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ON CONNECTED SPACES VIA GENERALIZED CLOSED SETS

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Abstract. In this paper, we use g-closure operator to define g-connected as a property which is weaker than τ^*-connectedness and stronger than τ-connectedness. We give some results regarding the union of g-connected. Also, we introduce and investigate the cut-point of g-connected.

1. Introduction

Let (X, τ) be a topological space with no separation axioms assumed. If $A \subseteq X$, $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively. The concept of generalized closed sets of a topological space (briefly g-closed) was introduced by Levine [11] in 1970. These sets were also considered by Dunham [10] in 1982 and by Dunham and Levine [8] in 1980. Recently papers [7, 12] have introduced some property of connected structure spaces. Throughout this paper (X, τ) (simply X) always means a topological space. A subset B of a topological space X is called g-closed in X [11] if $Cl(B) \subseteq G$ whenever $B \subseteq G$ and G is open in X. Hence the union of two g-closed sets is a g-closed set and the intersection of two g-closed sets is generally not a g-closed set. In a T_1-space g-closed sets are closed [10]. For each $x \in X$, either $\{x\}$ is closed or $X \setminus \{x\}$ is g-closed. A subset A is called g-open in X if its complement $X \setminus A$ or $X - A$ is g-closed, equivalently if $F \subseteq Int(A)$ whenever F is closed and $F \subseteq A$. The intersection of all g-closed sets containing a set A is called the g-closure of A [10] and is denoted by $Cl_g(A)$. This is, for any $A \subseteq X$, $Cl_g(A) = \cap\{F : A \subseteq F$ and F is g-closed in $X\}$. The collection of all g-closed (resp. g-open) subsets of X will be denoted by $GC(X)$ (resp. $GO(X)$). We set $GC(X, x) = \{V \in GC(X) : x \in V\}$ for

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$x \in X$. We define similarly $GO(X, x)$. The g-interior of A is the union of all g-open sets contained in A and is denoted by $Int_g(A)$. Cl_g is a Kuratowski closure operator on X and $\tau^* = \{A : Cl_g(X - A) = X - A\}$ is a topology on X generated by Cl_g in the usual manner and denoted by τ^*. A is said to be τ^*-closed if $Cl_g(A) = A$. The complement of a τ^*-closed set is called a τ^*-open set. For more on the previous notions and the following ones, one can see [1-6].

In this paper, we use g-closure operator to define g-connected as a property which is weaker than τ^*-connectedness and stronger than τ-connectedness. We give some results regarding the union of g-connected. Moreover we introduce and investigate the cut-point of g-connected.

2. Preliminaries

Definition 2.1. [11] Let (X, τ) be a topological space and $A \subseteq X$. Then

(1) A is generalized closed (briefly g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(2) A is generalized open(briefly g-open) if $X \setminus A$ is g-closed.
(3) X is $T_{\frac{1}{2}}$-space if every g-closed set is closed.

Let us start by the following Lemma:

Lemma 2.2. [10] *Let (X, τ) be a topological space and $A \subseteq X$. Then*

(1) $A \subseteq Cl_g(A) \subseteq Cl(A)$.
(2) $Cl_g(A)$ need not be g-closed.
(3) Cl_g is a Kuratowski closure operator on X and $\tau^* = \{A : Cl_g(X - A) = X - A\}$ is a topology on X generated by Cl_g in the usual manner.
(4) $\tau \subseteq GO(X) \subseteq \tau^*$ with equality if and only if X is $T_{\frac{1}{2}}$-space.

Theorem 2.3. [9] *For a topological space (X, τ) is $T_{\frac{1}{2}}$-space if and only if every singleton in X is either open or closed if and only if every subset of X is the intersection of all open sets and all closed sets containing it.*

Levine proves in [11] that every T_1-space is $T_{\frac{1}{2}}$-space and $T_{\frac{1}{2}}$-space is T_0-space, although neither implication is reversible.

Definition 2.4. Let (X, τ) be a topological space. A pair (A, B) of non-empty subsets of X is said to be g-separated if $Cl_g(A) \cap B = A \cap Cl(B) = \emptyset$.

Every τ-separated sets are g-separated and every g-separated sets are τ^*-separated but the converse is not true as shown in the following example.

