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A NOTE ON OPERATOR VALUED FRAMES

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Abstract. In this paper, we prove that the Riesz wavelet type basis in $L^2(\mathbb{R}^d)$ is image of an orthonormal wavelet basis under a bounded invertible operator and prove that an orthonormal wavelet basis and an orthonormal wavelet type basis in $L^2(\mathbb{R}^d)$ are unitarily equivalent. Also, it is proved that the image of an orthonormal wavelet basis under unitary operator is an orthonormal wavelet type basis. Further, we prove that dual of Riesz wavelet basis always exists and is biorthogonal to it. Finally, we study dual wavelet frames for a given wavelet frame.

1. Introduction

Wavelet frames were studied considerably by many researchers [1, 2, 3, 4, 6, 8]. Below we give the formal definition of wavelet frame for $L^2(\mathbb{R}^d)$.

Definition 1.1. Let $d \in \mathbb{N}$, $\{\psi^1, \psi^2, ..., \psi^N\} \subseteq L^2(\mathbb{R}^d)$. Define

$$\psi^l_{j,\bar{k}}(\bar{x}) = D_{2^j} T_{\bar{k}} \psi^l(\bar{x}) = 2^{j/2}\psi^l(2^j\bar{x} - \bar{k}),$$

for $\bar{x} \in \mathbb{R}^d, \bar{k} \in \mathbb{Z}^d, j \in \mathbb{Z}$, $l = 1, 2, 3, ..., N$. The system $\{\psi^l_{j,\bar{k}}\}$ is called a *wavelet frame* for $L^2(\mathbb{R}^d)$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{l=1}^{N} \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} |\langle f, \psi^l_{j,\bar{k}}\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars A and B are called the *lower* and *upper wavelet frame bounds* of the wavelet frame, respectively. They are not unique. If $A = B$, then $\{x_n\}$ is called an *A-tight wavelet frame* and if $A = B = 1$, then $\{x_n\}$ is called a *Parseval wavelet frame*. The inequality in (1.1) is called the *wavelet frame inequality* of the wavelet frame. If $\{\psi^l_{j,\bar{k}}\}$ satisfies the upper frame inequality in (1.1), then $\{\psi^l_{j,\bar{k}}\}$ is called a Bessel wavelet sequence. The system defined in Definition 1.1 is called the homogeneous system and

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for such systems, N can be as small as one. There is another more fundamental system, called a nonhomogeneous system:

$$\{\phi(x-k) \mid k \in \mathbb{Z}^d\} \cup \{\psi^l_{j,k} \mid j \geq 0, k \in \mathbb{Z}^d, l = 1, \ldots, N\}.$$

For such systems, indeed one must have $N \geq 2^d - 1$.

Kaftal et al. [7] defined Operator Valued frames (OPV frames) in Hilbert spaces and studied various properties of OPV frames. More importantly, treating multi wavelet frames as OPV frames permits to parametrize them in an explicit and transparent ways. In [10], Poumai et al. gave the applications of OPV frames to the theory of quantum channel.

Definition 1.2. ([5, 7, 9]) Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathbb{J}$ be a countable or finite set. Let $F_j \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, for $j \in \mathbb{J}$. If there exist positive constants A and B such that

$$AI \leq \sum_{j \in \mathbb{J}} F_j^* F_j \leq BI, \tag{1.2}$$

then $\{F_j\}_{j\in\mathbb{J}}$ is called an *Operator Valued Frame (OPV-frame)* for $\mathcal{H}$ with range in $\mathcal{K}$. The constants A and B are called lower and upper OPV frame bounds. $\{F_j\}_{j\in\mathbb{J}}$ is *Parseval OPV-frame* for $\mathcal{H}$ with range in $\mathcal{K}$ if $A = B$.

Definition 1.3. ([10]) Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces. Let $k \in \mathcal{K}$ and define a map $|k\rangle_{\mathcal{K}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ as

$$|k\rangle_{\mathcal{K}}(h) = h \otimes k, \text{ for } h \in \mathcal{H}.$$

Also, define ${}_{\mathcal{K}}\langle k| : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H}$ as

$${}_{\mathcal{K}}\langle k|(h_1 \otimes k_1) = \langle k_1, k\rangle h_1, \text{ for } h_1 \in \mathcal{H},\ k_1 \in \mathcal{K}.$$

Remark 1.4. Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces and let $k \in \mathcal{K}$. Then, $|k\rangle^*_{\mathcal{K}} =_{\mathcal{K}} \langle k|$.

Let $\{F_j\}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and $\{e_j\}$ be an orthonormal basis of $\ell^2(\mathbb{J})$. The *analysis operator* $T_V : \mathcal{H} \to \mathcal{K} \otimes \ell^2(\mathbb{J})$ of the OPV-frame $\{F_j\}$ is given by

$$T_V(h) = \sum_{j\in\mathbb{J}} |e_j\rangle_{\ell^2(\mathbb{J})} F_j(h) = \sum_{j\in\mathbb{J}} F_j(h) \otimes e_j, \text{ for } h \in \mathcal{H}.$$

Also, the *synthesis operator* $T_V^* : \mathcal{K} \otimes \ell^2(\mathbb{J}) \to \mathcal{H}$ of the OPV-frame $\{F_j\}$ is given by

$$T_V^* = \sum_{j\in\mathbb{J}} F_j^* \ {}_{\ell^2(\mathbb{J})}\langle e_j|.$$

Note that for $k \otimes a \in \mathcal{K} \otimes \ell^2(\mathbb{J})$ $(a = \{a_n\} \in \ell^2(\mathbb{J}))$, we have

$$T_V^*(k \otimes a) = \sum_{j\in\mathbb{J}} a_j F_j^*(k).$$

Definition 1.5. ([10]) Let $\{F_j\}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. If $T_V(\mathcal{H}) = \mathcal{K} \otimes \ell^2(\mathbb{J})$, then $\{F_n\}$ is called a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. A Parseval Riesz OPV-frame is called an orthonormal OPV-frame.

