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BOUNDS ON ZALCMAN CONJECTURE AND $H_2(2)$ OF λ-PSEUDO-STARLIKE FUNCTIONS RELATED TO A SHELL-LIKE DOMAIN

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Abstract. The current work aims at estimating the bounds on few initial coefficients, Fekete-Szegö inequality, Second order Hankel determinant, Zalcman inequality for the functions belonging to the subclasses $\pounds_{\lambda}(q)$ and $\pounds_{\lambda,g}(q)$ of holomorphic functions.

1. Introduction

The family $\mathcal{A}$ of holomorphic functions f defined on the disc $\Delta = \{z \in C : |z| < 1\}$ are of the form $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ with $a_1 = 1$.

The subfamily of $\mathcal{A}$ whose members are holomorphic and injective in Δ is designated by S following the rule $f = 0,\ f' = 1,$ at $z = 0$.

The geometrical properties between two holomorphic functions G and H can be studied by the concept of subordination, which perhaps expressed as $G \prec H$ and defined by $G(z) = H(w(z))$, $z \in \Delta$, where w is holomorphic function such that $w(0) = 0$ and $|w(z)| < 1, z \in \Delta$. Moreover if the function G is injective in Δ, then the image of Δ under G is contained in the image of Δ under H with $G(0) = H(0)$.

Lawrence Zalcman conjectured that the members of S satisfy the following property

$$|a_m^2 - a_{2m-1}| \leq (m-1)^2, \forall m > 2.$$

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Krushkal [8] has proved this conjecture for $3 \leq m \leq 6$. Authors like Ma [11], Ravichandran [21] and Sharma et al. [24] had determined the Zalcman inequality for functions in certain subclasses of S.

A function $f \in \mathfrak{L}_\lambda(\beta)$ is called λ-pseudo-starlike function of order β if

$$\Re\Big(\frac{z(f'(z))^\lambda}{f(z)}\Big) > \beta,\ z \in \Delta.$$

In 2013, Babalola [2] estimated the bounds for λ-pseudo-starlike function as a subclass of S and remarked as below

Remark 1.1. If $\beta = 0$ then the above set reduced to the function class $\mathfrak{L}_\lambda$, given by

$$\Re\Big(\frac{z(f'(z))^\lambda}{f(z)}\Big) > 0,\ z \in \Delta.$$

Murugusundaramoorthy et al.[14, 15] also studied pseudo-starlike functions.
The Hankel determinant of f for $l, n \in \mathcal{Z}^+$ was defined by Pommerenke [16]. It has the following representation

$$H_l(n) = \begin{vmatrix} a_n & a_{n+1} & ------ & a_{n+l-1} \\ a_{n+1} & a_{n+2} & ------ & a_{n+l} \\ -- & -- & ------ & --- \\ -- & -- & ------ & --- \\ a_{n+l-1} & a_{n+l} & ------ & a_{n+2l-2} \end{vmatrix}. \tag{1.1}$$

The computation of this determinant for functions in various subfamilies of S was considered by several authors. For distinct values of l and n one can obtain Hankel determinant of different orders. For the choice of $l = 2$ and $n = 1$, we have $H_2(1) = a_3 - a_2^2$, which is a particular case of Fekete-Szegö functional [4] for $\mu = 1$. The Fekete-Szegö inequality has been estimated by several authors ([1], [6], [19], [23]). $H_2(2)$ is termed as Hankel determinant of second kind, obtained by taking $l = n = 2$ in equation (1.1), and employing modulus on either side, we get

$$\mid H_2(2) \mid = |a_2a_4 - a_3^2|. \tag{1.2}$$

One can view the work on $H_2(2)$ in [3, 17, 9, 16, 20, 25].
The function $q(z) = z + \sqrt{1 + z^2}$ is holomorphic, injective on $C \setminus \{i, -i\}$ with $\Re(q(z)) > 0$ and $q(0) = q'(0) = 1$. It has a nice property of mapping unit disc Δ onto a shell like domain which is symmetric about the real axis bounded between the points 0.4 to 2.41 on real axis and from -1 to 1 on y-axis. (Figure 1).

