Poincare Journal of Analysis & Applications
Vol. 9, No. 1 (2022), 11-20

THE SPECTRAL AND BOUNDEDNESS RADII DEFINING AN EXTREMAL TOPOLOGY

HUGO ARIZMENDI-PEIMBERT AND ANGEL CARRILLO-HOYO[†]

Date of Receiving : 16. 10. 2021
Date of Acceptance : 03. 05. 2022

Abstract. It is characterized when the spectral radius in a Hausdorff locally A-convex algebra is an m-convex norm that defines the weakest m-convex topology stronger than the original one. The same is done for the boundedness radius on A-normed algebras.

1. Introduction

Throughout this paper X will be a complex associative algebra. It is called almost commutative if X is commutative modulo its Jacobson radical. When X is endowed with a topology τ we shall write (X, τ). The linear topology generated by a family $\mathcal{P}$ of seminorms on X will be denoted by $\sigma(\mathcal{P})$. Then $(X, \sigma(\mathcal{P}))$ is a locally convex linear space. When $\mathcal{P}$ consists of only one seminorm $\|\cdot\|$ we simply write $(X, \|\cdot\|)$.

The algebra X with a linear topology τ for which multiplication is separately continuous (respectively, jointly continuous) is called a *semitopological algebra* (respectively, *topological algebra*).

A *locally convex algebra* is a semitopological algebra whose topology is defined by a family of seminorms

The concepts of absorbing seminorm and locally absorbing convex algebra were introduced in [3]. They are called in short form A-convex seminorm and A-convex algebra, respectively. The m-convex seminorms and m-convex algebras are special cases of these concepts. Their definitions of all of them are recalled in the next section.

Every A-convex algebra is a locally convex algebra. While every m-convex algebra is a locally convex algebra with jointly continuous multiplication and therefore, a topological algebra.

When (X, τ) is an A-convex algebra, then there always exists an m-convex topology on X stronger than τ. This fact was proved, for X unital, by M. Oudadess in [12] using

---

2010 *Mathematics Subject Classification.* 46H05 .
Key words and phrases. Locally m-convex algebras, locally A-convex algebras, extremal topology, Oudadess topology, spectral radius, radius of boundedness.
Communicated by. Abdellah El Kinani
[†]*Corresponding author*

the operator topology $Op(\mathcal{P})$, where $\mathcal{P}$ is any family of A-convex seminorms such that $\sigma(\mathcal{P}) = \tau$. L. Oubbi generalizes this to nonunital algebras in [9], using the topology $\mathcal{Q}(\mathcal{P})$. The definitions of these two topologies are also recalled in the next section. When X is unital, then $\mathcal{Q}(\mathcal{P}) = Op(\mathcal{P})$.

We use $\mathcal{Q}(\mathcal{P})$ instead of $\mathcal{Q}(\tau)$ because it could happen that $\mathcal{Q}(\mathcal{P}) \neq \mathcal{Q}(\mathcal{P}')$ even for two systems of A-convex seminorms satisfying $\sigma(\mathcal{P}) = \tau = \sigma(\mathcal{P}')$ ([10, p. 458, Proposition 3.3]). The same can be said about the operator topology.

When for an A-convex algebra (X, τ) there exists the *weakest m-convex topology stronger than τ*, we shall call it the *Oudadess topology* on X and it will be denoted by $\ell(\tau)$. This is the extremal topology that we refer to in the title of this work.

The first two examples that disproved the general existence of $\ell(\tau)$ were given by L. Oubbi in [10] and [11]. The A-convex algebras considered there are nonunital, but for its unitizations, with the usual topologies τ_1 determined by τ, the topologies $\ell(\tau_1)$ neither exist.

It is an open problem to find necessary and sufficient conditions on the existence of the Oudadess topology. In [2], we gave two sufficient conditions for the existence of $\ell(\tau)$ in any commutative A-convex algebra. As application we recovered there (Corollary 1, p.73) the result of L. Oubbi [11, Proposition 2.3, p.323] about the existence of $\ell(\tau)$ in any A-convex algebra contained in a weighted space of continuous functions, in the sense of Nachbin. We also presented in that work an example due to V. Müller (personal communication) of a unital A-normed algebra (X, τ) in which the topology $\ell(\tau)$ exists and it is different from any topology $Op(\mathcal{P})$, with $\tau = \sigma(\mathcal{P})$.

A. El Kinani showed in Theorem 1 of [5] that under certain conditions the radius of boundedness in a unital locally A-convex algebra is exactly the least upper bound of "the associated locally m-convex topology" (this is the operator topology) whenever the latter is square preserving.

Here we split this theorem in Propositions 3.7 and 3.8 without assuming that the algebra is unital. In the second one we put a weaker hypothesis than one of those imposed in the theorem (see Remark 3.9). With the aid of this last proposition and using the notion of spectral algebra we establish necessary and sufficient conditions for the spectral radius in a locally A-convex algebra to be an m-convex seminorm defining the Oudadess topology. The same is done for the radius of boundedness in the frame of A-normed algebras.

