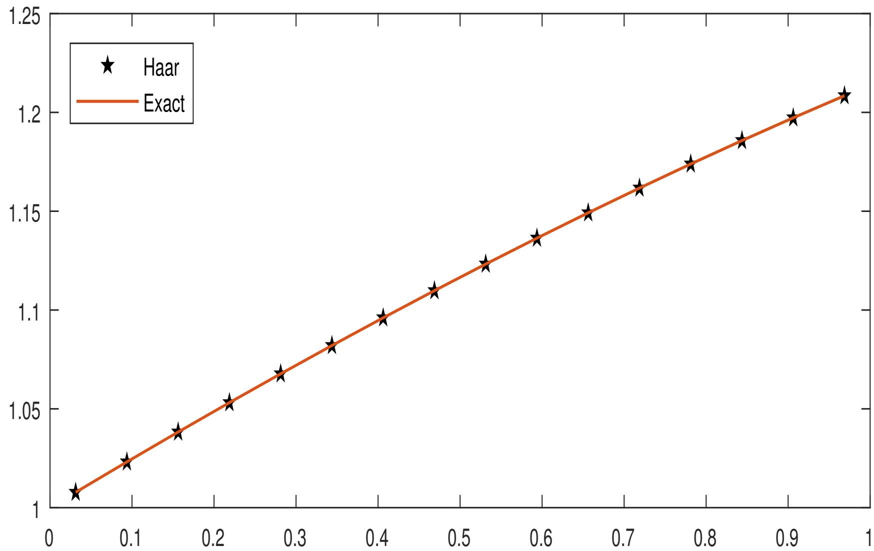

FIGURE 7. Comparison Between Exact and Numerical Solution at J=3

theoretical results mentioned in [18]. Comparison between exact solution and numerical solution at $J = 4$ is shown in Fig.8.

TABLE 9. Comparison Between Exact Solution and Haar Solution for Problem 5 at J=3

$t_\rho = (\rho - 0.5)/2^{J+1}$ $t_\rho = (1/32)$	Haar solution	Exact	Error
1	1.007751	1.007781	0.303515E-004
3	1.023128	1.023158	0.300143E-004
5	1.038250	1.038279	0.296578E-004
7	1.053109	1.053138	0.292823E-004
9	1.067697	1.067726	0.288881E-004
11	1.082009	1.082038	0.284758E-004
13	1.096036	1.096065	0.280460E-004
15	1.109773	1.109801	0.275987E-004
17	1.123212	1.123239	0.271345E-004
19	1.136346	1.136373	0.266540E-004
21	1.149170	1.149196	0.261577E-004
23	1.161677	1.161702	0.256458E-004
25	1.173860	1.173885	0.251189E-004
27	1.185715	1.185740	0.245777E-004
29	1.197235	1.197259	0.240227E-004
31	1.208414	1.208438	0.234542E-004

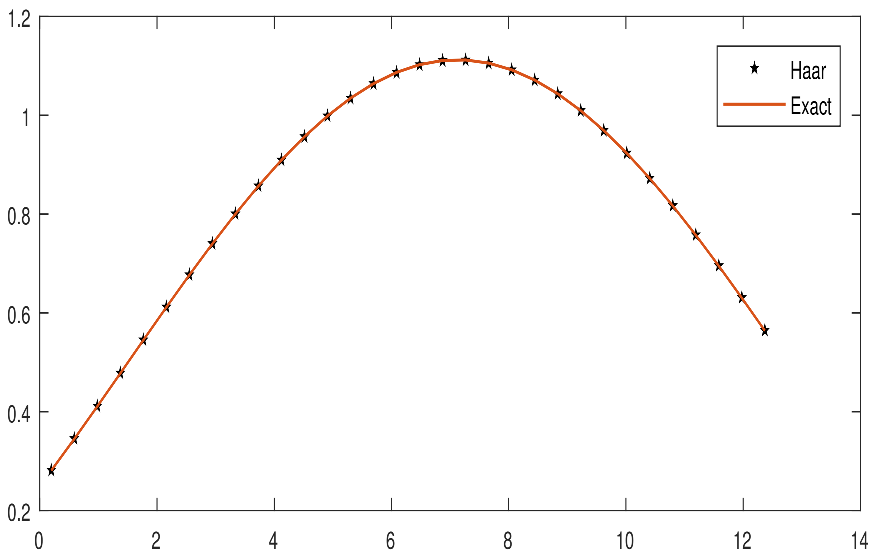

FIGURE 8. Comparison Between Exact and Numerical Solution at J=4

5. Conclusion:

In this manuscript, Haar wavelet method has been successfully applied to find the numerical solution of RDEs and proportional-delay variant of RDEs and several problems from literature has been solved. Numerical results are compared with Bezier Curves method and comparison between exact solution and numerical solutions clearly indicate that our computation is much better in terms of accuracy and convergence.

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