Figure 1. Shell shaped region

Raina and Sokół [18] have estimated bounds and certain coefficient inequalities for the function f in the class $S^\star(q)$. Very recently Sharma and Hari Priya [22] introduced the new subclass $M_\alpha(q)$ and calculated the inequalities associated with the coefficients, for the quotient function and inverse function, the inequality due to Fekete-Szegö through convolution and bound on $|H_2(2)|$.

2. Preliminaries

Here, we give some definitions and lemmas that help in proving main results.

Definition 2.1. A function $f \in \mathcal{L}_\lambda(q)$, if

$$\Big[\frac{\mathrm{z}(f'(z))^\lambda}{f(z)}\Big] \prec z + \sqrt{1+z^2} = q(z),\ z \in \Delta, \tag{2.1}$$

where the subordinating function is such that $q(0) = 1$.

Definition 2.2. A function $f \in \mathcal{L}_{\lambda,g}(q)$ if

$$\Big(\frac{z[(f*g)'(z)]^\lambda}{(f*g)(z)}\Big) \prec z + \sqrt{1+z^2} = q(z),\ z \in \Delta, \tag{2.2}$$

where the subordinating function is such that $q(0) = 1$.

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots,\ z \in \Delta \tag{2.3}$$

be a member of the family $\mathcal{P}$, of regular functions in Δ, with $\Re(p(z)) > 0, p = 1$ at $z = 0$.

Lemma 2.3. [16] *Let $p \in \mathcal{P}$, then*

$$|\ p_n\ | \leq 2\ \ \forall\ n \in \mathbb{N}.$$

The Carathéodary function $p(z) = \dfrac{1+z}{1-z}$, is the extremal function.

Lemma 2.4. *([13]) Let $p \in \mathcal{P}$, then*

$$\Big|p_2 - \frac{\mu}{2}p_1^2\Big| \leq\ Max\{2, 2|\mu-1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2; \\ 2|\mu-1|, & otherwise. \end{cases}$$

Further we also have [12],

$$\Big|p_2 - \frac{p_1^2}{2}\Big| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.5. [7] *For a Schwarz function $w(z) = p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, for any $\mu \in \mathbb{C}$,*

$$\Big|p_2 - \mu p_1^2\Big| \leq Max\{1, |\mu|\}.$$

Lemma 2.6. *([5, 10]) Let $p \in \mathcal{P}$, then*

$$\begin{aligned} p_2 =& \frac{p_1^2 + (4-p_1^2)x}{2}, \\ p_3 =& \frac{1}{4}\Big(p_1^3 + 2x(4-p_1^2)p_1 - p_1 x^2(4-p_1^2) + 2(4-p_1^2)(1-|x|^2)z\Big), \end{aligned}$$

for any x, z satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

3. Main Results

Theorem 3.1. *If* $f \in \pounds_\lambda(q),\ \lambda \geq 1,$ *then*

$$| a_2 | \leq \frac{1}{2\lambda - 1}, \tag{3.1}$$

$$| a_3 | \leq \frac{1}{3\lambda - 1} Max\Big(1, \Big|\frac{1-4\lambda}{2(4\lambda^2 - 4\lambda + 1)}\Big|\Big), \tag{3.2}$$

$$| a_4 | \leq \frac{1}{(4\lambda - 1)}\Big(1 + \frac{12\lambda^2 - 16\lambda + 3}{(6\lambda^2 - 5\lambda + 1)} Max\Big[1, \Big|\frac{168\lambda^4 - 308\lambda^3 + 186\lambda^2 - 55\lambda + 6}{6(12\lambda^2 - 16\lambda + 3)(2\lambda - 1)^2}\Big|\Big]\Big). \tag{3.3}$$

Proof. As $f \in \pounds_\lambda(q)$ there is a Schwarz function w, with $w = 0$ at $z = 0$ and $|w(z)| - 1 \leq 0$ such that

$$\frac{z[f'(z)]^\lambda}{f(z)} = w(z) + \sqrt{1 + [w(z)]^2},\ z \in \Delta. \tag{3.4}$$