We also see that the existence of $\ell(\tau)$ is ensured by the existence of this type of extreme topology in each one of the quotients $X/\|\cdot\|_i$, for $i \in I$, when the A-convex seminorms $\|\cdot\|_i$ induced the topology τ on X.

2. Preliminaries

A seminorm $\|\cdot\|$ defined on X is called an *absorbing seminorm* (in short form *A-convex*) if for each $x \in X$ there exists $M(x) > 0$, depending on $\|\cdot\|$ and x, such that

$$\max(\|xy\|, \|yx\|) \leq M(x)\|y\| \text{ for every } y \in X. \tag{2.1}$$

In particular,

$$\|xy\| = \|yx\| = 0 \text{ if } \|y\| = 0 \text{ or } \|x\| = 0. \tag{2.2}$$

Thus, $\ker\|\cdot\|$ is a bilateral ideal of X. If also $\|x\| = 0$ implies $x = 0$, then $\|\cdot\|$ is called an *A-convex norm.*

If (2.1) holds with $M_x = \|x\|$ then it is said that the seminorm $\|\cdot\|$ is *submultiplicative, m-convex* or *algebra seminorm.* If also $\|x\| = 0$ implies $x = 0$, then $\|\cdot\|$ is called an *m-convex norm* or *algebra norm.*

A linear locally convex topology τ on X is referred to as *A-convex* (resp. *m-convex*) if it is defined by some family $\mathcal{P}$ of A-convex (resp. m-convex) seminorms on X.

An *A -convex*(resp. *m-convex*) *algebra* is a locally convex algebra whose topology is *A-convex* (resp. *m-convex*). For every A-convex seminorm (resp. A-convex norm) $\|\cdot\|$ on X the locally convex algebra $(X, \|\cdot\|)$ is called an *A-seminormed algebra* (resp. *A-normed algebra*). If $\|\cdot\|$ is an algebra seminorm (resp. algebra norm), then we say as usual that $(X, \|\cdot\|)$ is a *seminormed algebra* (resp. *normed algebra*).

Given a family $\mathcal{P} = \{\|\cdot\|_i : i \in I\}$ of seminorms on X we denote by $\mathcal{P}_s$ the saturated family of seminorms obtained from $\mathcal{P}$. That is, $\mathcal{P}_s = \left\{\max\limits_{i \in I'} \|\cdot\|_i : \emptyset \neq I' \subset I \text{ is finite}\right\}$. For a family $\mathcal{P}$ of A-convex or m-convex seminorms the saturated family $\mathcal{P}_s$ is of the same type than $\mathcal{P}$.

From now on when we suppose that $(X, \sigma(\mathcal{P}))$ is an A-convex algebra, the family $\mathcal{P}$ will be assumed to be formed by A-convex seminorms.

Let $\|\cdot\|$ be an A-convex seminorm in X. The *operator seminorm* $\|\cdot\|^{op}$ corresponding to $\|\cdot\|$ is defined on X by $\|x\|^{op} = \sup\limits_{\|y\| \leq 1} \|xy\|$.

The seminorm $\|\cdot\|^{op}$ satisfies that $\|xy\| \leq \|x\|^{op} \|y\|$ for all $x, y \in X$. Thus, $\|\cdot\|^{op}$ is an m-convex seminorm and $\|x^n\| \leq (\|x\|^{op})^{n-1} \|x\|$ for every $x \in X$ and $n \geq 1$. If X is unital then $\|x\| \leq \|x\|^{op} \|e\|$ for all $x \in X$ and $\|\cdot\|^{op}$ is stronger than $\|\cdot\|$.

L. Oubbi defines in [9] the functional

$$q_{\|\cdot\|}(x) = \max(\|x\|, \|x\|^{op}) \tag{2.3}$$

for any A-convex seminorm $\|\cdot\|$ on X. This functional $q_{\|\cdot\|}$ is an m-convex seminorm stronger than $\|\cdot\|$ and $\|x\|^{op}$. When X is unital, the inequality

$$q_{\|\cdot\|}(x) \leq \max(\|e\|, 1) \|x\|^{op} \tag{2.4}$$

holds for all $x \in X$.

Let X be an A-convex algebra whose topology τ is given by a family $\mathcal{P} = \{\|\cdot\|_i : i \in I\}$ of A-convex seminorms. The *operator topology* $Op(\mathcal{P})$, used by Oudadess in [12], is the one induced by the family $\mathcal{P}_{op} = \{\|\cdot\|_i^{op} : i \in I\}$ of m-convex seminorms. Oubbi defined $\mathcal{Q}(\mathcal{P})$ as the m-convex topology generated by the family $\mathcal{Q}_{\mathcal{P}}$ of the m-convex seminorms $q_i(\cdot) = \max(\|\cdot\|_i, \|\cdot\|_i^{op})$ for $i \in I$. The topology $\mathcal{Q}(\mathcal{P})$ is stronger than τ and the operator topology $Op(\mathcal{P})$.