Example 2.5. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$ and $A = \{a\}$, $B = \{b, c\}$, then the pair (A, B) is g-separated but not τ-separated.

Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, X\}$ and $A = \{a, b, c\}$, $B = \{d\}$, then the pair (A, B) is τ^*-separated but not g-separated.

In $T_{\frac{1}{2}}$-spaces we have τ-separated, g-separated and τ^*-separated are coincide.

Proposition 2.6. Let (X, τ) be a topological space. If A and B are nonempty disjoint subsets of X such that A is τ^*-open and B is open, then a pair (B, A) are g-separated.

Proof. Since $A \cap B = \emptyset$, $A \subseteq X \setminus B$ and so $Cl(A) \subseteq Cl(X \setminus B) = X \setminus B$. Then, $Cl(A) \cap B = \emptyset$. Again $B \subseteq X \setminus A$ and so $Cl_g(B) \subseteq Cl_g(X \setminus A) = X \setminus A$. Thus, $Cl_g(B) \cap A = \emptyset$. Therefore, a pair (B, A) is g-separated. $\square$

Corollary 2.7. Let (X, τ) be a topological space. Then the disjoint nonempty open sets of X are g-separated.

Proposition 2.8. Let a pair (A, B) be a g-separated sets in a topological space (X, τ). If C and D are nonempty subsets such that $C \subseteq A$ and $D \subseteq B$, then a pair (C, D) is also g-separated.

Proof. Since a pair (A, B) is g-separated, $Cl_g(A) \cap B = \emptyset = A \cap Cl(B)$. Now, $C \cap Cl(D) \subseteq A \cap Cl(B) = \emptyset$ and so $C \cap Cl(D) = \emptyset$. Similarly, $Cl_g(C) \cap D \subseteq Cl_g(A) \cap B = \emptyset$ and so $Cl_g(C) \cap D = \emptyset$. Hence a pair (C, D) is g-separated. $\square$

Theorem 2.9. *Let (X, τ) be a topological space. If the union of a pair (A, B) of a g-separated sets is closed set, then B is τ^*-closed and A is closed.*

Proof. Let (A, B) be a pair of g-separated sets such that $A \cup B$ is closed. Then $A \cap Cl_g(B) = \emptyset = Cl(A) \cap B$. Since $A \cup B$ is closed, $A \cup B = Cl(A) \cup Cl(B)$. Now, $Cl(A) = Cl(A) \cap [Cl(A) \cup Cl(B)] = Cl(A) \cap [A \cup B] = [Cl(A) \cap A] \cup [Cl(A) \cap B] = A \cup \emptyset = A$ and so A is closed. Also, $B \subseteq A \cup B$ implies that $Cl_g(B) \subseteq Cl_g[A \cup B] \subseteq Cl[A \cup B] = A \cup B$ and so $Cl_g(B) = Cl_g(B) \cap [A \cup B] = [Cl_g(B) \cap A] \cup [Cl(B) \cap B] = \emptyset \cup B = B$. Hence B is τ^*-closed.

$\square$

Corollary 2.10. Let (X, τ) be a topological space. If a pair (A, B) is g-separated sets of X and $A \cup B \in \tau$, then B and A are τ^*-open and open respectively.

Lemma 2.11. *Let (X, τ) be a topological space and $B \subseteq Y \subseteq X$. Then $Cl_g^Y(B) = Cl_g(B) \cap Y$.*

Lemma 2.12. *Let (X, τ) be a topological space and $A, B \subseteq Y \subseteq X$. The following are equivalent:*

(1) A pair (A, B) is g-separated in Y;
(2) A pair (A, B) is g-separated in X.

Proof. It follows form Lemma 2.11 that $Cl_g^Y(A) \cap B = \emptyset = A \cap Cl^Y(B)$ if and only if $Cl_g(A) \cap B = \emptyset = A \cap Cl(B)$. $\square$

3. g-connected spaces

In this section, we give the properties of g-separated sets and g-connected sets.

Definition 3.1. A subset A of a topological space (X, τ) is said to be g-connected if A is not the union of a pair of g-separated sets in (X, τ).