Lemma 1.6 ([10])**.** *Let $\{F_j\}$ be an OPV-frame of $\mathcal{H}$ with range in $\mathcal{K}$ and S_V be its frame operator. Then* *(i)* $\{F_j\}$ *is a Riesz OPV-frame of* $\mathcal{H}$ *with range in* $\mathcal{K}$ *if and only if* $F_i S_V^{-1} F_j^* = \delta_{ij} I_{\mathcal{K}},\ i, j \in \mathbb{J}$.

(ii) $\{F_n\}$ *is an orthonormal OPV-frame of* $\mathcal{H}$ *with range in* $\mathcal{K}$ *if and only if* $F_i F_j^* = \delta_{ij} I_{\mathcal{K}},\ i, j \in \mathbb{J}$ *and* $\{F_j\}$ *is a parseval OPV-frame.*

Ron and Shen [12] described that for a given countable subset $X \subset L^2(\mathbb{R}^d)$, the synthesis operator $T_X : \ell^2(X) \to L^2(\mathbb{R}^d)$ is defined by $T_X(c) = \sum_{x \in X} c(x)x$, for $c \in X$ and the analysis operator $T_X^* : L^2(\mathbb{R}^d) \to \ell^2(X)$ is given by $T_X^*(f) = \{\langle f, x\rangle\}$, for $f \in L^2(\mathbb{R}^d)$. They also defined that a Bessel system X is a frame if range of T_X is closed and a Bessel system X is a Riesz basis if X is a frame and T_X is one-one.

Let $M_N = \{1, 2, 3, ..., N\}$ and $\{\delta_l\}_{l \in M_n}$ be the unit vector basis of $\ell^2(M_N)$. Also, let $\{e_j\}$ be standard basis of $\ell^2(\mathbb{Z})$. Let $\bar{e}_{\bar{k}}$ takes 1 at $\bar{k} \in \mathbb{Z}^d$ and 0 elsewhere. Then, $\{\bar{e}_{\bar{k}}\}$ is a standard basis for $\ell^2(\mathbb{Z}^d)$. Also, we have

$$\ell^2(\mathbb{Z}^d) = \{\sum_{\bar{k} \in \mathbb{Z}^d} c_{\bar{k}} \bar{e}_{\bar{k}} : \sum_{\bar{k} \in \mathbb{Z}^d} |c_{\bar{k}}|^2 < \infty\}, \text{ where } c_{\bar{k}} \text{ are scalars.}$$

Next result discusses the frame property of translates of the function ψ^l, for $l \in M_N$ in wavelet frames.

Lemma 1.7 ([11])**.** *Let* $\{\psi^l_{j,\bar{k}}\}$ *be a wavelet frame. Then* $\Phi_{lj} : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^d)$ *given by* $\Phi_{lj}(f) = \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}}$ *is a well defined operator. Moreover, the adjoint of* Φ_{lj} *is given by* $\Phi^*_{lj}(\{a_{\bar{k}}\}) = \sum_{\bar{k} \in \mathbb{Z}^d} a_{\bar{k}} \psi^l_{j,\bar{k}}$.

The following result gives the frame properties of translates and dilations of the function ψ^l, for $l \in M_N$ in wavelet frames.

Lemma 1.8 ([11])**.** *Let* $\{\psi^l_{j,\bar{k}}\}$ *be a wavelet frame. Then* $\Phi_l : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z})$ *given by* $\Phi_l(f) = \sum_{j \in \mathbb{Z}} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}} \otimes e_j$ *is well defined operator. Moreover, the adjoint of* Φ_l *is given by* $\Phi^*_l(\sum_{j \in \mathbb{Z}} \sum_{\bar{k} \in \mathbb{Z}^d} a_{j,\bar{k}} \bar{e}_{\bar{k}} \otimes e_j) = \sum_{j \in \mathbb{Z}} \sum_{\bar{k} \in \mathbb{Z}^d} a_{j,\bar{k}} \psi^l_{j,\bar{k}}$.

Now, let us define the synthesis and analysis operators of a given wavelet frame.
The operator $\Phi : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(M_N)$ given by

$$\Phi(f) = \sum_{l=1}^{N} \sum_{j \in \mathbb{Z}} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l,$$

is called the *Analysis Operator* of wavelet frame $\{\psi^l_{j,\bar{k}}\}$.

The operator $\Phi^* : \ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(M_N) \to L^2(\mathbb{R}^d)$ given by

$$\Phi^*(\sum_{l=1}^{N} \sum_{j \in \mathbb{Z}} \sum_{\bar{k} \in \mathbb{Z}^d} a^l_{j,\bar{k}} \bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l) = \sum_{l=1}^{N} \sum_{j \in \mathbb{Z}} \sum_{\bar{k} \in \mathbb{Z}^d} a^l_{j,\bar{k}} \psi^l_{j,\bar{k}},$$

is called the *Synthesis Operator* of wavelet frame $\{\psi^l_{j,\bar{k}}\}$.
Using the synthesis and analysis operators of wavelet frames, let us define the frame operator of wavelet frames.