Therefore, for any such algebra X and family $\mathcal{P}$ the topology $\mathcal{Q}(\mathcal{P})$ is m-convex and stronger than topologies τ and $Op(\mathcal{P})$. If X is unital then $\mathcal{Q}(\mathcal{P}) = Op(\mathcal{P})$ since in this case we have that $\mathcal{Q}(\mathcal{P}) \subset Op(\mathcal{P})$ by inequality (2.4).

We consider for $x, y \in X$ the *circle product or quasi-product* defined as $x \circ y = x + y - xy$. An element x is said to be quasi-invertible in X if $x \circ y = y \circ x = 0$ for some $y \in A$. In this case y is called the *quasi-inverse* of x and is denoted by x^q.

A semitopological algebra X is a *Q-algebra* if the set $G_q(X)$ of all its quasi-invertible elements is open or, equivalently, if 0 has a neighborhood contained in $G_q(X)$.

For $x \in X$ the set $\Lambda(x) = \left\{\lambda \in \mathbb{C} : \frac{1}{\lambda}x \notin G_q(X)\right\}$ is the *spectrum* $Sp(x)$ of x if X is unital and x is invertible in X; otherwise, $Sp(x) = \Lambda(x) \bigcup \{0\}$. The *spectral radius of*

x is $\rho(x) = \sup\{|\lambda| : \lambda \in Sp(x)\}$, with $\sup \emptyset = -\infty$. If convenient we shall write $\rho_X(x)$ instead of simply $\rho(x)$.

In [1], G. R. Allan introduced the following concepts for a locally convex algebra (X, τ). An element $x \in X$ is called *bounded* if the set $\{(\lambda x)^n : n = 0, 1, ...\}$ is bounded on X for some non-zero complex number λ. Equivalently, if $(\lambda x)^n$ converges to 0 for some non-zero complex number λ.

The *radius of boundedness* or *boundedness radius* $\beta(x)$ of x is defined as

$$\beta(x) = \inf\left\{\lambda > 0 : \left\{\left(\lambda^{-1}x\right)^n\right\}_{n\geq 1} \text{ is bounded}\right\}.$$

Equivalently

$$\beta(x) = \inf\left\{\lambda > 0 : \left(\lambda^{-1}x\right)^n \to 0\right\},$$

with $\inf \emptyset = \infty$.

It is proved there that

$$\beta(x) = \sup_{\|\cdot\|\in\mathcal{P}} \limsup \|x^n\|^{\frac{1}{n}}. \tag{2.5}$$

where $\mathcal{P}$ is any basic set of seminorms which define the topology of X.

If convenient we shall write $\beta_\tau(x)$ instead of simply $\beta(x)$. Also we could write $\beta_{\|\cdot\|}(x)$ for the boundedness radius of x respect to the topology induced for some seminorm $\|\cdot\|$ on X.

Proposition 2.1. Every element in an A-convex algebra $(X, \sigma(\mathcal{P}))$ is bounded.

Proof. For $x \in X$ and $\|\cdot\| \in P$ we have $\|\lambda^n x^n\| \leq |\lambda|^{n-1} (\|x\|^{op})^{n-1} \|\lambda x\|$ $\forall \lambda \in \mathbb{C}$. Taking $\lambda \neq 0$ such that $|\lambda| \|x\|^{op} < 1$, $\|\lambda^n x^n\| \to 0$. □

Proposition 2.2. Let (X, τ) be a commutative A-convex algebra. Suppose there exists an m-convex topology τ_m on X such that

$$\begin{aligned}&\{x \in X : \{x^n : n \geq 1\} \text{ is } \tau\text{-bounded}\} \\ &= \{x \in X : \{x^n : n \geq 1\} \text{ is } \tau_m\text{-bounded}\}.\end{aligned} \tag{2.6}$$

Then the boundedness radii respect to the two topologies τ and τ_m coincide defining an m-convex seminorm.

Proof. By the equality (2.6), $\beta_\tau = \beta_{\tau_m}$. Since every element in X is τ-bounded (Proposition 2.1) the radius of boundedness β_τ respect to τ is finite valued on X. Thus, $C = \{x \in X : \{x^n : n \geq 1\} \text{ is } \tau_m\text{-bounded}\}$ is an absorbing set.

On the other hand, (X, τ_m) is a commutative locally convex algebra with continuous multiplication. Then C is an idempotent disk (see proof of Lemma II.9 in [6, p. 236]). Since β_{τ_m} is the Minkowski functional of C, it is an m-convex seminorm. □

T. W. Palmer [13, p. 188] defined a *spectral seminorm* on X as an algebra seminorm $\|\cdot\|$ satisfying $\rho(x) \leq \|x\|$ for all $x \in X$. Equivalently, $\rho(x) = \lim \|x^n\|^{\frac{1}{n}}$. When X admits a spectral seminorm, then it is called a *spectral algebra*.