Define a function $p(z)$ such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

$$w(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{3.5}$$

From relations (3.4), (3.5) along with the corresponding Taylor's series expansion of f, its derivative and series representation of w on the right side of (3.4), upon comparing the coefficients of $z^i,\ i = 1, 2, 3, \ldots$, we have

$$a_2 = \frac{p_1}{2(2\lambda - 1)}, \tag{3.6}$$

$$a_3 = \frac{1}{2(3\lambda - 1)}\Big[p_2 - \Big(\frac{8\lambda^2 - 12\lambda + 3}{2(2\lambda - 1)^2}\Big)\frac{p_1^2}{2}\Big], \tag{3.7}$$

$$a_4 = \frac{1}{2(4\lambda - 1)}\Big[p_3 - p_1 p_2\Big[\frac{12\lambda^2 - 16\lambda + 3}{12\lambda^2 - 10\lambda + 2}\Big] + \frac{p_1^3}{24(2\lambda - 1)^3(3\lambda - 1)}\Big[120\lambda^4 - 364\lambda^3 + 342\lambda^2 - 113\lambda + 12\Big]\Big]. \tag{3.8}$$

The bound on a_2 obtained by employing modulus on either side of (3.6), and using that $\big|p_n\big| \leq 2$, we have

$$| a_2 | \leq \frac{1}{2\lambda - 1}. \tag{3.9}$$

Taking the value of $\nu_1 = \Big(\frac{8\lambda^2 - 12\lambda + 3}{2(2\lambda - 1)^2}\Big)$ and by employing modulus on either side of (3.7), and using Lemma 2.4, we get

$$| a_3 | \leq \frac{1}{2(3\lambda - 1)} \,|\, p_2 - \nu_1 \frac{p_1^2}{2} \,|\,. \tag{3.10}$$

The sharpness of the result attains as follows

$$| a_3 |= \begin{cases} \dfrac{1}{3\lambda-1}, & \text{if } 0 \leq \nu_1 \leq 2; \\ \dfrac{| 1-4\lambda |}{2(3\lambda-1)(2\lambda-1)^2}, & \text{elsewhere.} \end{cases}$$

If a function $f_2(z) = z + \sum\limits_{n=2}^{\infty} d_n z^n$ and

$$\frac{z[f_2'(z)]^\lambda}{f_2(z)} = q(z^2) = z^2 + \sqrt{1+z^4}, \tag{3.11}$$

then $f_2 \in \pounds_\lambda(q)$. From relation (3.11), after computation and comparing the coefficients of like powers of z on either side of (3.11) few coefficients of the series are given by $d_2 = d_4 = 0$ and $d_3 = \dfrac{1}{(3\lambda-1)}$. The function $f_2(z)$ can be expressed as

$$f_2(z) = z + \frac{1}{3\lambda-1} z^3 + \cdots \tag{3.12}$$

The value of a_3 matches for the function $f_2(z)$. Similarly by employing Lemmas 2.3 and 2.4 on (3.8),we get equation (3.3). $\square$

Theorem 3.2. *If $f \in \pounds_\lambda(q)$, $\lambda \geq 1$, then for any $\mu \in \mathbb{R}$*

$$| a_3 - \mu a_2^2 |\leq \frac{1}{(3\lambda-1)} Max\Big\{1, \Big|\frac{1-4\lambda+2\mu(3\lambda-1)}{2(2\lambda-1)^2}\Big|\Big\} \tag{3.13}$$

and the sharpness is attained.

Proof. From equations (3.6) and (3.7), the functional

$$a_3 - \mu a_2^2 = \frac{1}{2(3\lambda-1)}\Big(p_2 - \nu_2 \frac{p_1^2}{2}\Big), \tag{3.14}$$

where $\nu_2 = \Big(\dfrac{8\lambda^2 - 12\lambda + 3 + 2\mu(3\lambda-1)}{2(2\lambda-1)^2}\Big)$. We obtain the bound by using Lemma 2.4 on the R.H.S. of the equation (3.14), we get (3.13). The sharpness of the result is as given below

$$| a_3 - \mu a_2^2 |= \begin{cases} \dfrac{1}{(3\lambda-1)}, & \mu \in \big[\eta_1, \eta_2\big]; \\ \dfrac{1}{2(4\lambda^2-4\lambda+1)(3\lambda-1)} | 1+2\mu(3\lambda-1)-4\lambda |, & \text{else where,} \end{cases} \tag{3.15}$$

where $\eta_1 = \dfrac{-8\lambda^2+12\lambda-3}{2(3\lambda-1)}$ and $\eta_2 = \dfrac{8\lambda^2-4\lambda+1}{2(3\lambda-1)}$. The sharpness of the results attains with respect to the following functions

