Poincare Journal of Analysis & Applications
Vol. 9, No. 1 (2022), 31-40

b-$\mathcal{H}_\sigma$-OPEN SETS IN HGTS

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Date of Receiving	:	02. 02. 2022
Date of Revision	:	12. 05. 2022
Date of Acceptance	:	23. 05. 2022

Abstract. In this paper, we introduce and study the new types of sets in hereditary generalized topological space. Also, we obtained decompositions of (μ,λ)-continuity.

1. Introduction and Preliminaries

In the year 2002, Csaszar [6] introduced very usefull notions of generalized topology and generalized continuity. Consider Z be a nonempty set and μ be a collection from the subsets of Z. Then μ is called a *generalized topology* (briefly GT) if $\emptyset \in \mu$ and an arbitrary union of elements from μ belongs to μ. The main purpose of this paper is to establish some new decompositions of (μ,λ)-continuous functions. Fistly, we introduce a new class of sets called b-$\mathcal{H}_\sigma$-open sets. Properties and the relationships of b-$\mathcal{H}_\sigma$-open sets are investigated. On the other hand, we introduce the notions of π^*-B-$\mathcal{H}_\sigma$-sets, α^*-B-$\mathcal{H}_\sigma$-sets and σ^*-B-$\mathcal{H}_\sigma$-sets. Finally, we obtain some new decompositions of (μ,λ)-continuous functions via these new concepts. A space Z is called a C_0-space [20], if $C_0 = Z$, where C_0 is the set of all representative elements of sets of μ. A subset L of a space (Z,μ) is called asμ-α-*open* [8] (resp. μ-σ-open [8], μ-π-open [8], μ-β-open [8], μ-b-open [19], μ-t-set [15], μ-t^*-set [15]), if $L \subset i_\mu c_\mu i_\mu(L)$ (resp. $L \subset c_\mu i_\mu(L)$, $L \subset i_\mu c_\mu(L)$, $L \subset c_\mu i_\mu c_\mu(L)$, $L \subset c_\mu i_\mu(L) \cup i_\mu c_\mu(L)$, $i_\mu c_\mu(L) = i_\mu(L)$, $i_\mu c_\mu i_\mu(L) = i_\mu(L)$). A subset L of Z is μ-locally closed set [13], $L = U \cap V$, where U is μ-open and V is μ-closed. A GTS (Z,μ) is called μ-extremally disconnected [7], if the μ-closure of every μ-open set is μ-open. A function $f : (Z,\mu) \to (W,\lambda)$ is said to be (μ,λ)-continuous [6], iff $M \in \lambda$ implies that $f^{-1}(M)$ is μ-open. A nonempty family $\mathcal{H}$ of subsets of Z is called as a *hereditary class* [9], if $L \in \mathcal{H}$ and $B \subset L$, then $B \in \mathcal{H}$. For each $L \subseteq Z$,

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2010 *Mathematics Subject Classification.* 54A05, 54D20.
Key words and phrases. hereditary generalized topology, α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open and π-$\mathcal{H}_\sigma$-open sets, b-$\mathcal{H}_\sigma$-open sets.
Communicated by. Erdal Ekici
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$L^*(\mathcal{H},\mu)$=$\{z \in Z : L \cap V \notin \mathcal{H}$ for All $V \in \mu$ such that $z \in V\}$[9]. For $L \subset Z$, define $c^*_\mu(L) = L \cup L^*(\mathcal{H},\mu)$ and $\mu^* = \{L \subset Z : Z - L = c^*_\mu(Z - L)\}$. If $\mathcal{H}$ is a hereditary class on Z then $(Z,\mu,\mathcal{H})$ is called a hereditary generalized topological space ($H.G.T.S.$).

Definition 1.1. [16] Consider L be a subset of H.G.T.S. $(Z,\mu,\mathcal{H})$. Then $L^*_\sigma(\mathcal{H},\mu)$=$\{z \in Z : L \cap V \notin \mathcal{H}$ for All V $\in \mu$-σ-open such that z $\in V\}$.

Definition 1.2. [9] A subset L of a H.G.T.S. $(Z,\mu,\mathcal{H})$ is said to be

(1) α-$\mathcal{H}$-open, if $L \subseteq i_\mu c^*_\mu i_\mu(L)$,
(2) σ-$\mathcal{H}$-open, if $L \subseteq c^*_\mu i_\mu(L)$,
(3) π-$\mathcal{H}$-open, if $L \subseteq i_\mu c^*_\mu(L)$,
(4) β-$\mathcal{H}$-open, if $L \subseteq c_\mu i_\mu c^*_\mu(L)$,
(5) strong β-$\mathcal{H}$-open, if $L \subseteq c^*_\mu i_\mu c^*_\mu(L)$,
(6) μ^*-closed, if $c^*_\mu(L) \subset L$.

