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θ_{e^*}-OPEN SETS AND θ_{e^*}-CONTINUITY OF MAPS IN THE PRODUCT SPACE

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Abstract. In this paper, we introduce the concept of θ_{e^*}-open set defined using e^*-closure operator. We then investigate the relationship of this set to the other known concepts in topology such as the classical open, θ-open, θ_s-open and θ_e-open sets. We also characterize the concepts of θ_{e^*}-continuous function from an arbitrary topological space into the product space.

1. Introduction and Preliminaries

The notion of open sets lies at the heart of topology. Since then, open sets have been developing from many variations with unique characteristics. In 1963, Levine [16] was the first to initiate and study the concept of semi-open set, semi-closed set, and semi-continuity of a function.

A subset O of a topological space X is semi-open [16] if $O \subseteq Cl(Int(O))$. Equivalently, O is semi-open if there exists an open set G in X such that $G \subseteq O \subseteq Cl(G)$. A subset F of X is semi-closed if its complement $X \setminus F$ is semi-open in X. Let A be a subset of X. A point $p \in X$ is a semi-closure point of A if for every semi-open set G in X containing x, $G \cap A \neq \varnothing$. We denote by $sCl(A)$ the set of all semi-closure points of A.

From this beneficial study onwards, many mathematicians made several attempts in generating new types of open sets to broaden the topology and its purpose in the real physical world.

In 1968, Veličko [22] introduced the concept of θ-continuity between topological spaces and defined the concepts of θ-closure and θ-interior of a set. The study of Veličko was pursued by Dickman and Porter [7, 8], Joseph [14], and Long and Herrington [17]. Moreover, several authors have obtained interesting results related to these sets, see [1, 6, 12, 13, 15, 19].

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Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The closure of A and interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. The θ-closure of A denoted by $Cl_\theta(A)$, is defined as the set of points $x \in X$ such that $Cl(U) \cap A \neq \varnothing$ for every open set U containing x and the θ-interior of A denoted by $Int_\theta(A)$, is defined as the set of points $x \in X$ such that $Cl(U) \subseteq A$ for some open set U containing x. A is θ-closed if $Cl_\theta(A) = A$ and θ-open if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser than $\mathcal{T}$.

In 2009, Ekici, E. [10] instituted the idea of e^*-open set which is a generalization of e-open set, δ-semiopen sets and δ-preopen sets. Moreover, he introduced the notions of $(\mathcal{D}, \mathcal{S})$-sets, $(\mathcal{DS}, \mathcal{E})$-sets, $(\mathcal{D}, \mathcal{S})^*$-sets, $(\mathcal{DS}, \mathcal{E})^*$-sets, e^*-continuity, $(\mathcal{D}, \mathcal{S})^*$-continuity, $(\mathcal{DS}, \mathcal{E})^*$-continuity and obtained new decompositions of continuous functions through these concepts. Different versions of continuity related to e^*-open sets were studied in [3, 4, 20]

A subset A of a topological space X is said to be regular open [21] (resp., regular closed [21]) if $A = Int(Cl(A))$ (resp., $A = Cl(Int(A))$). A point $x \in X$ is said to be δ-cluster point of A if $A \cap V \neq \varnothing$ for every regular open set V containing x. The set of all δ-cluster points of A is called the δ-closure [22] of A and is denoted by $Cl_\delta(A)$. If $A = Cl_\delta(A)$, then A is called δ-closed and the complement of A, $X \setminus A$ is δ-open. The δ-interior [22] of A denoted by $Int_\delta(A)$, is the union of all δ-open sets contained in A. A subset A of X is called an e^*-open [10] set (resp., e-open [9] set) if $A \subseteq Cl(Int(Cl_\delta(A)))$ (resp., $A \subseteq Cl(Int_\delta(A)) \cup Int(Cl_\delta(A)))$. A subset F of X is e^*-closed [10] (resp., e-closed [9]) its complement $X \setminus F$ is e^*-open (resp., e-open). The e^*-closure [10]) (resp., e-closure [9]) of A denoted by e^*-$Cl(A)$ (resp., e-$Cl(A)$), is defined as the intersection of all e^*-closed (resp., e-closed) sets containing A and the e^*-interior [10] (resp., e-interior [9]) of A denoted by e^*-$Int(A)$ (resp., e^*-$Int(A)$), is defined as the union of all e^*-open (resp., e-open) sets contained in A.