Note that $\Phi^*(\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l) = \psi^l_{j,\bar{k}}$. Thus, for $f \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\Phi^*\Phi(f) &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \psi^l_{j,\bar{k}}\rangle \Phi^*(\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l)\\ &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \psi^l_{j,\bar{k}}\rangle \psi^l_{j,\bar{k}}.\end{aligned}$$

The operator $S_W = \Phi^*\Phi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given by

$$S_W(f) = \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \psi^l_{j,\bar{k}}\rangle \psi^l_{j,\bar{k}}, f \in L^2(\mathbb{R}^d)$$

is called the *Frame Operator* of the wavelet frame $\{\psi^l_{j,\bar{k}}\}$. The wavelet frame operator S_W is a bounded, positive, self-adjoint and invertible. The *reconstruction formula* is given by

$$f = S_W S_W^{-1} f = \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, S_W^{-1}\psi^l_{j,\bar{k}}\rangle \psi^l_{j,\bar{k}}, \text{ for all } f \in L^2(\mathbb{R}^d).$$

Next, we define orthonormal wavelet basis and Riesz wavelet basis in $L^2(\mathbb{R}^d)$.

Definition 1.9. $\{\psi^l_{j,\bar{k}}\}$ is an orthonormal wavelet basis for $L^2(\mathbb{R}^d)$ if it is a Parseval wavelet frame and $\langle \psi^{l_1}_{j_1,\bar{k}_1}, \psi^{l_2}_{j_2,\bar{k}_2}\rangle = \delta_{j_1 j_2}\delta_{\bar{k}_1\bar{k}_2}\delta_{l_1 l_2}$.

Definition 1.10. A wavelet frame $\{\psi^l_{j,\bar{k}}\}$ is a Riesz wavelet basis for $L^2(\mathbb{R}^d)$ if $\langle \psi^{l_1}_{j_1,\bar{k}_1}, S_W^{-1}\psi^{l_2}_{j_2,\bar{k}_2}\rangle = \delta_{j_1 j_2}\delta_{\bar{k}_1\bar{k}_2}\delta_{l_1 l_2}$, where S_W^{-1} is the frame operator of the wavelet frame $\{\psi^l_{j,\bar{k}}\}$.

The following result discusses the interplay between OPV frames and wavelet frames.

Theorem 1.11 ([11]). *Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet Bessel sequence for $L^2(\mathbb{R}^d)$. Then*

(a) $\{\psi^l_{j,\bar{k}}\}$ is a wavelet frame for $L^2(\mathbb{R}^d)$ with bounds A and B if and only if $\{\Phi_l\}_{l\in M_N}$ is an OPV frame for $L^2(\mathbb{R}^d)$ with range in $\ell^2(\mathbb{Z}^d)\otimes\ell^2(\mathbb{Z})$ with same bounds.

(b) $\{\psi^l_{j,\bar{k}}\}$ is a Parseval wavelet frame for $L^2(\mathbb{R}^d)$ if and only if $\{\Phi_l\}_{l\in M_N}$ is a parseval OPV frame for $L^2(\mathbb{R}^d)$ with range in $\ell^2(\mathbb{Z}^d)\otimes\ell^2(\mathbb{Z})$.

(c) $\{\psi^l_{j,\bar{k}}\}$ is a Riesz wavelet basis for $L^2(\mathbb{R}^d)$ if and only if $\{\Phi_l\}_{l\in M_N}$ is a Riesz OPV frame for $L^2(\mathbb{R}^d)$ with range in $\ell^2(\mathbb{Z}^d)\otimes\ell^2(\mathbb{Z})$.

(d) $\{\psi^l_{j,\bar{k}}\}$ is an orthonornal wavelet frame for $L^2(\mathbb{R}^d)$ if and only if $\{\Phi_l\}_{l\in M_N}$ is a orthonormal OPV frame for $L^2(\mathbb{R}^d)$ with range in $\ell^2(\mathbb{Z}^d)\otimes\ell^2(\mathbb{Z})$.

2. Main Results

In the following result, we prove the canonical dual property of a given wavelet frame.

Theorem 2.1. *Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet frame for $L^2(\mathbb{R}^d)$ with bounds A and B. Then, $\{S_W^{-1}\psi^l_{j,\bar{k}}\}$ is also a wavelet frame for $L^2(\mathbb{R}^d)$ with bounds B^{-1} and A^{-1}.*

Proof. Let $\Gamma_l = \Phi_l S_W^{-1}$. Then

$$\Gamma_l(f) = \Phi_l S_W^{-1}(f) = \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, S_W^{-1}\psi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}} \otimes e_j, f \in L^2(\mathbb{R}^d).$$

Also, $\Gamma_l^* = S_W^{-1}\Phi_l^*$. So, $\Gamma_l^*(\bar{e}_{\bar{k}} \otimes e_j) = S_W^{-1}\psi^l_{j,\bar{k}}$. Thus, we obtain

$$\Gamma_l^*\Gamma_l(f) = \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, S_W^{-1}\psi^l_{j,\bar{k}}\rangle S_W^{-1}\psi^l_{j,\bar{k}}, f \in L^2(\mathbb{R}^d).$$