If a function $f_1(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ is in S and is such that

$$\frac{z[f_1'(z)]^\lambda}{f_1(z)} = q(z) = z + \sqrt{1+z^2}, \tag{3.16}$$

then $f_1 \in \pounds_\lambda(q)$. From relation (3.16), after computation and comparing the coefficients of like powers of z on either side of (3.16) few coefficients of the series are given by

$$\begin{aligned} b_2 &= \frac{1}{(2\lambda-1)}, \\ b_3 &= \frac{4\lambda-1}{2(4\lambda^2-4\lambda+1)(3\lambda-1)}, \\ b_4 &= \frac{1}{6(4\lambda-1)(3\lambda-1)(2\lambda-1)^3}(-24\lambda^4+44\lambda^3-6\lambda^2+\lambda). \end{aligned}$$

Hence, the function $f_1(z)$ can be expressed as

$$f_1(z) = z + \frac{1}{(2\lambda-1)}z^2 + \Big(\frac{4\lambda-1}{2(4\lambda^2-4\lambda+1)(3\lambda-1)}\Big)z^3 + \frac{(-24\lambda^4+44\lambda^3-6\lambda^2+\lambda)}{6(4\lambda-1)(3\lambda-1)(2\lambda-1)^3}z^4 + \cdots . \tag{3.17}$$

Now,

$$|b_3 - \mu b_2^2| = \Big|\frac{2\mu(3\lambda-1)+1-4\lambda}{2(2\lambda-1)^2(3\lambda-1)}\Big|.$$

Thus $| b_3 - \mu b_2^2 | = \dfrac{1}{(3\lambda-1)}$ for the function $f_2(z)$ and $|b_3-\mu b_2^2| = \Big|\dfrac{2\mu(3\lambda-1)+1-4\lambda}{2(2\lambda-1)^2(3\lambda-1)}\Big|$ for the function $f_1(z)$. □

Remark 1: For $\lambda = 1$, we get the result of Raina and Sokół [18].

Theorem 3.3. *If $f \in \pounds_{\lambda,g}(q)$, then for any $\mu \in \mathbb{R}$*

$$| a_3 - \mu a_2^2 | \leq \frac{1}{g_3(3\lambda-1)} Max\Big\{1, \Big|\frac{(1-4\lambda)g_2^2 + (2\mu(3\lambda-1))g_3}{2g_2^2(2\lambda-1)^2}\Big|\Big\} \tag{3.18}$$

and the sharpness is attained.

Proof. From the definition of the class $\pounds_{\lambda,g}(q)$, along with the principle of subordination using a similar methodology as in theorem 3.2 the result follows. □

Theorem 3.4. *If $f \in \pounds_\lambda(q)$, $\lambda \geq 1$, then the bound on $H_2(2)$ is given by*

$$\Big|H_2(2)\Big| \leq \frac{1}{(3\lambda-1)^2}. \tag{3.19}$$

Proof. From relations (3.6), (3.7) and (3.8), consider

$$a_2a_4 - a_3^2 = \Big(\frac{p_3p_1}{4(2\lambda-1)(4\lambda-1)} - \frac{p_2^2}{4(3\lambda-1)^2} - \xi p_1^4\Big) - \Big(\frac{\lambda}{8(3\lambda-1)^2(4\lambda-1)}\Big)p_2p_1^2, \tag{3.20}$$

where $\xi = \Big(\dfrac{48\lambda^5 - 72\lambda^4 + 100\lambda^3 - 78\lambda^2 + 26\lambda - 3}{192(2\lambda-1)^4(3\lambda-1)^2(4\lambda-1)}\Big)$. Now taking modulus on both sides and upon simplification we obtain,

$$\begin{aligned}|a_2a_4 - a_3^2| =& \frac{1}{192(2\lambda-1)^4(3\lambda-1)^2(4\lambda-1)}\times \\ &\Big|48(2\lambda-1)^3(3\lambda-1)^2p_1p_3 - 24\lambda(2\lambda-1)^4p_1^2p_2 - 48(2\lambda-1)^4(4\lambda-1)p_2^2 \\ &- p_1^4(48\lambda^5 - 72\lambda^4 + 100\lambda^3 - 78\lambda^2 + 26\lambda - 3)\Big|. \end{aligned} \tag{3.21}$$