Variations of open sets related to semi-open and e-open sets were also studied in [2, 11]. A subset A of a topological space X is said to be a θ_s-open [11] (resp., θ_e-open [2]) if for every $x \in A$, there exists an open set U containing x such that $sCl(U) \subseteq A$ (resp., e-$Cl(U) \subseteq A$). A subset F of X is called a θ_s-closed (resp., θ_e-closed) if its complement $X \setminus F$ is a θ_s-open (resp., θ_e-open). The θ_s-closure (resp., θ_e-closure) of A denoted by $Cl_{\theta_s}(A)$ (resp., $Cl_{\theta_e}(A)$), is defined as the intersection of all θ_s-closed (resp., θ_e-closed) sets containing A and the θ_s-interior (resp., θ_e-interior) of A denoted by $Int_{\theta_s}(A)$ (resp., $Int_{\theta_e}(A)$), is defined as the union of all θ_s-open (resp., θ_e-open) sets contained in A. A is θ_s-closed (resp., θ_e-closed) if and only if $Cl_{\theta_s}(A) = A$ (resp., $Cl_{\theta_e}(A) = A$) and θ_s-open (resp., θ_e-open) if and only if $Int_{\theta_s}(A) = A$ (resp., $Int_{\theta_e}(A) = A$). Let $\mathcal{T}_{\theta_s}$ (resp., $\mathcal{T}_{\theta_e}$) be the topology containing all θ_s-open (resp., θ_e-open) sets on X. It is known [2, p.135] that $\mathcal{T}_\theta \subsetneq \mathcal{T}_{\theta_s} \subsetneq \mathcal{T}_{\theta_e} \subsetneq \mathcal{T}$.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1}\rangle \cap \langle U_{\alpha_2}\rangle \cap \cdots \cap \langle U_{\alpha_n}\rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha\rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta\neq\alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k}\rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta\rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we introduce θ_{e^*}-open set, a new type of open set, defined via e^*-closure operator and investigate its connection with the classical open, θ-open, θ_s-open and θ_e-open sets.

2. θ_{e^*}-Open and θ_{e^*}-Closed Functions

This section presents the concepts of θ_{e^*}-open sets and θ_{e^*}-open and θ_{e^*}-closed functions.

Definition 2.1. Let X be a topological space. A subset A of X is said to be θ_{e^*}*-open* if for every $x \in A$, there exists an open set U containing x such that $e^*\text{-}Cl\,(U) \subseteq A$. A subset F of X is called a θ_{e^*}-closed if its complement $X \setminus F$ is a θ_{e^*}-open.

Since $A \subseteq e^*\text{-}Cl(A) \subseteq e\text{-}Cl(A) \subseteq Cl(A)$ (see [9, 18]) for every subset A of a topological space X, the following remark holds.

Remark 2.2. The following diagram holds for any subset of a topological space.

$$\begin{tikzcd}
\theta_s\text{-open} \arrow[d] & \theta\text{-open} \arrow[l] & & e^*\text{-open} \\
\theta_e\text{-open} \arrow[r] & \theta_{e^*}\text{-open} \arrow[r] & \text{open} \arrow[r] \arrow[d] & e\text{-open} \arrow[u] \\
 & & \text{semi-open} & \delta\text{-semi-open} \arrow[l] \arrow[u]
\end{tikzcd}$$

We remark that the above diagram is also true for their respective closed sets. The reverse implications for θ_{e^*}-open and θ_e-open sets and for open and θ_{e^*}-open sets are not true as shown in the following example. Counterexamples for the other reverse implications are found in [2, 9, 11, 18].

Example 2.3. Let $X = \{a, b, c, d, e\}$ with topology

$$\mathcal{T} = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, c, e\}\}.$$

Then $\{a, c, d\}$ and $\{a, b, c, d\}$ are θ_{e^*}-open but not θ_e-open. Also, the set $\{a, b, c, e\}$ is open but not θ_{e^*}-open.

The proof of the following theorem is analogous to the proof of Theorem 2.6 in [2].

Theorem 2.4. *Let $\mathcal{T}_{\theta_{e^*}}$ be the family of θ_{e^*}-open subsets of a topological space X. Then $\mathcal{T}_{\theta_{e^*}}$ forms a topology on X.*

In view of Remark 2.2, Example 2.3, and Theorem 2.4, the following remark holds.

Remark 2.5. $\mathcal{T}_{\theta_e} \subsetneq \mathcal{T}_{\theta_{e^*}} \subsetneq \mathcal{T}$.

Definition 2.6. Let X be a topological space and $A \subseteq X$.

(i) The θ_{e^*}*-interior* of A is denoted and defined by $Int_{\theta_{e^*}}(A)$=$\cup\{U : U$ is θ_{e^*}-open and $U \subseteq A\}$. We note that in view of Theorem 2.4, $Int_{\theta_{e^*}}(A)$ is the largest θ_{e^*}-open set contained in A. Moreover, $x \in Int_{\theta_{e^*}}(A)$ if and only if there exists a θ_{e^*}-open set U containing x such that $U \subseteq A$.