Theorem 3.2. *A topological space (X, τ) is g-connected if and only X cannot be written as the disjoint union of a nonempty τ^*-open set and a nonempty open set.*

Proof. Suppose exist nonempty disjoint τ^*-open and open sets A, B in X such that $X = A \cup B$. Since A and B are disjoint, then $Cl(A) \cap B = \phi = A \cap Cl_g(B)$. This implies that the pair (A, B) is g-separated sets in X. Thus, X is not g-connected. This is a contradiction.
Conversely, suppose that X is not g-connected. There exist a pair (A, B) of g-separated sets such that $X = A \cup B$. By Theorem 2.10, A and B are τ^*-open and open in X, respectively. Since A and B are g-separated in X, then A and B are nonempty disjoint. This is a contradiction. $\square$

Every g-connected space is connected but the converse is not true as shown in the following example.

Example 3.3.
(1) Let X be an uncountable set with $\tau = \{\emptyset, X\}$. Then (X, τ) connected and g-connected, while (X, τ^*), being discrete topology then disconnected.
(2) Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$ and $\tau^* = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then (X, τ) connected and not g-connected, while (X, τ^*) is disconnected.
(3) Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a, b\}, X\}$ and $\tau^* = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then (X, τ) g-connected, also (X, τ^*) is connected.

Theorem 3.4. *Let (X, τ) be a topological space, X not g-connected and A be a subset of X such that (i) $A \neq \emptyset$, X and (ii) A is open and τ^*-closed in X. If Y is a nonempty g-connected subset of X, then either $Y \subseteq A$ or $Y \subseteq X \setminus A$.*

Proof. Let $X = A \cup B$, where $B = X \setminus A$. Then $Y = X \cap Y = [A \cup B] \cap Y = (A \cap Y) \cup (B \cap Y)$. Also $[A \cap Y] \cap Cl[B \cap Y] \subseteq A \cap Cl(B) = \emptyset$, since $A \cap B = \emptyset$ and A is open. This implies that $[A \cap Y] \cap Cl[B \cap Y] = \emptyset$. Similarly, $Cl_g[A \cap Y] \cap [B \cap Y] \subseteq Cl_g(A) \cap B = A \cap B = \emptyset$ implies that $Cl_g[A \cap Y] \cap [B \cap Y] = \emptyset$. It is given that Y is g-connected subspace of X. Hence it cannot happen that $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$, Since Y is a g-connected subspace of X, Y cannot admit any g-separation. Hence $A \cap Y = \emptyset$ or $B \cap Y = \phi$ implies that $Y \subseteq A$ or $Y \subseteq X - A$. $\square$

Theorem 3.5. *Let (X, τ) be a topological space. If A is a g-connected set of X and a pair (G, H) is g-separated sets of X with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.*

Proof. Let $A \subseteq H \cup G$. Since $A = [A \cap H] \cup [A \cap G]$, then $[A \cap H] \cap Cl_g[A \cap G] \subseteq H \cap Cl_g(G) = \emptyset$. By similar way, we have $[A \cap G] \cap Cl[A \cap H] \subseteq G \cap Cl(H) = \emptyset$. Then $A \cap H$ and $A \cap G$ are g-separated sets. Suppose that $A \cap H$ and $A \cap G$ are nonempty. Then A is not g-connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subseteq H$ or $A \subseteq G$.

□

Theorem 3.6. *Let A be an g-connected set of a topological space (X, τ). If $A \subseteq B \subseteq Cl_g(A)$, then B is g-connected.*

Proof. Suppose that B is not g-connected. There exist a pair (G, H) of g-separated such that $B = H \cup G$. This implies that H and G are nonempty and $G \cap Cl(H) = \emptyset = H \cap Cl_g(G)$. By Theorem 3.5, we have either $A \subseteq H$ or $A \subseteq G$. Suppose that $A \subseteq G$. Then $Cl_g(A) \subseteq Cl_g(G)$ and $H \cap Cl_g(A) = \emptyset$. This implies that $H \subseteq B \subseteq Cl_g(A)$ and $H = Cl_g(A) \cap H = \emptyset$. Thus H is an empty set. Since H is nonempty, this is a contradiction. Suppose that $A \subseteq H$. By similar way, it follows that G is empty. This is a contradiction. Hence, B is g-connected.

□

Corollary 3.7. Let (X, τ) be a topological space. If A is a g-connected set, then $Cl_g(A)$ is g-connected.