Definition 1.3. A subset L of a H.G.T.S. $(Z,\mu,\mathcal{H})$ is said to be δ-$\mathcal{H}$-open [14], if $i_\mu c^*_\mu(L) \subseteq c^*_\mu i_\mu(L)$.

Definition 1.4. A subset L of a H.G.T.S. $(Z,\mu,\mathcal{H})$ is said to be b-$\mathcal{H}$-open [17], if $L \subseteq i_\mu c^*_\mu(L) \cup c^*_\mu i_\mu(L)$.

Definition 1.5. [16] A subset L of a H.G.T.S. $(Z,\mu,\mathcal{H})$ is said to be $\sigma\mu^*$-closed, if $L^*_\sigma \subseteq L$.

Proposition 1.6. [16] Let L be a μ-σ-closed. Then $L^*_\sigma \subset L$.

Let $(Z,\mu,\mathcal{H})$ be a hereditary generalized topological space. For $L \subset Z$, define $c^*_\sigma(L) = L \cup L^*_\sigma$ [16] and $c^*_\sigma(L)$ is enlarging, monotone and idempotent.

Definition 1.7. [18] A subset L of a H.G.T.S. $(Z,\mu,\mathcal{H})$ is said to be

(1) α-$\mathcal{H}_\sigma$-open, if $L \subseteq i_\mu c^*_\sigma i_\mu(L)$,
(2) σ-$\mathcal{H}_\sigma$-open, if $L \subseteq c^*_\sigma i_\mu(L)$,
(3) π-$\mathcal{H}_\sigma$-open, if $L \subseteq i_\mu c^*_\sigma(L)$,
(4) β-$\mathcal{H}_\sigma$-open, if $L \subseteq c_\mu i_\mu c^*_\sigma(L)$.

Definition 1.8. [18] A function $f : (Z,\mu,\mathcal{H}) \to (W,\lambda)$ is said to be $(\alpha$-$\mathcal{H}_\sigma,\lambda)$-continuous (resp.$(\sigma$-$\mathcal{H}_\sigma,\lambda)$-continuous, $(\pi$-$\mathcal{H}_\sigma,\lambda)$-continuous), $f^{-1}(N)$ is α-$\mathcal{H}_\sigma$-open (resp. σ-$\mathcal{H}_\sigma$-open, π-$\mathcal{H}_\sigma$-open) for each λ-open set N in (W,λ).

Recently, as related spaces with those of the present paper, many authors studied the notions of e-open [10] , a-open [11] and e^*-open [12]. Also papers [1, 2, 3, 4] have introduced some property related to b-$\mathcal{H}_\sigma$-open sets in HGTS.

A subset L of a space (Z,μ) is said to be μ-regular open (resp. μ-regular closed) if $L = i_\mu c_\mu(L))$ (respectively $L = c_\mu i_\mu(L))$). The μ-δ-interor of subset L of Z is the union of all μ-regular open sets of Z containing L and it is denoted by δ-$i_\mu(L)$. A subset L is called μ-δ-*open* if $L = \delta$-$i_\mu(L)$. The complement of μ-δ-open set is called μ-δ-closed. The μ-δ-closure of a set (Z,μ) is defined by δ-$c_\mu(L) = \{x \in Z : L \cap i_\mu c_\mu(A)) \neq \emptyset : A \in \mu, x \in A\}$ and it is denoted by δ-$c_\mu(L)$.

Definition 1.9. Consider L be a subset of H.G.T.S. $(Z, \mu, \mathcal{H})$. Then $L^*_\delta(\mathcal{H}, \mu)$=$\{z \in Z : L \cap V \notin \mathcal{H}$ for All V $\in \mu$-δ-open such that z $\in V\}$. And if L be a μ-δ-closed. Then $L^*_\delta \subset L$.

For $L \subset Z$, define $c^*_\delta(L) = L \cup L^*_\delta$ and $c^*_\delta(L)$ is enlarging, monotone and idempotent. And $i^*_\delta(L) = Z - c^*_\delta(Z - L)$. Analogous to the present paper as generalization of e-open, a-open and e^*-open we can investigate their basic properties and several characterizations of the following notions.