(ii) The θ_{e^*}*-closure* of A is denoted and defined by $Cl_{\theta_{e^*}}(A)$=$\cap\{F : F$ is θ_{e^*}-closed and $A \subseteq F\}$. Again, by Theorem 2.4, $Cl_{\theta_{e^*}}(A)$ is the smallest θ_{e^*}-closed set containing A. Moreover, $x \in Cl_{\theta_{e^*}}(A)$ if and only if for every θ_{e^*}-open set U containing x, $U \cap A \neq \varnothing$.

Remark 2.7. Let X be a topological space and $A, B \subseteq X$. Then the following statements hold:

(i) If $A \subseteq B$, then $Cl_{\theta_{e^*}}(A) \subseteq Cl_{\theta_{e^*}}(B)$.
(ii) A is θ_{e^*}-closed if and only if $A = Cl_{\theta_{e^*}}(A)$.
(iii) $Cl_{\theta_{e^*}}(Cl_{\theta_{e^*}}(A)) = Cl_{\theta_{e^*}}(A)$.
(iv) $Cl_{\theta_{e^*}}(A \cup B) = Cl_{\theta_{e^*}}(A) \cup Cl_{\theta_{e^*}}(B)$.
(v) If $A \subseteq B$, then $Int_{\theta_{e^*}}(A) \subseteq Int_{\theta_{e^*}}(B)$.
(vi) A is θ_{e^*}-open if and only if $A = Int_{\theta_{e^*}}(A)$.
(vii) $Int_{\theta_{e^*}}(A) = Int_{\theta_{e^*}}(Int_{\theta_{e^*}}(A))$.
(viii) $Int_{\theta_{e^*}}(A \cap B) = Int_{\theta_{e^*}}(A) \cap Int_{\theta_{e^*}}(B)$.
(ix) $Cl_{\theta_{e^*}}(X \setminus A) = X \setminus Int_{\theta_{e^*}}(A)$.
(x) $Int_{\theta_{e^*}}(X \setminus A) = X \setminus Cl_{\theta_{e^*}}(A)$.
(xi) $x \in Int_{\theta_{e^*}}(A)$ if and only if there exists an open set U containing x such that e^*-$Cl(U) \subseteq A$.
(xii) $x \in Cl_{\theta_{e^*}}(A)$ if and only if for each open set U containing x, e^*-$Cl(U) \cap A \neq \varnothing$.
(xiii) $A \subseteq Cl(A) \subseteq Cl_{\theta_{e^*}}(A) \subseteq Cl_{\theta_e}(A) \subseteq Cl_{\theta_s}(A) \subseteq Cl_\theta(A)$.
(xiv) If A is open, then $Cl(A) = Cl_{\theta_{e^*}}(A) = Cl_{\theta_e}(A) = Cl_{\theta_s}(A) = Cl_\theta(A)$.
(xv) $Int_\theta(A) \subseteq Int_{\theta_s}(A) \subseteq Int_{\theta_e}(A) \subseteq Int_{\theta_{e^*}}(A) \subseteq Int(A) \subseteq A$.
(xvi) If A is closed, then $Int_\theta(A) = Int_{\theta_s}(A) = Int_{\theta_e}(A) = Int_{\theta_{e^*}}(A) = Int(A)$.

Next, we characterize the concepts of θ_{e^*}-open and θ_{e^*}-closed functions.

Definition 2.8. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be θ_{e^*}-open (resp., θ_{e^*}-closed) on X if $f(G)$ is θ_{e^*}-open (resp., θ_{e^*}-closed) in Y for every open (resp., closed) set G in X.

Remark 2.9. Let X, Y, and Z be topological spaces. If $f : X \to Y$ is open on X and $g : Y \to Z$ is θ_{e^*}-open on Y, then the composition $g \circ f : X \to Z$ is θ_{e^*}-open on X.

We note that θ_{e^*}-open and θ_{e^*}-closed functions are equivalent if f is bijective.

The proofs of the following two results are standard [11, p.185], hence omitted.