Theorem 3.8. *Let $\{N_i : i \in I\}$ is a nonempty family of g-connected sets of a topological space (X, τ). If $\cap_{i \in I} N_i \neq \emptyset$, then $\cup_{i \in I} N_i$ is g-connected.*

Proof. Suppose that $\cup_{i \in I} N_i$ is not g-connected. Then we have $\cup_{i \in I} N_i = H \cup G$, where H and G are g-separated sets in X. Since $\cap_{i \in I} N_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} N_i$. Since $x \in \cup_{i \in I} N_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in N_i$ for each $i \in I$, then N_i and H intersect for each $i \in I$. By Theorem 3.5 $N_i \subseteq H$ or $N_i \subseteq G$. Since H and G are disjoint, $N_i \subseteq H$ for all $i \in I$ and hence $\cup_{i \in I} N_i \subseteq H$. This implies that G is empty. This is a contradiction. Suppose $x \in G$. By similar way, we have that H is empty. This is a contradiction. Thus $\cup_{i \in I} N_i$ is g-connected. □

Corollary 3.9. Let (X, τ, m_X) be a mixed space and $\{A_\alpha : \alpha \in \Delta\}$ be a family of m-connected subsets of X, and A be an m-connected subset of X. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \Delta$, then $A \cup (\cup A_\alpha)$ is m-connected.

Proof. By Theorem 3.8, $A \cup A_\alpha$ is m-connected for each $\alpha \in \Delta$ and $\cap(A \cup A_\alpha) \supseteq A \neq \emptyset$. Therefore, by Theorem 3.8 $\cup(A \cup A_\alpha) = A \cup (\cup A_\alpha)$ is m-connected. □

Theorem 3.10. *Let (X, τ) be a topological space. Every continuous image of an g-connected space is connected.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function and X is g-connected space. If possible suppose that $f(X)$ is not a connected subset of Y. Then, there exists nonempty separated sets A and B such that $f(X) = A \cup B$. Since f is continuous and $A \cap Cl(B) = \emptyset = Cl(A) \cap B$, $Cl(f^{-1}(A)) \cap f^{-1}(B) \subseteq f^{-1}(Cl(A)) \cap f^{-1}(B) = f^{-1}[Cl(A) \cap B] = \emptyset$, $f^{-1}(A) \cap Cl_g(f^{-1}(B)) \subseteq f^{-1}(A) \cap Cl(f^{-1}(B)) \subseteq f^{-1}(A) \cap f^{-1}(Cl(B)) = f^{-1}[A \cap Cl(B)] = \emptyset$. Since A and B are nonempty, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ are g-separated and $X = f^{-1}(A) \cup f^{-1}(B)$. This is contrary to the assumption that X is g-connected. Therefore, $f(X)$ is connected. □

Theorem 3.11. *Let (X, τ) be a topological space and H a subset of X. If every pair of distinct points of H are elements of some g-connected subset of H, then H is an g-connected subset of X.*

Proof. Suppose H is not g-connected. Then, there exist a pair (B, A) of g-separated which nonempty subset of X such that $Cl(A) \cap B = \emptyset = A \cap Cl_g(B)$ and $H = A \cup B$. Since A and B are nonempty, there exists a point $a \in A$ and a point $b \in B$. By hypothesis, a and b must be elements of a g-connected subset C of H. Since $C \subseteq A \cup B$, by Theorem 3.5, either $C \subseteq A$ or $C \subseteq B$. Consequently, either a and b are both in A or both in B. Let $a, b \in A$. Then $A \cap B \neq \emptyset$. This is contrary to the fact that A and B are disjoint. Similarly, if we suppose that $a, b \in B$, then we have a contradiction. Therefore, H must be g-connected. □

Theorem 3.12. *Let (X, τ) be a topological space and X is g-connected. If A is a g-connected subset of X such that $X \setminus A$ is the union of two g-separated sets B and C, then $A \cup B$ and $A \cup C$ are g-connected.*