Let X be a locally convex algebra. A net $(x_i)_{i\in I}$ in X is called *advertible convergent* respect to $x \in X$ if $xx_i \to 0$ and $x_i x \to 0$ in X. The algebra X is *advertibly complete* if every Cauchy net advertible convergent respect to x converges in X (necessarily to the quasi- inverse of x). Every Q-algebra is advertibly complete [7, Theorem 6.5, p. 71].

A seminorm $\|\cdot\|$ on X is called *square preserving* if $\|x^2\| = \|x\|^2$ for all $x \in X$. Any square preserving seminorm is m-convex ([4, Theorem, p.52]). It is clear that $\beta_{\|\cdot\|}(x) = \lim \|x^n\|^{\frac{1}{n}} = \|x\|$ if $\|\cdot\|$ is a square preserving seminorm on X.

A Hausdorff m-convex algebra is called a *uniform algebra* if its topology is generated by a family of square preserving seminorms.

Proposition 2.3. [8, Lema 5.1, p. 275] Every uniform algebra is commutative and semisimple.

3. Results

Using the terminology in [13], we call an m-convex seminorm $\|\cdot\|$ on X *nontrivial* unless X has an identity at which $\|\cdot\|$ is zero. We now say that an A-convex seminorm $\|\cdot\|$ is nontrivial if its associated m-convex seminorm $q_{\|\cdot\|}$ (see (2.3)) is nontrivial. If X has an identity, then the identically zero seminorm is the unique A-convex seminorm that is trivial on X (recall (2.2)). Then any Hausdorff A-convex algebra posses a nontrivial A-convex seminorm.

Proposition 3.1. Let X be a Hausdorff A-convex algebra. Then any element x in X has a nonempty spectrum and $\limsup (\|x^n\|)^{\frac{1}{n}} \leq \rho(x)$ for some A-convex seminorm $\|\cdot\|$ on X.

Proof. Let $\|\cdot\|$ be a nontrivial A-convex seminorm on X. Then its associated m-convex seminorm $q_{\|\cdot\|}$ is nontrivial. By [13, Theorem 2.2.2, p.210], we have $Sp(x) \neq \emptyset$ and $\lim \left(q_{\|\cdot\|}(x^n)\right)^{\frac{1}{n}} \leq \rho(x)$ for all $x \in X$. Then $\limsup (\|x^n\|)^{\frac{1}{n}} \leq \rho(x)$ since $q_{\|\cdot\|}$ is stronger than $\|\cdot\|$. ☐

Corollary 3.2. The spectral radius is a real valued function in any Hausdorff spectral A-convex algebra.

The next two theorems collect part of what is established in ([13, Theorem 2.2.5, p. 212 and Theorem 3.1.5, p. 309]).

Theorem 3.3. *The following are equivalent for an algebra seminorm $\|\cdot\|$ on X.*

(a) $\|\cdot\|$ is a spectral seminorm.

(b) The set of quasi-invertible elements of X is open with respect to $\|\cdot\|$, i.e. $(X, \|\cdot\|)$ is a seminormed Q-algebra.

Theorem 3.4. *The following are equivalent.*

(a) X is an almost commutative spectral algebra.

(b) The spectral radius on X is an algebra seminorm.

Lemma 3.5. *An A-convex (resp. m-convex) algebra $(X, \sigma(\mathcal{P}))$ is a Q-algebra if and only there exists a seminorm $\|\cdot\|$ in the saturated family $\mathcal{P}_s$ for which $(X, \|\cdot\|)$ is an A-seminormed (resp. seminormed) Q-algebra.*

Proof. Any seminorm in $\mathcal{P}_s$ is A-convex (resp. m-convex). X is a Q-algebra $\Leftrightarrow$ there are $\|\cdot\| \in \mathcal{P}_s$ and $\epsilon > 0$ such that $x \in G_q(X)$ if $\|x\| < \epsilon \Leftrightarrow 0$ is an interior point of $G_q(X)$ respect to $\|\cdot\| \Leftrightarrow (X, \|\cdot\|)$ is an A-seminormed (resp. seminormed) Q-algebra. ☐

Proposition 3.6. Suppose (X,τ) is an A-convex algebra and that X becomes to be a Q-algebra when it is endowed with any m-convex topology stronger than τ.

(a) If $\mathcal{P}'$ is a family of m-convex seminorms such that $\sigma(\mathcal{P}')$ is stronger than τ, then there is a seminorm $\|\cdot\|$ in $\mathcal{P}'_s$ that is spectral on X.

(b) If τ is generated by a family $\mathcal{P}$ of A-convex seminorms, then there is a seminorm in $(\mathcal{Q}_{\mathcal{P}})_s$ that is a spectral seminorm on X.