The above equation can be further rewritten as

$$|a_2a_4 - a_3^2| = \frac{|g_1p_1p_3 + g_2p_1^2p_2 + g_3p_2^2 + g_4p_1^4|}{192(2\lambda-1)^4(3\lambda-1)^2(4\lambda-1)}, \tag{3.22}$$

where

$$\begin{aligned} g_1 =& 48(2\lambda-1)^3(3\lambda-1)^2, \\ g_2 =& -24\lambda(2\lambda-1)^4, \\ g_3 =& -48(2\lambda-1)^4(4\lambda-1), \\ g_4 =& -(48\lambda^5 - 72\lambda^4 + 100\lambda^3 - 78\lambda^2 + 26\lambda - 3). \end{aligned}$$

According to Lemma 2.6, expressing p_2, p_3 in terms of p_1 on the R.H.S of the equation (3.22) and notice that $|z| < 1$, an application of triangle inequality yields to

$$|g_1p_1p_3 + g_2p_1^2p_2 + g_3p_2^2 + g_4p_1^4| \leq \tag{3.23}$$

$$\begin{aligned}&\frac{1}{4}\Big|(g_1 + 2g_2 + g_3 + 4g_4)p_1^4 + 2g_1p_1(4-p_1^2) + 2(g_1+g_2+g_3) \\ &(4-p_1^2)p_1^2|y| - [(g_1+g_3)p_1^2 + 2g_1p_1 - 4g_3](4-p_1^2)|y|^2\Big|. \end{aligned} \tag{3.24}$$

Using g_1, g_2, g_3 and g_4 values, we obtain

$$g_1 + 2g_2 + g_3 + 4g_4 = -576\lambda^5 + 1248\lambda^4 - 1264\lambda^3 + 648\lambda^2 - 152\lambda + 12, \tag{3.25}$$

$$g_1 + g_2 + g_3 = 24\lambda(2\lambda-1)^3, \tag{3.26}$$

$$(g_1+g_3)p_1^2 + 2g_1p_1 - 4g_3 = 48(2\lambda-1)^3[\lambda^2p_1^2 + 4(2\lambda-1)(4\lambda-1) + 2(3\lambda-1)^2p_1]. \tag{3.27}$$

Now we consider the expression
$[\lambda^2p_1^2 + 4(2\lambda-1)(4\lambda-1) + 2(3\lambda-1)^2p_1]$

$$\begin{aligned} &= \lambda^2\Big[p_1 + \Big[\Big(\frac{3\lambda-1}{\lambda}\Big)^2 + \sqrt{\frac{49\lambda^4 - 84\lambda^3 + 50\lambda^2 - 4\lambda + 1}{\lambda^4}}\Big]\Big]\times \\ &\Big[p_1 + \Big[\Big(\frac{3\lambda-1}{\lambda}\Big)^2 - \sqrt{\frac{49\lambda^4 - 84\lambda^3 + 50\lambda^2 - 4\lambda + 1}{\lambda^4}}\Big]\Big]. \end{aligned} \tag{3.28}$$

As $p_1 \in [0,2]$, for any two positive real numbers l, m, we observe that
$(p_1 - l)(p_1 - m) \leq (p_1 + l)(p_1 + m)$. Thus we have from (3.28)

$$[\lambda^2p_1^2 + 4(2\lambda-1)(4\lambda-1) - 2(3\lambda-1)^2p_1] \leq [\lambda^2p_1^2 + 4(2\lambda-1)(4\lambda-1) + 2(3\lambda-1)^2p_1]. \tag{3.29}$$