Theorem 2.10. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_{e^*}-open on X.*

(ii) *$f(Int(A)) \subseteq Int_{\theta_{e^*}}(f(A))$ for every $A \subseteq X$.*
(iii) *$f(B)$ is θ_{e^*}-open for every basic open set B in X.*
(iv) *For each $x \in X$ and every open set U in X containing x, there exists an open set V in Y containing $f(x)$ such that e^*-$Cl(V) \subseteq f(U)$.*

Theorem 2.11. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*
(i) *f is θ_{e^*}-closed on X.*
(ii) *$Cl_{\theta_{e^*}}(f(G)) \subseteq f(Cl(G))$ for each $G \subseteq X$.*

Theorem 2.12. *Let X, Y, and Z be topological spaces. If $f : X \to Y$ is open on X, $g : Y \to Z$ is open on Y and Z is regular, then the composition $g \circ f : X \to Z$ is θ_{e^*}-open on X.*

Proof. Let $x \in X$ and U be open in X containing x. By assumption, there exists an open set V_Y in Y containing $f(x)$ such that $V_Y \subseteq f(U)$. Also, there exists an open set V_Z in Z containing $g(f(x)) = (g \circ f)(x)$ such that $V_Z \subseteq g(V_Y)$. Since Z is regular, there exists an open set W_Z in Z containing $(g \circ f)(x)$ such that $W_Z \subseteq Cl(W_Z) \subseteq V_Z$. From Remark 2.2, e^*-$Cl(W_Z) \subseteq Cl(W_Z) \subseteq V_Z \subseteq g(V_Y) \subseteq (g \circ f)(U)$. Thus, g is θ_{e^*}-open on X. □

Theorem 2.13. *Let X and Y be topological spaces and $\mathcal{T}_A$ be the subspace topology on $A \subseteq X$. If $f : X \to Y$ is θ_{e^*}-open and A is open in X, then $f\big|_A : A \to Y$ is θ_{e^*}-open on A.*

Proof. Let $x \in A$ and O be open in A containing x. Then $O = A \cap U$, where U is open in X. Since A is open in X, O is also open in X. By assumption, there exists an open set V in Y containing $f(x) = f\big|_A(x)$ such that e^*-$Cl(V) \subseteq f(O) = f\big|_A(O)$. In view of Theorem 2.10, $f\big|_A : A \to Y$ is θ_{e^*}-open on A. □

Theorem 2.14. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $\mathcal{T}_A$ be the subspace topology on $A \subseteq X$. If $X = B \cup C$ and $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a function such that $f\big|_B : (B, \mathcal{T}_B) \to (Y, \mathcal{T}_Y)$ and $f\big|_C : (C, \mathcal{T}_C) \to (Y, \mathcal{T}_Y)$ are θ_{e^*}-open, then $f : X \to Y$ is θ_{e^*}-open on X.*

Proof. Let $x \in X$ and $U \in \mathcal{T}_X$ containing x. Then $x \in B$ or $x \in C$. If $x \in B$, then $B \cap U \in \mathcal{T}_A$ containing x. By assumption, there exists $V \in \mathcal{T}_Y$ containing $f\big|_B(x) = f(x)$ such that e^*-$Cl(V) \subseteq f\big|_B(B \cap U) \subseteq f(U)$. This means that $f : X \to Y$ is θ_{e^*}-open on X. The same conclusion holds if $x \in C$. □

3. θ_{e^*}-Continuous Functions in the Product Space

This section presents the concept of θ_{e^*}-continuous functions from an arbitrary topological space into the product space.

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *θ_{e^*}-continuous* (resp., θ_e-continuous [2], θ_s-continuous [11], θ-continuous [22]) if $f^{-1}(G)$ is θ_{e^*}-open (resp., θ_e-open, θ_s-open, θ-open) in X for every open set G in Y.

Remark 3.2. Let X, Y, and Z be topological spaces. If $f : X \to Y$ is θ_{e^*}-continuous on X and $g : Y \to Z$ is continuous on Y, then the composition $g \circ f : X \to Z$ is θ_{e^*}-continuous on X.

Remark 3.3. The following diagram holds for a function $f : X \to Y$.

$$\begin{array}{ccccc}
\theta\text{-continuous} & & & & \text{continuous} \\
\downarrow & & & & \uparrow \\
\theta_s\text{-continuous} & \longrightarrow & \theta_e\text{-continuous} & \longrightarrow & \theta_{e^*}\text{-continuous}
\end{array}$$

The proofs of the following result is standard [5, p.83], hence omitted.

Theorem 3.4. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ_{e^*}-continuous on X.*
(ii) *$f^{-1}(F)$ is θ_{e^*}-closed in X for each closed subset F of Y.*
(iii) *$f^{-1}(B)$ is θ_{e^*}-open for each (subbasic) basic open set B in Y.*
(iv) *For every $x \in X$ and every open set V of Y containing $f(x)$, there exists a θ_{e^*}-open set U containing x such that $f(U) \subseteq V$.*
(v) *For every $x \in X$ and every open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(e^*\text{-}Cl(U)) \subseteq V$.*
(vi) *$f(Cl_{\theta_{e^*}}(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.*
(vii) *$Cl_{\theta_{e^*}}(f^{-1}((B)) \subseteq f^{-1}(Cl(B))$.*

Definition 3.5. Let X and Y be topological spaces. A bijective function $f : X \to Y$ is said to be a θ_{e^*}-*homeomorphism* if f and $f^{-1} : Y \to X$ are θ_{e^*}-continuous on X and Y, respectively.