(c) X is a spectral algebra.

(d) If X is almost commutative, then the spectral radius ρ is an m-convex seminorm on X.

Proof. (a) Since $\tau \subset \sigma(\mathcal{P}')$ then $(X,\sigma(\mathcal{P}'))$ is a Q-algebra by hypothesis. It follows from Lemma 3.5 that there exists an m-convex seminorm $\|\cdot\|$ in $\mathcal{P}'_s$ such that $(X,\|\cdot\|)$ is a Q-algebra. Therefore, $\|\cdot\|$ is a spectral seminorm on X (Theorem 3.3).

(b) The result follows from (a) since $(\mathcal{Q}_{\mathcal{P}})_s$ is a family of m-convex seminorms on X and its generated topology $\mathcal{Q}(\mathcal{P})$ is stronger than τ.

(c) From (b), X is a spectral algebra.

(d) Since X is a spectral algebra almost commutative, then ρ defines an m-convex seminorm on X (Theorem 3.4). □

Proposition 3.7. Let (X,τ) be an A-convex algebra that is advertibly complete. Then $\rho \le \beta_{\tau'}$ on X for every topology τ' stronger than τ that makes X into a locally convex algebra.

Proof. It is enough to prove the inequality for $\beta = \beta_\tau$. Let $\{\|\cdot\|_i : i \in I\}$ be a family of A-convex seminorms inducing the topology τ and suppose $\rho(x) > \beta(x)$ for some $x \in X$. Then there are $\lambda \in Sp(x)$, with $\lambda > 0$, and $r > 0$ such that $\beta\left(\frac{1}{\lambda}x\right) < r < 1$. By equality (2.5), for each $i \in I$, $\left\|\left(\frac{1}{\lambda}x\right)^n\right\|_i \le r^n$ for all n (depending on i) large enough. Then, the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\lambda}x\right)^n$ is Cauchy in (X,τ) and $\frac{1}{\lambda}x \circ \left(-\sum\limits_{k=1}^{n}\left(\frac{1}{\lambda}x\right)^k\right) = \left(-\sum\limits_{k=1}^{n}\left(\frac{1}{\lambda}x\right)^k\right) \circ \frac{1}{\lambda}x = \left(\frac{1}{\lambda}x\right)^{n+1} \to 0$. Since (X,τ) is advertibly complete, $-\sum\limits_{k=1}^{\infty}\left(\frac{1}{\lambda}x\right)^k = \left(\frac{1}{\lambda}x\right)^q$ contradicting that $\lambda \in Sp(x)$. □

Proposition 3.8. Let (X,τ) be a locally convex algebra. If there exists a family $\mathcal{P}_0 = \{p_j : j \in J\}$ of square preserving seminorms on X such that $\tau \subset \sigma(\mathcal{P}_0)$, then $\beta_\tau \le \sup\limits_{j\in J} p_j \le \rho$ on X.

Proof. We may assume that $\mathcal{P}_0$ is saturated. Suppose the topology τ is induced by the family $\{\|\cdot\|_i : i \in I\}$ of seminorms. For each $i \in I$ there exist $j \in J$ and $M > 0$ such that $\|x\|_i \le Mp_j(x)$ for all $x \in X$. Since p_j is a square preserving seminorm it is submultiplicative. Then $\limsup \|x^n\|_i^{\frac{1}{n}} \le p_j(x)$ and $\beta_\tau(x) \le \sup\limits_{j\in J} p_j(x)$ by equality (2.5).

For each $j \in J$ let $\widetilde{p}_j$ be the norm on the corresponding Banach algebra X_j of the Arens-Michael analysis of $(X,\mathcal{P}_0)$, i.e., X_j is the completion of the normed algebra $X/\ker p_j$. It is clear that $\rho(x) \ge \rho_{X_j}([x]_j)$ for all $x \in X$ and $j \in J$, where $[x]_j$ is the class of x in $X/\ker p_j$. Since $\widetilde{p}_j\left([x]_j\right) = p_j(x)$ for all $x \in X$, then $\widetilde{p}_j$ is square preserving seminorm on X_j and $\rho_{X_j}([x]_j) = \widetilde{p}_j\left([x]_j\right)$. Thus, $\rho(x) \geq \sup_{j\in J} p_j(x)$ for all $x \in X$. □

Remark 3.9. In [5, Theorem 1] it is assumed for the above result that X is a unital A-convex algebra and precisely $\mathcal{P}_{op} = \{\|\cdot\|_i^{op} : i \in I\}$ is a family of square preserving seminorms on X for some system $\mathcal{P} = \{\|\cdot\|_i : i \in I\}$ of A-convex seminorms generating τ.