Proof. Suppose $A \cup B$ is not g-connected. Then there exist a pair of nonempty subsets $(H, G))$ of g-separated such that $A \cup B = G \cup H$. Since A is g-connected, $A \subseteq A \cup B = G \cup H$, by Theorem 3.5, either $A \subseteq G$ or $A \subseteq H$. Suppose $A \subseteq G$. Since $A \cup B = G \cup H$, $A \subseteq G$ implies that $A \cup B \subseteq G \cup B$ and so $G \cup H \subseteq G \cup B$. Hence $H \subseteq B$. Since B and C are g-separated, H and C are also g-separated. Thus, H is g-separated from G as well as C. Now, $Cl_g(H) \cap [G \cup C] = [Cl_g(H) \cap G] \cup [Cl_g(H) \cap C] = \emptyset$ and $H \cap Cl[G \cup C] = H \cap [Cl(G) \cup Cl(C)] = [H \cap Cl(G)] \cup [H \cap Cl(C)] = \emptyset$. Therefore, H is g-separated from $G \cup C$. Since $X \setminus A = B \cup C$, $X = A \cup [B \cup C] = [A \cup B] \cup C = [G \cup H] \cup C$, since $A \cup B = G \cup H$ and so $X = [G \cup C] \cup H$. Thus, X is union of two nonempty g-separated sets $G \cup C$ and H, which is a contradiction. Similar contradiction will arise if $A \subseteq H$. Hence, $A \cup B$ is g-connected. Similarly, we can prove that $A \cup C$ is g-connected. □

Theorem 3.13. *Let (X, τ) be a topological space If A and B are g-connected sets of X such that none of them is g-separated, then $A \cup B$ is g-connected.*

Proof. Let A and B be g-connected in X. Suppose $A \cup B$ is not g-connected. Then, there exist two nonempty g-separated sets G and H such that $A \cup B = G \cup H$. Since A and B are g-connected, by Theorem 3.5, either $A \subseteq G$ and $B \subseteq H$ or $B \subseteq G$ and $A \subseteq H$. Let $A \subseteq G$ and $B \subseteq H$. Then, since G and H are g-separated, by Proposition 2.8 A and B are g-separated. This is a contradiction. Similarly, let $B \subseteq G$ and $A \subseteq H$. Then B and A are g-separated. This is a contradiction. Hence $A \cup B$ is g-connected. $\square$

Definition 3.14. Let (X, τ) be a topological space and $x \in X$. The union of all g-connected subsets of X containing x is called the g-component of X containing x.

Lemma 3.15. *The g-component of each point x of a topological space X is the maximal g-connected set of X that contains x.*

Lemma 3.16. *The set of all distinct g-components of a topological space X forms a partition of X.*

Proof. Let A and B be two distinct g-components of X. Suppose A and B intersect. Then, by Theorem 3.8, $A \cup B$ is g-connected in X. Since $A \subseteq A \cup B$, then A is not maximal. Thus, A and B are disjoint. $\square$

Lemma 3.17. *Each g-component of a topological space X is a τ^*-closed.*

Proof. Let A be an g-component of X. By Corollary 3.7, $Cl_g(A)$ is g-connected and $A = Cl_g(A)$. Hence A is τ^*-closed in X. $\square$

Theorem 3.18. *Let (X, τ) be a topological space. Then each g-connected subset of X which both open and τ^*-closed is g-component of X.*

Proof. Let A be a g-connected subset of X which both open and τ^*-closed. Let $x \in A$. Since A is a g-connected subset of X containing x, if C is the g-component containing x, then $A \subseteq C$. Let A be a proper subset of C. Then C is nonempty and $C \cap (X \setminus A) \neq \emptyset$. Since A is open and τ^*-closed, $X \setminus A$ is closed and τ^*-open and $[A \cap C] \cap [(X \setminus A) \cap C] = \emptyset$. Also $[A \cap C] \cup [(X \setminus A) \cap C] = [A \cup (X \setminus A)] \cap C = C$. Again A and $X \setminus A$ are two nonempty disjoint open and τ^*-open set respectively, such that $A \cap Cl(X \setminus A) = \emptyset = Cl_g(A) \cap (X \setminus A)$. This implies $(A \cap C) \cap Cl[(X \setminus A) \cap C] = \emptyset = Cl_g(A \cap C) \cap [(X \setminus A) \cap C]$. This shows that A and $C \setminus A$ are g-separated sets. This is a contradiction. Hence, A is not a proper subset of C and $A = C$. Therefore, A is a g-component of X.