Combining Theorems 2.10, 2.10, and 3.4, we have the following result.

Theorem 3.6. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is a θ_{e^*}-homeomorphism.*
(ii) *f is θ_{e^*}-continuous and θ_{e^*}-open on X.*
(iii) *f is θ_{e^*}-continuous and θ_{e^*}-closed on X.*
(iv) *$f(Cl_{\theta_{e^*}}(A)) = Cl(f(A)) = Cl_{\theta_{e^*}}(f(A)) = f(Cl(A))$ for each $A \subseteq X$.*
(v) *f is a homeomorphism.*

Theorem 3.7. *Let X and Y be topological spaces and $f : X \to Y$ be a function. If Y is a regular space, then f is θ-continuous on X if and only if f is continuous on X.*

Proof. From Remark 3.3, θ-continuity implies continuity. Suppose that f is continuous on X. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W containing $f(x)$ such that $W \subseteq Cl(W) \subseteq V$. By assumption, there exists an open set Z containing x such that $f(Z) \subseteq W$. Next, we show that $f(Cl(Z)) \subseteq Cl(W)$ Let $y \in f(Cl(Z))$. Then there exists $x \in Cl(Z)$ such that $f(x) = y$. Let G be an open set in Y containing y. Since f is continuous, $f^{-1}(G)$ is an open set in X containing x. Since $x \in Cl(Z)$, $f^{-1}(G) \cap Z \neq \varnothing$. It follows that $\varnothing \neq f\left(f^{-1}(G) \cap Z\right) = G \cap f(Z) \subseteq G \cap W$ so that $y \in Cl(W)$. Thus, $f(Cl(Z)) \subseteq Cl(W) \subseteq V$. Accordingly, f is θ-continuous on X. □

Corollary 3.8. Let X and Y be topological spaces and $f : X \to Y$ be a function. If Y is a regular space, then the following statements are equivalent:

(i) f is θ-continuous on X.

(ii) f is θ_s-continuous on X.
(iii) f is θ_e-continuous on X.
(iv) f is θ_{e^*}-continuous on X.
(v) f is continuous on X.

Theorem 3.9. *Let X and Y be topological spaces, $f : X \to Y$ be a function, and $g : X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is θ_{e^*}-continuous on X, then f is θ_{e^*}-continuous on X.*

Proof. Suppose that g is θ_{e^*}-continuous on X. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $X \times V$ is open in $X \times Y$ containing $g(x)$. By Theorem 3.4, there exists a θ_{e^*}-open set U containing x such that $g(U) \subseteq X \times V$. Since $g(x) = (x, f(x))$ for every $x \in X$, $f(U) \subseteq V$. Thus, f is θ_{e^*}-continuous on X. $\square$

Theorem 3.10. *Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of $A \subseteq X$, where $\mathcal{D}$ is the set $\{0, 1\}$ with discrete topology. Then f_A is θ_{e^*}-continuous if and only if A is both θ_{e^*}-open and θ_{e^*}-closed.*

In the following results, if $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \ldots, A_{\alpha_n}\rangle$, where $K = \{\alpha_1, \ldots, \alpha_n\}$.

If $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ is a finite product, denote $A_1 \times \cdots \times A_n$ by $\langle A_1, \ldots, A_n\rangle$.

The proof of the following result is similar with the proof of [2, Theorem 3.4].

Theorem 3.11. *Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is θ_{e^*}-continuous on X if and only if $p_\alpha \circ f$ is θ_{e^*}-continuous on X for every $\alpha \in \mathcal{A}$.*

Corollary 3.12. Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x)\rangle$. Then f is θ_{e^*}-continuous on X if and only if each f_α is θ_{e^*}-continuous on X for each $\alpha \in \mathcal{A}$.

Theorem 3.13. *Let $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ be a finite product space and $\varnothing \neq O_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $O = \langle O_1, \ldots, O_n\rangle$ is e^*-open if and only if each O_i is e^*-open.*

Proof. Suppose that $O = \langle O_1, \ldots, O_n\rangle$ is e^*-open. Then by [2, Corollary 3.8],

$$\begin{aligned} O &\subseteq Cl(Int(Cl_\delta(O))) \\ &= Cl(Int(\langle Cl_\delta(O_1), \ldots, Cl_\delta(O_n)\rangle)) \\ &= \langle Cl(Int(Cl_\delta(O_1))), \ldots, Cl(Int(Cl_\delta(O_n)))\rangle \end{aligned}$$

Thus, each O_i is e^*-open.