In Proposition 3.8, $\sup_{j\in J} p_j$ is a square preserving seminorm on X if ρ is finite valued. Thus the existence of such family $\mathcal{P}_0 = \{p_j : j \in J\}$ is equivalent to the existence of a single square preserving seminorm p_0 such that $\tau \subset \sigma(p_0)$ when X is a spectral algebra; in particular, when (X, τ) is a locally convex Q-algebra.

Remark 3.10. Let (X, τ) be an A-convex algebra. If $\ell(\tau)$ exists and it is induced by a family of square preserving seminorms p_j, with $j \in J$, then $\beta_\tau \leq \sup_{j\in J} p_j \leq \rho$ on X.

We do not know of any example of an A-convex algebra, other than m-convex, in which $\ell(\tau)$ is not given by a family of square preserving seminorms.

Corollary 3.11. Let (X, τ) be an A-convex Q-algebra. If there exists a family $\mathcal{P}_0 = \{p_j : j \in J\}$ of square preserving seminorms on X such that $\tau \subset \sigma(\mathcal{P}_0)$, then $\rho = \beta_\tau = \sup_{j\in J} p_j$ on X.

Theorem 3.12. *Let (X, τ) be a Hausdorff A-convex algebra. The spectral radius ρ is an algebra norm and defines the Oudadess topology $\ell(\tau)$ if and only if the following two conditions hold.*

(a) X is a Q-algebra for any m-convex topology stronger than τ.

(b) There exists a square preserving norm p_0 generating on X a stronger topology than τ.

Proof. Part "if". By (b), (X, p_0) is a normed uniform algebra and then X is a commutative algebra (Proposition 2.3). Hence it follows from hypothesis (a) and item (d) of Proposition 3.6 that ρ is an m-convex seminorm. Since $p_0 \leq \rho$ by Proposition 3.8, ρ is an algebra norm and the topology $\sigma(\rho)$ determined by ρ is stronger than $\sigma(p_0)$ and therefore, stronger than τ. Thus, ρ is an algebra norm that defines an m-convex topology stronger than τ.

Suppose τ' is an m-convex topology on X stronger than τ defined by a saturated family $\mathcal{P}'$ of m-convex seminorms. By (a) of Proposition 3.6, there is some seminorm $\|\cdot\| \in \mathcal{P}'$ that is spectral on X. Then $\rho(x) = \lim \|x^n\|^{\frac{1}{n}} \leq \|x\|$ $\forall x \in X$ and $\sigma(\rho)$ is weaker than τ'. Therefore, $\ell(\tau) = \sigma(\rho)$.

Part "only if". Suppose ρ is an algebra norm and $\sigma(\rho) = \ell(\tau)$. Then ρ is a square preserving norm generating on X a stronger topology than τ and we get condition (b). Also ρ is obviously a spectral norm and then $(X, \rho) = (X, \ell(\tau))$ is a Q-algebra (Theorem 3.3). Thus, X is a Q-algebra for any m-convex topology stronger than τ and condition (a) holds. □

Remark 3.13. For the part "only if" we cannot change condition (a) for (a') (X, τ) is Q-algebra. In [2, Example 2, p.70] is established that in the commutative unital algebra $X = C([0, 1])$ of all complex continuous functions on $[0, 1]$, endowed with the topology defined by the countable family of A-convex seminorms $\|f\|_i = \left(\int_0^1 |f(x)|^i \, dx\right)^{\frac{1}{i}}$ for $i \geq 1$, the Oudadess topology on (X, τ) is induced by the algebra norm $\|f\|_{[0,1]} = \max_{x \in [0,1]} |f(x)|$. That is, in this algebra the spectral radius $\rho(x) = \max_{x \in [0,1]} |f(x)|$ is an m-convex seminorm that defines the Oudadess topology $\ell(\tau)$, but (X, τ) is not a Q-algebra.

However, from part "if" of Theorem 3.12 and Corollary 3.11 we get the following result.

Corollary 3.14. Let (X, τ) be a Hausdorff A-convex Q-algebra for which there exists a square preserving seminorm p_0 on X such that $\tau \subset \sigma(p_0)$. Then the Oudadess topology $\ell(\tau)$ exists and it is given by the spectral radius ρ. Moreover, $\rho = \beta_\tau$.

We summarize a couple of simple facts in the next result.

Lemma 3.15. *Let $\|\cdot\|$ be an A-convex (resp. m-convex) seminorm on X.*

(a) The quotient norm $|[x]| = \|x\|$ on $X \diagup \ker \|\cdot\|$ is an A-convex norm (resp. algebra norm). Hence the algebra $X \diagup \ker \|\cdot\|$ endowed with the quotient topology τ is an A-normed (resp. normed) algebra .

(b) Suppose $\mathfrak{q}$ is an m-convex norm on $X \diagup \ker \|\cdot\|$ stronger than $|\cdot|$. Then $p(x) = \mathfrak{q}([x])$ is an m-convex seminorm on X stronger than $\|\cdot\|$.