Also, for $f \in L^2(\mathbb{R}^d)$, we have

$$\sum_{l=1}^N \Gamma_l^*\Gamma_l(f) = \sum_{l=1}^N S_W^{-1}\Phi_l^*\Phi_l(S_W^{-1}f) = S_W^{-1}\sum_{l=1}^N \Phi_l^*\Phi_l(S_W^{-1}f) = S_W^{-1}S_W S_W^{-1}(f).$$

Therefore, $S_W^{-1}(f) = \sum\limits_{l=1}^N \Gamma_l^*\Gamma_l(f)$, for $f \in L^2(\mathbb{R}^d)$. But, $B^{-1}I \leq S_W^{-1} \leq A^{-1}I$. Thus

$$B^{-1}\|f\|^2 \leq \langle \sum_{l=1}^N \Gamma_l^*\Gamma_l(f), f\rangle \leq A^{-1}\|f\|^2, f \in L^2(\mathbb{R}^d).$$

Hence, we conclude that

$$B^{-1}\|f\|^2 \leq \sum_{l=1}^N \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} |\langle f, S_W^{-1}\psi^l_{j,\bar{k}}\rangle|^2 \leq A^{-1}\|f\|^2, f \in L^2(\mathbb{R}^d).$$

□

Next, we show that Riesz wavelet type basis in $L^2(\mathbb{R}^d)$ is image of an orthonormal wavelet basis in $L^2(\mathbb{R}^d)$ under a bounded invertible operator.

Theorem 2.2. *Let $\{\psi^l_{j,\bar{k}}\}$ be a orthonormal wavelet basis for $L^2(\mathbb{R}^d)$ and $V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a bounded invertible operator. Then $\{V\psi^l_{j,\bar{k}}\}$ is a Riesz wavelet type basis.*

Proof. Let $\Gamma_l = \Phi_l V^*$. Then, $\Gamma_l^* = V\Phi_l^*$. So that

$$\Gamma_l^*(\bar{e}_{\bar{k}} \otimes e_j) = V\Phi_l^*(\bar{e}_{\bar{k}} \otimes e_j) = V(\psi^l_{j,\bar{k}}).$$

Note that

$$\sum_{l=1}^N \Gamma_l^*\Gamma_l(f) = \sum_{l=1}^N V\Phi_l^*\Phi_l(V^*f) = V\sum_{l=1}^N \Phi_l^*\Phi_l(V^*f) = VV^*(f), f \in L^2(\mathbb{R}^d).$$

Since V is bounded invertible operator, VV^* is a bounded invertible operator. Thus, $\{\Gamma_l\}$ is an OPV frame and $S_U = VV^*$ is the OPV frame operator of $\{\Gamma_l\}$. Notice that

$$\Gamma_l(f) = \Phi_l V^*(f) = \sum_{j\in\mathbb{Z}} \sum_{\bar{k}} \langle f, V(\psi^l_{j,\bar{k}})\rangle \bar{e}_{\bar{k}} \otimes e_j, f \in L^2(\mathbb{R}^d).$$

Therefore, by Theorem 1.11, $\{V(\psi^l_{j,\bar{k}})\}$ is a wavelet type frame for $L^2(\mathbb{R}^d)$. Since $\{\psi^l_{j,\bar{k}}\}$ is an orthonormal wavelet basis, $\{\Phi_l\}$ is an orthonormal OPV frame. Also, we have

$$\Gamma_{l_1} S_U^{-1}\Gamma_{l_2}^* = \Phi_{l_1} V^*(VV^*)^{-1}\Phi_{l_2}^* = \Phi_{l_1}\Phi_{l_2} = \delta_{l_1 l_2} I.$$

Thus, $\{\Gamma_l\}$ is a Riesz OPV frame. Hence, by Theorem 1.11, $\{V(\psi^l_{j,\bar{k}})\}$ is a Riesz wavelet type basis for $L^2(\mathbb{R}^d)$. □

Towards the converse of Theorem 2.2, we have the following result.

Theorem 2.3. *Let $\{\psi^l_{j,\bar{k}}\}$ be a Riesz wavelet type basis for $L^2(\mathbb{R}^d)$. Then, there exists a bounded invertible operator $V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and an orthonormal wavelet basis $\{\phi^l_{j,\bar{k}}\}$ such that $\psi^l_{j,\bar{k}} = V(\phi^l_{j,\bar{k}})$.*

Proof. Let $\Gamma_l = \Phi_l S_W^{-1/2}$, where S_W is the wavelet frame operator of $\{\psi^l_{j,\bar{k}}\}$. Then

$$\Gamma_l(f) = \sum_{j\in\mathbb{Z}}\sum_{\bar{k}}\langle f, S_W^{-1/2}(\psi^l_{j,\bar{k}})\rangle \bar{e}_{\bar{k}} \otimes e_j, f \in L^2(\mathbb{R}^d).$$

Also, we have

$$\begin{aligned}\sum_{l=1}^{N}\Gamma_l^*\Gamma_l(f) &= \sum_{l=1}^{N} S_W^{-1/2}\Phi_l^*\Phi_l S_W^{-1/2}(f)\\ &= S_W^{-1/2}\sum_{l=1}^{N}\Phi_l^*\Phi_l S_W^{-1/2}(f)\\ &= S_W^{-1/2} S_W S_W^{-1/2}(f)\\ &= f, f\in L^2(\mathbb{R}^d).\end{aligned}$$