Conversely, assume that each O_i is e^*-open. Then for every $i = 1, 2, \ldots, n$, $O_i \subseteq Cl(Int(Cl_\delta(O_i)))$. Hence,

$$\begin{aligned} O &\subseteq \langle Cl(Int(Cl_\delta(O_1))), \ldots, Cl(Int(Cl_\delta(O_n)))\rangle \\ &= Cl(Int(Cl_\delta(O))). \end{aligned}$$

Thus, O is e^*-open. $\square$

Theorem 3.14. *Let $Y = \Pi\{Y_i : 1 \le i \le n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $e^*\text{-}Cl(\langle A_1, \ldots, A_n\rangle) \subseteq \langle e^*\text{-}Cl(A_1), \ldots, e^*\text{-}Cl(A_n)\rangle$.*

Proof. In view of [10, Theorem 2.3] and [2, Corollary 3.8] we have

$$\begin{aligned}
& e^*\text{-}Cl(\langle A_1, \ldots, A_n\rangle) \\
= \;& \langle A_1, \ldots, A_n\rangle \cup [Int(Cl(Int_\delta(\langle A_1, \ldots, A_n\rangle)))] \\
= \;& \langle A_1, \ldots, A_n\rangle \cup [Int(Cl(\langle Int_\delta(A_1), \ldots, Int_\delta(A_n)\rangle))] \\
= \;& \langle A_1, \ldots, A_n\rangle \cup \langle Int(Cl(Int_\delta(A_1)), \ldots, Int(Cl(Int_\delta(A_n))\rangle \\
\subseteq \;& \langle A_1 \cup Int(Cl(Int_\delta(A_1)), \ldots, A_n \cup Int(Cl(Int_\delta(A_n))\rangle\,,
\end{aligned}$$

thereby completing the proof. □

Theorem 3.15. *Let $Y = \Pi\{Y_i : 1 \le i \le n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then*

(i) *$Cl_{\theta_{e^*}}(\langle A_1, \ldots, A_n\rangle) \subseteq \langle Cl_{\theta_{e^*}}(A_1), \ldots, Cl_{\theta_{e^*}}(A_n)\rangle$ and*

(ii) *$\langle Int_{\theta_{e^*}}(A_1), \ldots, Int_{\theta_{e^*}}(A_n)\rangle \subseteq Int_{\theta_{e^*}}(\langle A_1, \ldots, A_n\rangle)$.*

Proof. (i) Let $x = \langle a_i\rangle \in Cl_{\theta_{e^*}}(\langle A_1, \ldots, A_n\rangle)$. Suppose that there exists an open set $O_j \ni a_j$ such that $e^*\text{-}Cl(O_j) \cap A_j = \varnothing$. Then $\langle O_1, \ldots, O_j, \ldots, O_n\rangle$ is open containing x and

$$\begin{aligned}
& e^*\text{-}Cl(\langle O_1, \ldots, O_j, \ldots, O_n\rangle) \cap \langle A_1, \ldots, A_j, \ldots, A_n\rangle \\
\subseteq \;& \langle e^*\text{-}Cl(O_1) \cap A_1, \ldots, e^*\text{-}Cl(O_j) \cap A_j, \ldots, e\text{-}Cl(O_n) \cap A_n\rangle \\
= \;& \varnothing,
\end{aligned}$$

a contradiction. Thus, $x \in \langle Cl_{\theta_{e^*}}(A_1), \ldots, Cl_{\theta_{e^*}}(A_n)\rangle$.

(ii) Let $x = \langle a_i\rangle \in \langle Int_{\theta_{e^*}}(A_1), \ldots, Int_{\theta_{e^*}}(A_n)\rangle$. Then $a_i \in Int_{\theta_{e^*}}(A_i)$ for all $i = 1, 2, \ldots, n$. This means that there exists $O_i \ni a_i$ such that $e^*\text{-}Cl(O_i) \subseteq A_i$. Then $\langle O_1, \ldots, O_n\rangle$ is open containing x and by Theorem 3.14,

$$e^*\text{-}Cl(\langle O_1, \ldots, O_n\rangle) \subseteq \langle e^*\text{-}Cl(O_1), \ldots, e\text{-}Cl(O_n)\rangle \subseteq \langle A_1, \ldots, A_n\rangle\,.$$

Hence, $x \in Int_{\theta_{e^*}}(\langle A_1, \ldots, A_n\rangle)$. □

Following the same argument as in Theorem 3.15 (ii), we have the following result.