Proof. (a) For $x \in X$ there exists $M(x)$ such that $\max(\|xy\|, \|yx\|) \leq M(x)\|y\|$ for every $y \in X$. Then $\max(|[x][y]|, |[y][x]|) \leq M(x)|[y]|$ and $|\cdot|$ is A-convex. When $\|\cdot\|$ is m-convex $M(x)$ can be chosen equal to $\|x\| = |[x]|$ and then $|\cdot|$ is m-convex.

(b) It is clear that p is an m-convex seminorm. Since there exists $K > 0$ such that $\|x\| = |[x]| \leq K\mathfrak{q}([x]) = Kp(x)$, p is stronger than $\|\cdot\|$. □

Theorem 3.16. *Let (X, τ) be an A-convex algebra, with $\tau = \sigma(\mathcal{P})$ and $\mathcal{P} = \{\|\cdot\|_i : i \in I\}$. The topology $\ell(\tau)$ exists if $\ell(\tau_i)$ exists for each A-normed algebra $X_i = (X \diagup \ker \|\cdot\|_i, \tau_i)$, where τ_i is the quotient topology. Moreover, the topology $\ell(\tau)$ is defined by the seminorms given by $p_i(x) = \mathfrak{q}_i([x]_i)$ if $\ell(\tau_i)$ is determined by the m-convex norm $\mathfrak{q}_i$ on X_i for each $i \in I$.*

Proof. We use the notation of the above Lemma 3.15. By hypothesis, for each $i \in I$ the Oudadess topology $\ell(\tau_i)$ on the A-normed algebra $(X_i, |\cdot|_i)$ exists. It is determined by an m-convex norm $\mathfrak{q}_i$ on X_i.

The seminorm on X defined by $p_i(x) = \mathfrak{q}_i([x]_i)$ is m-convex and stronger than $\|\cdot\|_i$ for each $i \in I$ (Lemma 3.15 (b)). Thus the family $\mathcal{P}_0 = \{p_i : i \in I\}$ defines on X an m-convex topology stronger than τ.

Suppose τ' is an m-convex topology on X stronger than τ defined by a saturated family $\mathcal{P}'$ of m-convex seminorms. For proving that $\sigma(\mathcal{P}_0) \subset \sigma(\mathcal{P}')$ we may assume that for each seminorm $\|\cdot\|$ in $\mathcal{P}'$ there is $i \in I$ such that $\|x\|_i \leq M_i \|x\|$ for some $M_i > 0$ and all $x \in X$. Thus, for $\|\cdot\| \in \mathcal{P}'$ there exists $i \in I$ and an epimorphism of algebras $\varphi_i : X \diagup \ker \|\cdot\| \to X \diagup \ker \|\cdot\|_i$ given by $\varphi_i([x]) = [x]_i$. This function is continuous for the topologies given by the quotient algebra norm $|\cdot|$ and the quotient A-norm $|\cdot|_i$, respectively (Lemma 3.15 (a)). The identification topology $\mathcal{I}_i$ determined by φ_i on $X \diagup \ker \|\cdot\|_i$ is m-convex. In fact, $\mathcal{I}_i$ is determined by the algebra norm $\mathfrak{q}'_i([x]_i) = \inf\{|[x]| = \|x\| : \varphi_i([x]) = [x]_i\}$ and $\mathcal{I}_i$ is stronger than τ_i because φ_i is $|\cdot| - |\cdot|_i$ continuous.

Therefore, $\sigma(\mathfrak{q}_i) = \ell(\tau_i) \subset \mathcal{I}_i$ and then $p_i(x) = \mathfrak{q}_i([x]_i) \leq K_i \mathfrak{q}'_i([x]_i) \leq K_i \|x\|$ for some $K_i > 0$ and all $x \in X$. This proves that $\sigma(\mathcal{P}_0) \subset \sigma(\mathcal{P}')$ and then $\ell(\tau) = \sigma(\mathcal{P}_0)$. □

Corollary 3.17. Let (X, τ) be an A-convex algebra, with $\tau = \sigma(\mathcal{P})$ and $\mathcal{P} = \{\|\cdot\|_i : i \in I\}$. The topology $\ell(\tau)$ exists on X if for every $i \in I$, $X_i = \left(X \diagup \ker \|\cdot\|_i, (|\cdot|_i)_{i \in I}\right)$ is a Q-algebra and $|\cdot|_i$ is weaker than some square preserving norm on X_i. In fact, $\ell(\tau)$ is defined by the seminorms given by $p_i(x) = \rho_i([x]_i) = \beta_{\tau_i}([x]_i)$ for $i \in I$ and $x \in X$, where ρ_i is the spectral radius and β_{τ_i} is the radius of boundedness on X_i.