So, $\{\Gamma_l\}$ is a Parseval OPV frame for $L^2(\mathbb{R}^d)$. Now, let $\phi^l_{j,\bar{k}} = S_W^{-1/2}(\psi^l_{j,\bar{k}})$. Then, by Theorem 1.11, $\{\phi^l_{j,\bar{k}}\}$ is a Parseval wavelet frame for $L^2(\mathbb{R}^d)$. Moreover

$$\Gamma_{l_1}\Gamma_{l_2}^* = \Phi_{l_1} S_W^{-1/2} S_W^{-1/2}\Phi_{l_2}^* = \Phi_{l_1} S_W^{-1}\Phi_{l_2}^* = \delta_{l_1 l_2} I.$$

Thus, $\{\Gamma_l\}$ is a Parseval OPV frame. Hence, by Theorem 1.11, $\{\phi^l_{j,\bar{k}}\}$ is an orthonormal wavelet basis for $L^2(\mathbb{R}^d)$ and $Sw^{1/2}(\phi^l_{j,\bar{k}}) = \psi^l_{j,\bar{k}}$. □

In the following result, we prove that an orthonormal wavelet basis and an orthonormal wavelet type basis in $L^2(\mathbb{R}^d)$ are unitarily equivalent. More precisely, we prove the following result:

Theorem 2.4. *Let $\{\psi^l_{j,\bar{k}}\}$ be an orthonormal wavelet type basis for $L^2(\mathbb{R}^d)$ and $\{\phi^l_{j,\bar{k}}\}$ be an orthonormal wavelet basis for $L^2(\mathbb{R}^d)$. Then, there exists a unitary operator $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $\psi^l_{j,\bar{k}} = U(\phi^l_{j,\bar{k}})$.*

Proof. Let $\Phi_l(f) = \sum_{j\in\mathbb{Z}}\sum_{\bar{k}}\langle f, \psi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}}\otimes e_j$ and $\Gamma_l(f) = \sum_{j\in\mathbb{Z}\in\mathbb{Z}^d}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \phi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}}\otimes e_j$, for $f \in L^2(\mathbb{R}^d)$. By Theorem 1.11, we know that $\{\Phi_l\}$ and $\{\Gamma_l\}$ are orthonormal OPV frames for $L^2(\mathbb{R}^d)$. Let Φ and Γ be analysis operators of $\{\Phi_l\}$ and $\{\Gamma_l\}$. So, we have $\Phi^*\Phi = \Phi\Phi^* = I = \Gamma^*\Gamma = \Gamma\Gamma^*$. Also, we know that $\Phi_l = {}_{\ell^2(M_N)}\langle\delta_l|\Phi$ and $\Gamma_l = {}_{\ell^2(M_N)}\langle\delta_l|\Gamma$. Take $U = \Phi^*\Gamma$. Then

$$\Phi_l(f) = {}_{\ell^2(M_N)}\langle\delta_l|\Phi = {}_{\ell^2(M_N)}\langle\delta_l|\Phi\Phi^*\Phi = {}_{\ell^2(M_N)}\langle\delta_l|\Gamma\Gamma^*\Phi = \Gamma_l U^*.$$

Also, $U^*U = \Gamma^*\Phi\Phi^*\Gamma = I$ and $UU^* = \Phi^*\Gamma\Gamma^*\Phi = I$. Thus, U is unitary operator on $L^2(\mathbb{R}^d)$. Moreover, $\Phi^* = U\Gamma^*$ and therefore

$$\psi^l_{j,\bar{k}} = \Phi_l^*(\bar{e}_{\bar{k}}\otimes e_j) = U\Gamma^*(\bar{e}_{\bar{k}}\otimes e_j) = U(\phi^l_{j,\bar{k}}).$$

□

Next, we show that the image of an orthonormal wavelet basis under unitary operator is an orthonormal wavelet type basis.

Theorem 2.5. *Let $\{\psi^l_{j,\bar{k}}\}$ be an orthonormal wavelet basis for $L^2(\mathbb{R}^d)$ and $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be unitary operator. Then, $\{U(\psi^l_{j,\bar{k}}\})$ is an orthonormal wavelet type basis.*

Proof. Let $\phi^l_{j,\bar{k}} = U(\psi^l_{j,\bar{k}})$ and $f \in L^2(\mathbb{R}^d)$. Then

$$\begin{aligned}\sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f,\phi^l_{j,\bar{k}}\rangle\phi^l_{j,\bar{k}\in\mathbb{Z}^d} &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f,U(\psi^l_{j,\bar{k}})\rangle U(\psi^l_{j,\bar{k}})\\ &= U\sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle U^*f,\psi^l_{j,\bar{k}}\rangle\psi^l_{j,\bar{k}}\\ &= UU^*(f) = f.\end{aligned}$$

Thus, $\{\phi^l_{j,\bar{k}}\}$ is a Parseval wavelet type frame for $L^2(\mathbb{R}^d)$. Moreover, we have

$$\langle\phi^{l_1}_{j_1,\bar{k_1}},\phi^{l_2}_{j_2,\bar{k_2}}\rangle = \langle U\psi^{l_1}_{j_1,\bar{k_1}},U\psi^{l_2}_{j_2,\bar{k_2}}\rangle = \langle U^*U\psi^{l_1}_{j_1,\bar{k_1}},\psi^{l_2}_{j_2,\bar{k_2}}\rangle = \delta_{\bar{k_1}\bar{k_2}}\delta_{j_1j_2}\delta_{l_1l_2}.$$

Hence, $\{\phi^l_{j,\bar{k}}\}$ is an orthonormal wavelet type basis for $L^2(\mathbb{R}^d)$. □

Next, we prove that the image of Riesz wavelet basis under unitary operator is a Riesz wavelet type basis.