Theorem 3.16. *Let $Y = \Pi\{Y_i : 1 \le i \le n\}$ be a finite product space and $\varnothing \ne O_i \subseteq Y_i$ for each $i = 1, 2, \ldots, n$. If each O_i is θ_{e^*}-open in Y_i, then $O = \langle O_1, \ldots, O_n\rangle$ is θ_{e^*}-open in Y.*

Theorem 3.17. *Let $X = \Pi\{X_i : 1 \le i \le n\}$ and $Y = \Pi\{Y_i : 1 \le i \le n\}$ be finite product spaces and for each $i = 1, 2, \ldots, n$, let $f_i : X_i \to Y_i$ be a function. If each f_i is θ_{e^*}-continuous on X_i, then the function $f : X \to Y$ defined by $f(\langle x_i\rangle) = \langle f_i(x_i)\rangle$ is θ_{e^*}-continuous on X.*

Proof. Let $\langle V_1, \ldots, V_n\rangle$ be a basic open set in Y. Then

$$f^{-1}(\langle V_1, \ldots, V_n\rangle) = \left\langle f_1^{-1}(V_1), \ldots, f_n^{-1}(V_n)\right\rangle.$$

Since each f_i is θ_{e^*}-continuous, $f_i^{-1}(V_i)$ is θ_{e^*}-open in X_i. Let $x = \langle x_i\rangle \in f^{-1}(\langle V_1, \ldots, V_n\rangle)$. Then $x_i \in f_i^{-1}(V_i)$ for all $i = 1, 2, \ldots, n$. This means that there exists an open set $O_i \ni x_i$ such that $e^*\text{-}Cl(O_i) \subseteq f_i^{-1}(V_i)$. Then $\langle O_1, \ldots, O_n \rangle$ is open in X containing x and by Theorem 3.14,

$$\begin{aligned} e^*\text{-}Cl(\langle O_1, \ldots, O_n \rangle) &\subseteq \langle e^*\text{-}Cl(O_1), \ldots, e\text{-}Cl(O_n) \rangle \\ &\subseteq \left\langle f_1^{-1}(V_1), \ldots, f_n^{-1}(V_n) \right\rangle \\ &= f^{-1}(\langle V_1, \ldots, V_n \rangle). \end{aligned}$$

This implies that $f^{-1}(\langle V_1, \ldots, V_n \rangle)$ is θ_{e^*}-open in X. Thus, f is θ_{e^*}-continuous on X. $\square$

4. θ_{e^*}-Connected Space and Versions of Separation Axioms

This section characterizes the concepts of θ_{e^*}-connected space and some versions of separation axioms.

Definition 4.1. A topological space X is said to be a θ_{e^*}-connected (resp., θ-connected, θ_s-connected, θ_e-connected, connected) if it is not the union of two nonempty disjoint θ_{e^*}-open (resp., θ-open, θ_s-open, θ_e-open, open) sets. Otherwise, X is θ_{e^*}-disconnected (resp., θ-disconnected, θ_s-disconnected, θ_e-disconnected, disconnected). A subset B of X is θ_{e^*}-connected (resp., θ-connected, θ_s-connected, θ_e-connected, connected) if it is θ_{e^*}-connected (resp., θ-connected, θ_s-connected, θ_e-connected, connected) as a subspace of X.

Theorem 4.2. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_{e^*}-connected.*
(ii) *The only subsets of X that are both θ_{e^*}-open and θ_{e^*}-closed are $\varnothing$ and X.*
(iii) *No θ_{e^*}-continuous function $f : X \to \mathcal{D}$ is surjective.*

By Remark 2.2 and the equivalence of connected and θ-connected spaces [12], the following result holds.

Theorem 4.3. *A topological space X is θ_{e^*}-connected if and only if X is connected.*

Remark 4.4. The following diagram holds for a subset of a topological space.

$$\begin{array}{ccccc} \theta\text{-connected} & & & & \text{connected} \\ \updownarrow & & & & \updownarrow \\ \theta_s\text{-connected} & \longleftrightarrow & \theta_e\text{-connected} & \longleftrightarrow & \theta_{e^*}\text{-connected} \end{array}$$

Definition 4.5. A topological space X is said to be

(i) *θ_{e^*}-Hausdorff* if given any pair of distinct points p, q in X there exist disjoint θ_{e^*}-open sets U and V such that $p \in U$ and $q \in V$;
(ii) *θ_{e^*}-regular* if for each closed set F and each point $x \notin F$, there exist disjoint θ_{e^*}-open sets U and V such that $x \in U$ and $F \subseteq V$;
(iii) *θ_{e^*}-normal* if for every pair of disjoint closed sets E and F of X, there exist disjoint θ_{e^*}-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

In view of Remark 2.2, every θ_{e^*}-Hausdorff (resp., θ_{e^*}-regular, θ_{e^*}-normal) space is Hausdorff (resp., regular, normal).

The proofs of the following three results are standard [2], hence omitted.