Using (3.29) and (3.27), we see that

$$[(g_1+g_3)p_1^2+2g_1p_1-4g_3] \leq 48(2\lambda-1)^3[\lambda^2p_1^2+4(2\lambda-1)(4\lambda-1)-2(3\lambda-1)^2p_1]. \tag{3.30}$$

On replacement of the values in (3.24) from (3.25), (3.26) and (3.30) on the R.H.S., we get

$$|g_1p_1p_3+g_2p_1^2p_2+g_3p_2^2+g_4p_1^4| \leq \tag{3.31}$$

$$\begin{aligned}&\frac{1}{4}\Big|(-576\lambda^5+1248\lambda^4-1264\lambda^3+648\lambda^2-152\lambda+12)p_1^4+\\ &96p_1(4-p_1^2)(2\lambda-1)^3(3\lambda-1)^2+48(4-p_1^2)p_1^2|y|\lambda(2\lambda-1)^3+\\ &48(2\lambda-1)^3[\lambda^2p_1^2+4(2\lambda-1)(4\lambda-1)-2(3\lambda-1)^2p_1]\\ &(4-p_1^2)|y|^2\Big|.\end{aligned} \tag{3.32}$$

Letting $p_1=p\in[0,2]$, with the use of triangle inequality once again and writing $|y|=\varrho$ on the right hand side of the above inequality, gives

$$4|g_1p_1p_3+g_2p_1^2p_2+g_3p_2^2+g_4p_1^4| \leq \tag{3.33}$$

$$\begin{aligned}&\Big[(-576\lambda^5+1248\lambda^4-1264\lambda^3+648\lambda^2-152\lambda+12)p^4+\\ &96p(4-p^2)(2\lambda-1)^3(3\lambda-1)^2+48(4-p^2)p^2\varrho\lambda(2\lambda-1)^3+\\ &48(2\lambda-1)^3[\lambda^2p^2+4(2\lambda-1)(4\lambda-1)-2(3\lambda-1)^2p]\\ &(4-p^2)\varrho^2\Big]\\ &=L(p,\varrho)\ \text{(say)}.\end{aligned} \tag{3.34}$$

To estimate the bound, it is necessary that $L(p,\varrho)$ is to be maximized on the rectangular region $[0,2]\times[0,1]$. We start by assuming that the maximum exists at any point interior of the region $[0,2]\times[0,1]$. The partial derivative of $L(p,\varrho)$ with respect to ϱ, is given by

$$\frac{\partial L}{\partial \varrho}=48(2\lambda-1)^3(4-p^2)[\lambda^2p_1^2+4(2\lambda-1)(4\lambda-1)-2(3\lambda-1)^2p].$$

Thus $L(p,\varrho)$ accelerates as ϱ on the interval $(0,1)$. For $\lambda\geq 1$, with fixed p lying between 0 and 2 , it is clear that $\frac{\partial L}{\partial \varrho}>0$. Hence $L(p,\varrho)$ is non decreasing function of ϱ and cannot attain maximum value for any point inside of the region $[0,2]\times[0,1]$. Hence the greatest value exists on the boundary for $\varrho=1$. On writing $\varrho=1$ in equation (3.34), for fixed $p\in[0,2]$, we get

$$Max\{L(p,\varrho):0\leq\varrho\leq 1\}=L(p,1)=H(p)\ \text{(say)}.$$

On replacing $\varrho=1$ in (3.34), along with the above consideration,

$$\begin{aligned}H(p)=&-(960\lambda^5-1440\lambda^4+976\lambda^3-408\lambda^2+104\lambda-12)p^4-\\ &192(2\lambda-1)^3(7\lambda^2-7\lambda+1)p^2+768(2\lambda-1)^4(4\lambda-1).\end{aligned} \tag{3.35}$$

It can be easily seen that

$$H'(p) = -\Big(4p^3[960\lambda^5 - 1440\lambda^4 + 976\lambda^3 - 408\lambda^2 + 104\lambda - 12] + 2p(2\lambda-1)^3(384\lambda^2 - 384\lambda + 192)\Big). \quad (3.36)$$