Theorem 2.6. *Let $\{\psi^l_{j,\bar{k}}\}$ be a Riesz wavelet basis for $L^2(\mathbb{R}^d)$ and $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a bounded invertible linear operator. Then, $\{T(\psi^l_{j,\bar{k}})\}$ is a Riesz wavelet type basis.*

Proof. Let $\Gamma_l = \Phi_l T^*$. Then

$$\Gamma_l(f) = \Phi_l T^*(f) = \sum_{j\in\mathbb{Z}}\sum_{\bar{k}}\langle f, T\psi^l_{j,\bar{k}}\rangle\bar{e}_{\bar{k}}\otimes e_j,\, f\in L^2(\mathbb{R}^d).$$

Also, for $f \in L^2(\mathbb{R}^d)$, we have

$$\sum_{l=1}^{N}\Gamma_l^*\Gamma_l(f) = \sum_{l=1}^{N}T\Phi_l^*\Phi_l T^*(f) = T\sum_{l=1}^{N}\Phi_l^*\Phi_l T^*(f) = T\Phi^*\Phi T^*(f).$$

Let $S_V = T\Phi^*\Phi T^*$. Then, one may observe that S_V is a bounded invertible operator. Moreover, $\{\Gamma_l\}$ is an OPV frame for $L^2(\mathbb{R}^d)$ with OPV frame operator S_V. Thus, $\{T(\psi^l_{j,\bar{k}})\}$ is a wavelet type frame for $L^2(\mathbb{R}^d)$. Also, we have

$$\Gamma_{l_1}S_V^{-1}\Gamma_{l_2}^* = \Phi_{l_1}T^*(T^*)^{-1}\Phi^{-1}(\Phi^*)^{-1}T^{-1}T\Phi_{l_2}^* = \Phi_{l_1}S_W^{-1}\Phi_{l_2}^* = \delta_{l_1l_2}.$$

Therefore, $\{\Gamma_l\}$ is a Riesz OPV frame for $L^2(\mathbb{R}^d)$. Hence, by Theorem 1.11, $\{T(\psi^l_{j,\bar{k}})\}$ is a Riesz wavelet type basis for $L^2(\mathbb{R}^d)$. □

In the following result, it is proved that the Riesz wavelet basis in $L^2(\mathbb{R}^d)$ and the Riesz wavelet type basis in $L^2(\mathbb{R}^d)$ are equivalent.

Theorem 2.7. *Let $\{\psi^l_{j,\bar{k}}\}$ and $\{\phi^l_{j,\bar{k}}\}$ be the Riesz wavelet type basis and Riesz wavelet basis for $L^2(\mathbb{R}^d)$, respectively. Then, there exists a bounded invertible linear operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $\psi^l_{j,\bar{k}} = T(\phi^l_{j,\bar{k}})$.*

Proof. By Theorem 2.3, there exist an orthonormal wavelet basis $\{\theta^l_{j,\bar{k}}\}$ and bounded invertible operators T_ψ, T_ϕ such that $\psi^l_{j,\bar{k}} = T_\psi(\theta^l_{j,\bar{k}})$ and $\phi^l_{j,\bar{k}} = T_\phi(\theta^l_{j,\bar{k}})$. So

$$\psi^l_{j,\bar{k}} = T_\psi(\theta^l_{j,\bar{k}}) = T_\psi T_\phi^{-1} T_\phi(\theta^l_{j,\bar{k}}) = T_\psi T_\phi^{-1}(\phi^l_{j,\bar{k}}).$$

Take $T = T_\psi T_\phi^{-1}$. Then T is bounded invertible operator such that $\psi^l_{j,\bar{k}} = T(\phi^l_{j,\bar{k}})$. □

Next, we prove that wavelet type frames are the images of Parseval wavelet frames under a bounded invertible operator.

Theorem 2.8. *Let $\{\psi^l_{j,\bar{k}}\}$ be a Parseval wavelet frame for $L^2(\mathbb{R}^d)$ and $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a bounded invertible operator. Then, $\{T(\psi^l_{j,\bar{k}})\}$ is a wavelet type frame for $L^2(\mathbb{R}^d)$.*

Proof. Write $\Gamma_l = \Phi_l T^*$. Then

$$\Gamma_l(f) = \Phi_l T^*(f) = \sum_{j\in\mathbb{Z}} \sum_{\bar{k}} \langle f, T\psi^l_{j,\bar{k}}\rangle \bar{e}_{\bar{k}} \otimes e_j, f \in L^2(\mathbb{R}^d).$$

Also, we have

$$\sum_{l=1}^{N} \Gamma_l^* \Gamma_l(f) = T \sum_{l=1}^{N} \Phi_l^* \Phi_l(T^* f) = TT^*(f), f \in L^2(\mathbb{R}^d).$$