Theorem 4.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_{e^*}-Hausdorff.*
(ii) *Let $x \in X$. For $y \neq x$, there exists a θ_{e^*}-open set U containing x such that $y \notin Cl_{\theta_{e^*}}(U)$.*
(iii) *For each $x \in X$, $C = \cap\{Cl_{\theta_{e^*}}(U) : U$ is θ_{e^*}-open containing $x\}=\{x\}$.*

Theorem 4.7. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_{e^*}-regular.*
(ii) *For each $x \in X$ and an open set U containing x, there exists θ_{e^*}-open set V such that $x \in V \subseteq Cl_{\theta_{e^*}}(V) \subseteq U$.*
(iii) *For each $x \in X$ and closed set F with $x \notin F$, there exists a θ_{e^*}-open set V containing x such that $F \cap Cl_{\theta_{e^*}}(V) = \varnothing$.*

Theorem 4.8. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ_{e^*}-normal.*
(ii) *For each closed set A and an open set $U \supseteq A$, there exists a θ_{e^*}-open set V containing A such that $Cl_{\theta_{e^*}}(V) \subseteq U$.*
(iii) *For each pair of disjoint closed sets A and B, there exists a θ_{e^*}-open set V containing A such that $Cl_{\theta_{e^*}}(V) \cap B = \varnothing$.*

A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist an open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 4.9. *Let X be a T_1-space. Then the following statements hold:*

(i) *If X is θ_{e^*}-regular, then X is θ_{e^*}-Hausdorff.*
(ii) *If X is θ_{e^*}-normal, then X is θ_{e^*}-regular.*

Proof. (i): Suppose that X is θ_{e^*}-regular. Since X is a T_1-space, for each $x, y \in X$ with $x \neq y$, there exist open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$. This implies that $x \notin X \setminus U$ and $y \notin X \setminus V$. Since X is θ_{e^*}-regular, there exist disjoint θ_{e^*}-open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Note that $y \in X \setminus U$. Hence, $y \in B$. Thus, X is θ_{e^*}-Hausdorff.

We can prove (ii) by following the same argument used in (i). □

Next, we investigate some properties related to θ_{e^*}-regular spaces.

Theorem 4.10. *Let X and Y be topological spaces and $f : X \to Y$ be a function. If X is θ_{e^*}-regular, then f is continuous on X if and only if f is θ_{e^*}-continuous on X.*

Proof. From Remark 3.3, θ_{e^*}-continuity implies continuity. Suppose that f is continuous on X. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then there exists an open set U containing x such that $f(U) \subseteq V$. By Theorem 4.7, there exists a θ_{e^*}-open set W such that $x \in W \subseteq Cl_{\theta_{e^*}}(W) \subseteq U$. It follows that $f(W) \subseteq f(U) \subseteq V$. In view of Theorem 3.4, f is θ_{e^*}-continuous on X. □

Similarly, using [2, Theorem 4.7], the following result holds.

Theorem 4.11. *Let X and Y be topological spaces and $f : X \to Y$ be a function. If X is θ_e-regular, then f is θ_{e^*}-continuous on X if and only if f is θ_e-continuous on X.*

The following result states that the converse of Theorem 3.9 holds if X is θ_{e^*}-regular.

Theorem 4.12. *Let X and Y be topological spaces, $f : X \to Y$ be a function, and $g : X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If f is θ_{e^*}-continuous on X and X is θ_{e^*}-regular, then g is θ_{e^*}-continuous on X.*

Proof. Assume that f is θ_{e^*}-continuous on X. Let $x \in X$ and $U \times V$ be an open set in $X \times Y$ containing $g(x) = (x, f(x))$. Then $f(x) \in V$. By assumption, there exists a θ_{e^*}-open set W containing x such that $f(W) \subseteq V$. Since X is θ_{e^*}-regular, by Theorem 4.7, there exist a θ_{e^*}-open sets G and F such that $x \in G \subseteq Cl_{\theta_{e^*}}(G) \subseteq W$ and $x \in F \subseteq Cl_{\theta_{e^*}}(F) \subseteq U$. Let $O = G \cap F$, which is θ_{e^*}-open containing x. Moreover, $g(O) \subseteq U \times f(W) \subseteq U \times V$. Thus, g is θ_{e^*}-continuous on X. □

5. Conclusion and Recommendations

The paper has introduced the concept of θ_{e^*}-open set and described its connection to the other well-known concepts such as the classical open, θ-open, θ_s-open and θ_e-open sets. The paper has also defined and characterized the concepts of θ_{e^*}-open and θ_{e^*}-closed functions, θ_{e^*}-continuous function, θ_{e^*}-connected space, and some versions of separation axioms. A worthwhile direction for further investigation is to establish versions of Urysohn's Lemma and Tietze Extension Theorem with respect to θ_{e^*}-open sets.

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