It is clear that $H'(p) \leq 0$, for each fixed value of $p \in [0,2]$ and for $\lambda \geq 1$. Hence $H(p)$is non increasing function of p in $[0,2]$, the maximum attains at $p = 0$. Upon writing $p = 0$, we have

$$Max\ H(p) = H(0) = 768(2\lambda-1)^4(4\lambda-1). \quad (3.37)$$

From (3.37) and (3.34), we get

$$|g_1p_1p_3 + g_2p_1^2p_2 + g_3p_2^2 + g_4p_1^4| \leq 192(2\lambda-1)^4(4\lambda-1). \quad (3.38)$$

Hence we have from (3.38) and (3.22) that $|a_2a_3 - a_4^2| \leq \dfrac{1}{(3\lambda-1)^2}$. The sharpness can be achieved for $f_2(z)$ mentioned as in (3.16),

$$|d_2d_4 - d_3^2| = \Big|\frac{1}{(3\lambda-1)^2}\Big|.$$

□

Remark 2: On writing $\lambda = 1$, $|d_2d_4 - d_3^2| = \dfrac{1}{4}$, holds for $f \in S^\star(q)$.

Theorem 3.5. *If $f \in \pounds_\lambda(q)$, $\lambda \geq 1$, then the coefficient functional is given by*

$$\Big|a_2a_3 - a_4\Big| \leq \frac{1}{4\lambda-1}. \quad (3.39)$$

Proof. The proof of the theorem is similar to that of theorem 3.4, hence the details are omitted. □

Theorem 3.6. *If $f \in \pounds_\lambda(q), (\lambda \geq 1)$, then*

$$|a_3^2 - a_5| \leq \frac{\theta_1 - 9}{\theta_2}, \quad (3.40)$$

where

$$\theta_1 = 62208\lambda^7 - 130752\lambda^6 + 108576\lambda^5 - 10896\lambda^4 - 23712\lambda^3 + 6954\lambda^2 + 150\lambda, \quad (3.41)$$

$$\theta_2 = 72\big(2\lambda-1\big)^4\big(3\lambda-1\big)^2\big(4\lambda-1\big)\big(5\lambda-1\big). \quad (3.42)$$

Proof. As $f \in \pounds_\lambda(q)$, by the principle of subordination, we have

$$\frac{z\big[f'(z)\big]^\lambda}{f(z)} = w(z) + \sqrt{1 + w^2(z)}, \quad (3.43)$$

where $w(z)$ is the Schwarz function represented by

$$w(z) = \sum_{n\geq 1} p_nz^n; \forall n \in N, \text{with } \big|p_n\big| \leq 1. \quad (3.44)$$

Upon simplification on both sides of (3.43) along with (3.44) and comparing the coefficients of like terms of $z^i, i = 1, 2, 3, ...$, we get

$$a_2 = \frac{p_1}{\big(2\lambda - 1\big)}, \tag{3.45}$$

$$a_3 = \frac{1}{\big(3\lambda - 1\big)}\Big[p_2 + \frac{p_1^2}{2}\Big(\frac{4\lambda - 1}{\big(2\lambda - 1\big)^2}\Big)\Big], \tag{3.46}$$

$$a_4 = \frac{1}{\big(4\lambda - 1\big)}\Big[p_3 + \frac{\big(6\lambda - 1\big)}{\big(2\lambda - 1\big)\big(3\lambda - 1\big)}p_1p_2 - \Big(\frac{8\lambda^4 - \frac{44}{3}\lambda^3 + 2\lambda^2 - \frac{\lambda}{3}}{2\big(2\lambda - 1\big)^3\big(3\lambda - 1\big)}\Big)p_1^3\Big], \tag{3.47}$$

$$a_5 = \frac{1}{\big(5\lambda - 1\big)}\Big[\Big(\frac{9\lambda - 1}{\big(3\lambda - 1\big)^2}\Big)\frac{p_2^2}{2} + \frac{\big(8\lambda - 1\big)}{\big(2\lambda - 1\big)\big(4\lambda - 1\big)}p_1p_3 + p_4 - \sigma p_1^2p_2 + \Big[\frac{P}{Q}\Big]p_1^4\Big], \tag{3.48}$$

where

$$P = -6912\lambda^6 + 15840\lambda^5 - 13008\lambda^4 + 6000\lambda^3 - 1518\lambda^2 + 183\lambda - 9,$$

$$\sigma = \Big[\frac{144\lambda^5 - 276\lambda^4 + 64\lambda^3 - \lambda^2 + \lambda}{2\big(2\lambda - 1\big)^2\big(3\lambda - 1\big)^2\big(4\lambda - 1\big)}\Big],$$

$$Q = 72\big(2\lambda - 1\big)^4\big(3\lambda - 1\big)^2\big(4\lambda - 1\big).$$