Proof. It follows from Theorem 3.16 and Corollary 3.14 applied to each A-normed algebra $X_i = (X \diagup \ker \|\cdot\|_i, \tau_i)$. □

The algebra $C(\mathbb{R})$ of all the complex continuous functions on $\mathbb{R}$, endowed with the topology given by the A-convex seminorms $\|f\|_i = \int_{-i}^{i} |f(x)|\, dx$ for $i \geq 1$ ([2, Example 1, p.70]) is an example of the situation dealt with in the previous Corollary 3.17. In fact, $\ell(\tau_i)$ is generated by the uniform norm on $C([-i,i]) = C(\mathbb{R}) \diagup \ker \|\cdot\|_i$ that coincides with the spectral radius and the radius of boundedness on $C(\mathbb{R}) \diagup \ker \|\cdot\|_i$.

Theorem 3.18. *Let $(X, \|\cdot\|)$ be an A-normed algebra. The radius of boundedness $\beta_{\|\cdot\|}$ is an m-convex norm and defines the Oudadess topology on X if and only if there exists a square preserving norm p_0 on X stronger than $\|\cdot\|$ such that the sets of sequences (x^n) in X that are bounded respect each of these norms are equal.*

Proof. Part "if". Let p_0 be such square preserving norm on X, then $\beta_{p_0} = p_0$. By Proposition 2.2, $\beta_{\|\cdot\|} = \beta_{p_0}$. Thus $\beta_{\|\cdot\|}(x) = \limsup \|x^n\|^{\frac{1}{n}}$ is an m-convex norm stronger than $\|\cdot\|$ and clearly it is weaker than any other m-convex norm stronger than $\|\cdot\|$.

Part "only if". By hypothesis $\beta_{\|\cdot\|}$ is a square preserving norm on X stronger than $\|\cdot\|$. From this and equality 2.5, we get that a sequence (x^n) is $\|\cdot\|$-bounded if and only if it is $\beta_{\|\cdot\|}$-bounded. □

For the algebra $\left(\mathbb{C}[t], \|\cdot\|_{[0,1]}\right)$ of complex polynomials with the sup-norm on $[0,1]$ it is obvious, by the above Theorem 3.18, that the Oudadess topology is given by the radius of boundedness and we have $\rho(t) \neq \beta(t)$. This algebra is a particular case of the A-convex algebras of weighted functions. In each one of these the Oudadess topology is given by a family of radii of boundedness respect to certain collection of A-convex seminorms ([2, Corollary 1, p.73]).

Questions. Let $(X, \|\cdot\|)$ be an A-normed algebra which is not a normed algebra.

1. Does the Oudadess topology on X, if it exists, make X into a uniform algebra?
2. What are necessary and sufficient conditions for the existence on X of a square preserving norm stronger than $\|\cdot\|$?

References

[1] G.R. Allan, A spectral theory for locally convex algebras, Proc. London Math. Soc., 15(1965), no. 3, 399-421.

[2] H. Arizmendi-Peimbert, A. Carrillo-Hoyo and P. Tejada-Bassols, On the existence of extremal m-convex topology in A-convex algebras, in *International Conference on Topological Algebras and Their Applications ICTAA 2013*, 65–77, Math. Stud. (Tartu), 6, Est. Math. Soc., Tartu.
[3] A.C. Cochran, R. Keown, and C.R. Williams, On a class of topological algebras, Pacific J. Math., 34(1970), no. 1, 17-25.
[4] H.V. Dedania, A seminorm with square property is automatically submultiplicative, Proc. Indian Acad. Sci. Math. Sci., 108(1998), no. 1, 51-53.
[5] A. El Kinani, The regularity radius in locally A-convex algebras, Rend. Circ. Mat. Palermo 2, 63(2014), no. 1, 155-160.
[6] A. El Kinani, L. Oubbi, and M. Oudadess, Spectral and boundedness radii in locally convex algebras, Georgian Math. J., 5(1998), no. 3, 233-241.
[7] M. Fragoulopoulou, *Topological algebras with involution*, North-Holland Mathematics Studies, 200, Elsevier Science B.V., Amsterdam, 2005.
[8] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, 1986.
[9] L. Oubbi, Topologies m-convexes dans les algèbres A-convexes, Rend. Circ. Mat. Palermo (2)(1992), no. XLI, 397-406.
[10] L. Oubbi, The weakest m-convex topology stronger than an A-convex topology need not exist, Rend. Circ. Mat. Palermo (2), LIV (2005), 451-462.
[11] L. Oubbi, Locally A-convex algebras revisited, Contemp. Math., 427 (2007), 319-329.
[12] M. Oudadess, Unité et semi-normes dans les algèbres localment convexes, Rev. Colombiana Mat., XVI (1982), 141-150.
[13] T. W. Palmer, *Banach algebras and the general theory of *-algebras. Vol. I*, Encyclopedia of Mathematics and its Applications, 49, Cambridge University Press, Cambridge, 1994.

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad de México, 04510, México
E-mail address: hpeimbert@gmail.com

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad de México, 04510, México
E-mail address: angel8ch@yahoo.com