Write $S_T = TT^*$. Then, S_T is a bounded invertible operator. Therefore, $\{\Gamma_l\}$ is an OPV frame for $L^2(\mathbb{R}^d)$. Hence, by Theorem 1.11, $\{T(\psi^l_{j,\bar{k}})\}$ is a wavelet type frame for $L^2(\mathbb{R}^d)$. □

Definition 2.9. Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet frame for $L^2(\mathbb{R}^d)$. A wavelet frame $\{\phi^l_{j,\bar{k}}\}$ is called dual frame of $\{\psi^l_{j,\bar{k}}\}$ if

$$f = \sum_{l=1}^{N} \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, \psi^l_{j,\bar{k}}\rangle \phi^l_{j,\bar{k}} \text{ for all } f \in L^2(\mathbb{R}^d) \text{ or}$$

$$f = \sum_{l=1}^{N} \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, \phi^l_{j,\bar{k}}\rangle \psi^l_{j,\bar{k}} \text{ for all } f \in L^2(\mathbb{R}^d).$$

Remark 2.10. Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet frame for $L^2(\mathbb{R}^d)$. Then, $\{S_W^{-1}(\psi^l_{j,\bar{k}})\}$ is a dual frame of $\{\psi^l_{j,\bar{k}}\}$.

Next, we show that dual of a Riesz wavelet basis always exists and is biorthogonal to it.

Theorem 2.11. *Let $\{\psi^l_{j,\bar{k}}\}$ be a Riesz wavelet basis for $L^2(\mathbb{R}^d)$. Then there exists another Riesz wavelet basis $\{\phi^l_{j,\bar{k}}\}$ for $L^2(\mathbb{R}^d)$ such that*

$$f = \sum_{l=1}^{N} \sum_{j\in\mathbb{Z}} \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, \phi^l_{j,\bar{k}}\rangle \psi^l_{j,\bar{k}} \textit{ for all } f \in L^2(\mathbb{R}^d)$$

and $\{\psi^l_{j,\bar{k}}\}$, $\{\phi^l_{j,\bar{k}}\}$ are biorthogonal to each other.

Proof. Since $\{\psi^l_{j,\bar{k}}\}$ is a Riesz wavelet basis for $L^2(\mathbb{R}^d)$, by Theorem 2.3, there exist a bounded invertible operator $T_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and orthonormal wavelet basis $\{\theta^l_{j,\bar{k}}\}$ such that $\psi^l_{j,\bar{k}} = T_\psi(\theta^l_{j,\bar{k}})$. Let $\phi^l_{j,\bar{k}} = (T_\psi^{-1})^*\theta^l_{j,\bar{k}}$. By Theorem 2.2, it is clear that $\{\phi^l_{j,\bar{k}}\}$ be a wavelet Riesz basis for $L^2(\mathbb{R}^d)$. Moreover, for $f \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned} f = T_\psi T_\psi^{-1}(f) &= T_\psi \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle T_\psi^{-1}f, \theta^l_{j,\bar{k}}\rangle\theta^l_{j,\bar{k}} \\ &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, (T_\psi^{-1})^*\theta^l_{j,\bar{k}}\rangle T_\psi(\theta^l_{j,\bar{k}}) \\ &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \phi^l_{j,\bar{k}}\rangle\psi^l_{j,\bar{k}}. \end{aligned}$$

Moreover

$$\langle\phi^{l_1}_{j_1,\bar{k_1}}, \psi^{l_2}_{j_2,\bar{k_2}}\rangle = \langle(T_\psi^{-1})^*\theta^{l_1}_{j_1,\bar{k_1}}, T_\psi(\theta^{l_2}_{j_2,\bar{k_2}})\rangle = \langle\theta^{l_1}_{j_1,\bar{k_1}}, \theta^{l_2}_{j_2,\bar{k_2}}\rangle = \delta_{\bar{k_1}\bar{k_2}}\delta_{j_1j_2}\delta_{l_1l_2}.$$

□

In the following result, we give a result related to the existence of dual wavelet frames for a given wavelet frames.

Theorem 2.12. *Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet frame for $L^2(\mathbb{R}^d)$, Φ_W be the analysis operator of $\{\psi^l_{j,\bar{k}}\}$ and $\{\phi^1, \phi^2, ..., \phi^N\} \subseteq L^2(\mathbb{R}^d)$. The dual wavelet frames for $\{\psi^l_{j,\bar{k}}\}$ are preciously the families $\{\Gamma^*(\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l\}$, where $\Gamma : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^d) \otimes \ell(\mathbb{Z}) \otimes \ell^2(M_N)$ is a bounded operator, $\Gamma^*(\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l) = D_{2^j}T_{\bar{k}}\phi^l$ and $\Gamma^*\Phi = I$.*

Proof. Let $\phi^l_{j,\bar{k}} = \Gamma^*(\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l$. Since $\Gamma^*\Phi = I$, Γ^* is surjective. Thus, by Theorem 2.4, $\{\phi^l_{j,\bar{k}}\}$ is a wavelet frame for $L^2(\mathbb{R}^d)$. Moreover, we have

$$\begin{aligned} f = \Gamma^*\Phi(f) &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \psi^l_{j,\bar{k}}\rangle\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l \\ &= \sum_{l=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\bar{k}\in\mathbb{Z}^d}\langle f, \psi^l_{j,\bar{k}}\rangle\phi^l_{j,\bar{k}}. \end{aligned}$$

□

Lemma 2.13. *Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet frame for $L^2(\mathbb{R}^d)$. Let Φ be analysis operator of $\{\psi^l_{j,\bar{k}}\}$. More precisely, the left inverse of Φ are of the form $S_W^{-1}\Phi^* + \Gamma(I - \Phi S_W^{-1}\Phi^*)$, where $\Gamma : \ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(M_N) \to L^2(\mathbb{R}^d)$ is a bounded operator.*

Finally, we give the following characterization of all dual wavelet frames associated to a given wavelet frame.