From (3.46) and (3.48),

$$a_3^2 - a_5 = \Big(\frac{\lambda - 1}{2\big(3\lambda - 1\big)^2\big(5\lambda - 1\big)}\Big)p_2^2 - \frac{p_4}{\big(5\lambda - 1\big)} - \Big(\frac{\big(8\lambda - 1\big)}{\big(2\lambda - 1\big)\big(4\lambda - 1\big)\big(5\lambda - 1\big)}\Big)p_1p_3 + \kappa p_2p_1^2 + \Big[\frac{R}{T}\Big]p_1^4, \tag{3.49}$$

where

$$R = 6912\lambda^6 - 15840\lambda^5 + 18768\lambda^4 - 11472\lambda^3 + 3462\lambda^2 - 498\lambda + 273,$$

$$\kappa = \Big[\frac{144\lambda^5 - 276\lambda^4 + 224\lambda^3 - 113\lambda^2 + 27\lambda - 2}{2\big(4\lambda^2 - 4\lambda + 1\big)\big(3\lambda - 1\big)^2\big(4\lambda - 1\big)\big(5\lambda - 1\big)}\Big],$$

$$T = 72\big(2\lambda - 1\big)^4\big(9\lambda^2 - 6\lambda + 1\big)\big(4\lambda - 1\big)\big(5\lambda - 1\big).$$

From (3.44) and using lemma 2.5, we see that

$$|a_3^2 - a_5| \le \frac{\theta_1 - 9}{\theta_2}, \tag{3.50}$$

where θ_1 and θ_2 are given by (3.41)) and (3.42). Hence Zalcman inequality is vrified for $m = 3$. On writing $\lambda = 1$, leads to the result of starlike function as

$$|a_3^2 - a_5| \le \frac{12519}{3456} \approx 3.622395833 < 4, \tag{3.51}$$

which satisfies the inequality of Zalcman conjecture for $m = 3$. □

Suppose that Ψ is a rational function with representation

$$\Psi(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} q_n z^n, f \in \mathcal{L}_\lambda(q),\ z \in \Delta.$$

Theorem 3.7. *If $f \in \pounds_\lambda(q),\ \lambda \geq 1$ and $\Psi(z) = \dfrac{z}{f(z)}$, then for any $\mu \in \mathbb{R}$,*

$$| q_2 - \mu q_1^2 | \leq \frac{1}{(3\lambda - 1)} Max \left\{1, \left| \frac{(2\lambda - 1) - 2\mu(3\lambda - 1)}{2(2\lambda - 1)^2} \right| \right\}. \tag{3.52}$$

The sharpness is attained.

Proof. As $f \in \pounds_\lambda(q)$, using the representation of the function G along with the Taylor's series of f and following the methodology as in theorem 3.2 the result follows. □

Theorem 3.8. *If $f \in \pounds_\lambda(q),\ \lambda \geq 1$ and $F(w) = w + \sum\limits_{n=2}^{\infty} k_n w^n$ is the inverse mapping of f, where $|w| < r_0(f), r_0(f) \geq \frac{1}{4}$, we have*

$$| k_3 - \mu k_2^2 | \leq \frac{1}{(3\lambda - 1)} Max \left\{1, \left| \frac{8\lambda - 3 - 2\mu(3\lambda - 1)}{2(2\lambda - 1)^2} \right| \right\}, \tag{3.53}$$

where $\mu \in \mathbb{R}$, the sharpness is attained.

Proof. The proof of this theorem is obtained by a similar methodology followed in theorem 3.2 along with the definition of the inverse function F. Hence the details are omitted. □

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