$\square$

Theorem 3.19. *Let (X, τ) be a topological space and $A \subseteq X$. If C is a g-connected subset of X that intersects both A and $X \setminus A$, then C intersects $Bd(A)$, the boundary of A.*

Proof. Suppose $C \cap Bd(A) = \emptyset$. Then $C \cap Cl(A) \cap Cl(X \setminus A) = \emptyset$. Now, $C = C \cap X = C \cap (A \cup (X \setminus A)) = (C \cap A) \cup (C \cap (X \setminus A))$. Also, $Cl_g(C \cap A) \cap (C \cap (X \setminus A)) \subseteq Cl_g(C) \cap Cl_g(A) \cap C \cap (X \setminus A) = C \cap Cl_g(A) \cap (X \setminus A) = \emptyset$. and $(C \cap A) \cap Cl(C \cap (X \setminus A)) \subseteq C \cap A \cap Cl(C) \cap Cl(X \setminus A) = C \cap Cl(X \setminus A) \cap A = \emptyset$. Thus, $C \cap A$ and $C \cap (X \setminus A)$ form an g-separation for C, which is a contradiction. Hence, $C \cap Bd(A) \neq \emptyset$. □

4. Cut-point of g-connected

Definition 4.1. Let X be a nonempty connected topological space. A point x in X is said to be a g-cut point of X if $X - \{x\}$ is not a g-connected subset of X. A nonempty g-connected topological space X is said to be a g-cut-point space if every x in X is a g-cut point of X.

Theorem 4.2. *Let X be a g-connected topological space, and let x be a g-cut point of it. If $X \setminus \{x\} = B \cup A$ such that a pair (B, A) B is g-separated, then $A \cup \{x\}$ is g-connected.*

Proof. If $A \cup \{x\}$ is not g-connected, then there is a pair (C, D) is g-separated subsets of X such that $A \cup \{x\} = C \cup D$. Without loss of generality, we may assume that $x \in C$. Then $D \subseteq A$. Since $Cl_g(B \cup C) \cap D = (Cl_g(B) \cap D) \cup (Cl_g(C) \cap D) = Cl_g(B) \cap D \subseteq Cl_g(B) \cap A = \emptyset$, $Cl_g(B \cup C) \cap D = \emptyset$. Since $(B \cup C) \cap Cl(D) = (B \cap Cl(D)) \cup (C \cap Cl(D)) = (B \cap Cl(D)) \subseteq B \cap Cl(A) = \emptyset$, $(B \cup C) \cap Cl(D) = \emptyset$. Therefore $X = (B \cup C) \cup D$. This contradicts the g-connectedness of X □

Theorem 4.3. *Let X be a g-connected topological space, and let x and y be two g-cut points of it such that $X \setminus \{x\} = B \cup A$ such that a pair (B, A) is g-separated and $X \setminus \{y\} = C \cup D$ such that the pair (C, D) is g-separated. If $x \in C$ and $y \in A$, then $D \subseteq A$ and $B \subseteq C$.*

Proof. Since $D \cup \{y\}$ is g-connected by Theorem 4.2, and since $D \cup \{y\} \subseteq X \setminus \{x\}$, we have $D \cup \{y\} \subseteq A$ or $D \cup \{y\} \subseteq B$ by Theorem 3.5. Since $y \in A$, the second inclusion is not true. Hence $D \subseteq A$. A similar argument shows that $B \subseteq C$. □

Theorem 4.4. *Let (X, τ) be a topological space if every x in X is a g-cut point of X. Let $x \in X$, and $X \setminus \{x\} = B \cup A$ such that the pair (B, A) is g-separated. If A is not g-connected, then for every $y \in Y = A \cup \{x\}$ is a g-cut-point point of Y.*

Proof. Clearly x is a g-cut point of Y. Let y be an arbitrary point in A. Since $X \setminus \{y\} = (Y \setminus \{y\}) \cup (B \cup \{x\})$ is not g-connected, and since $x \in (Y \setminus \{y\}) \cap (B \cup \{x\})$, either $Y \setminus \{y\}$ or $B \cup \{x\}$ is not g-connected. By Theorem 4.2, $B \cup \{x\}$ is g-connected. Thus $Y \setminus \{y\}$ is not g-connected.

□

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