Theorem 2.14. *Let $\{\psi^l_{j,\bar{k}}\}$ be a wavelet frame for $L^2(\mathbb{R}^d)$. The dual wavelet frames are precisely the families $\{S_W^{-1}\psi^l_{j,\bar{k}} + \phi^l_{j,\bar{k}} - \sum_{l_1=1}^{N}\sum_{j_1\in\mathbb{Z}}\sum_{\bar{k}_1\in\mathbb{Z}^d}\langle\psi^l_{j,\bar{k}}, S_W^{-1}\psi^{l_1}_{j_1,\bar{k}_1}\rangle\phi^{l_1}_{j_1,\bar{k}_1}\}$, where $\{\phi^l_{j,\bar{k}}\}$ is a wavelet Bessel sequence.*

Proof. Let Γ be the analysis operator of the wavelet Bessel sequence $\{\phi^l_{j,\bar{k}}\}$. Then, $\Gamma^*(\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l) = \phi^l_{j,\bar{k}}$. Moreover, the left inverse of Φ are $S_W^{-1}\Phi^* + \Gamma^* - \Gamma^*\Phi S_W^{-1}\Phi^*$.

So, we have

$$
\begin{aligned}
&(S_W^{-1}\Phi^* + \Gamma^* - \Gamma^*\Phi S_W^{-1}\Phi^*)\bar{e}_{\bar{k}} \otimes e_j \otimes \delta_l \\
&\quad = S_W^{-1}\psi^l_{j,\bar{k}} + \phi^l_{j,\bar{k}} - \Gamma^*\Phi(S_W^{-1}\psi^l_{j,\bar{k}}) \\
&\quad = S_W^{-1}\psi^l_{j,\bar{k}} + \phi^l_{j,\bar{k}} - \Gamma^*\Big(\sum_{l_1=1}^{N}\sum_{j_1\in\mathbb{Z}}\sum_{\bar{k}_1\in\mathbb{Z}^d}\langle\psi^l_{j,\bar{k}}, S_W^{-1}\psi^{l_1}_{j_1,\bar{k}_1}\rangle\bar{e}_{\bar{k}_1} \otimes e_{j_1} \otimes \delta_{l_1}\Big) \\
&\quad = S_W^{-1}\psi^l_{j,\bar{k}} + \phi^l_{j,\bar{k}} - \sum_{l_1=1}^{N}\sum_{j_1\in\mathbb{Z}}\sum_{\bar{k}_1\in\mathbb{Z}^d}\langle\psi^l_{j,\bar{k}}, S_W^{-1}\psi^{l_1}_{j_1,\bar{k}_1}\rangle\phi^{l_1}_{j_1,\bar{k}_1}.
\end{aligned}
$$

Hence, $\{S_W^{-1}\psi^l_{j,\bar{k}} + \phi^l_{j,\bar{k}} - \sum_{l_1=1}^{N}\sum_{j_1\in\mathbb{Z}}\sum_{\bar{k}_1\in\mathbb{Z}^d}\langle\psi^l_{j,\bar{k}}, S_W^{-1}\psi^{l_1}_{j_1,\bar{k}_1}\rangle\phi^{l_1}_{j_1,\bar{k}_1}\}$ is a dual wavelet frame of $\{\psi^l_{j,\bar{k}}\}$. □

References

[1] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[2] I. Daubechies, B. Han, A. Ron, and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal., 14(2003), 1-46.

[3] M. Frazier, G. Garrigos, K. Wang and G. Weiss, A characterization of functions that generate wavelet and related expansions, J. Fourier Anal. Appl., 3(1997), 883-906.

[4] B. Han, On dual wavelet tight frames, App. Comput. Harmon. Anal., 4(1997), 380-413.

[5] D. Han, P. Li, B. Meng and W. Tang, Operator valued frames and structured quantum channels, Sci. China Math., 54(2011), no. 11, 2361-2372.

[6] C. Heil, *A basis theory primer*, Birkhäuser, Boston, 2011.

[7] V. Kaftal, D. Larson, S. Zhang, Operator-valued frames, Trans. Amer. Math. Soc., 361(2009), 6349-6385.

[8] S. Li, A Theory of Generalized Multiresolution Structure and Pseudoframes of Translates, J. Fourier Anal. Appl.,7(2001), 441-464.

[9] B. Meng, Operator-valued frame generators for group-like unitary systems, Oper. Matrices., 7(2013), 441-464.

[10] K. T. Poumai, S. K. Kaushik and S. V. Djordjević, Operator valued frames and applications to quantum channels, International Conference on Sampling Theory and Applications (SampTA)., IEEE (2017), 217-221.

[11] K. T. Poumai and S. K. Kaushik, Wavelet frames in $L^2(\mathbb{R}^d)$, Rocky Mountain J. Math., 50 (2020), no. 2, 677-692.

[12] A. Ron and Z. Shen, Affine Systems in $L^2(\mathbb{R}^d)$: The Analysis of the Analysis Operator, J. Func. Anal., 148(1997), 408-447.

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