Definition 1.10. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be

(1) e-$\mathcal{H}_\delta$-open, if $L \subseteq i_\mu c^*_\delta(L) \cup c^*_\delta i_\mu(L)$.
(2) a-$\mathcal{H}_\delta$-open, if $L \subseteq i_\mu c_\mu i^*_\delta(L)$.
(3) e^*-$\mathcal{H}_\delta$-open, if $L \subseteq c_\mu i_\mu c^*_\delta(L)$.

2. b-$\mathcal{H}_\sigma$-open sets

In this paper, we introduce and study the new types of sets in hereditary generalized topological space. Also basic properties and several characterizations of this notions are obtain.

Definition 2.1. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be b-$\mathcal{H}_\sigma$-open set, if $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$.

Proposition 2.2. In H.G.T.S. $(Z, \mu, \mathcal{H})$ all μ-open set is b-$\mathcal{H}_\sigma$-open but not conversely.

Proof. Let a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is μ-open. Then $L = i_\mu(L)$. Now $L \subseteq i_\mu(L) \subseteq i_\mu c^*_\sigma(L) \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$. Hence L is b-$\mathcal{H}_\sigma$-open. □

Example 2.3. Consider $Z = \{a, b, c, d, e\}$ $\mu = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{c, d, e\}, \{a, c, d, e\}, \{a, b, c\}, Z\}$, $\mathcal{H} = \{\emptyset, \{a\}, \}$. Then $L = \{a, d, e, \}$ is b-$\mathcal{H}_\sigma$-open but not μ-open.

Proposition 2.4. Every b-$\mathcal{H}_\sigma$-open is μ-b-open but not conversely.

Proof. Let L be a b-$\mathcal{H}_\sigma$-open. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\mu(L) \cup c^*_\mu i_\mu(L) \subseteq i_\mu c_\mu(L) \cup c_\mu i_\mu(L)$. Hence L is μ-b-open. □

Proposition 2.5. Every b-$\mathcal{H}_\sigma$-open is b-$\mathcal{H}$-open but not conversely.

Proof. Let L be a b-$\mathcal{H}_\sigma$-open. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\mu(L) \cup c^*_\mu i_\mu(L)$. Hence L is b-$\mathcal{H}$-open. □

Example 2.6. Consider $Z = \{a, b, c, d, e\}$ $\mu = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $L = \{e\}$ is μ-b-open but not b-$\mathcal{H}_\sigma$-open and $M = \{a, e\}$ is b-$\mathcal{H}$-open but not b-$\mathcal{H}_\sigma$-open.

Proposition 2.7. Every α-$\mathcal{H}_\sigma$-open (resp. σ-$\mathcal{H}_\sigma$-open, π-$\mathcal{H}_\sigma$-open)is b-$\mathcal{H}_\sigma$-open but not conversely.

Proof. (1) Let L be α-$\mathcal{H}_\sigma$-open. Then $L \subseteq i_\mu c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu c^*_\sigma(L)$. Hence L is b-$\mathcal{H}_\sigma$-open.

(2) Let L be σ-$\mathcal{H}_\sigma$-open. Then $L \subseteq c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu c^*_\sigma(L)$. Hence L is b-$\mathcal{H}_\sigma$-open.

(3) Let L be π-$\mathcal{H}_\sigma$-open. Then $L \subseteq i_\mu c^*_\sigma(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu c^*_\sigma(L)$. Hence L is b-$\mathcal{H}_\sigma$-open.

□

Example 2.8. Consider $Z = \{a, b, c, d, e\}$ $\mu = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{c, d, e\}, \{a, c, d, e\}, \{a, b, c, \}, Z\}$, $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $L = \{a, d, e, \}$ is b-$\mathcal{H}_\sigma$-open but neither σ-$\mathcal{H}_\sigma$-open nor α-$\mathcal{H}_\sigma$-open.

Theorem 2.9. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and μ-σ-open, then it is β-$\mathcal{H}_\sigma$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and μ-σ-open. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $L \subseteq c_\mu i_\mu(L)$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq c^*_\sigma(L)$, which implies $c_\mu i_\mu(L) \subseteq c_\mu i_\mu c^*_\sigma(L)$. So $L \subseteq c_\mu i_\mu(L) \subseteq c_\mu i_\mu c^*_\sigma(L)$. Hence L is β-$\mathcal{H}_\sigma$-open. □

Theorem 2.10. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and μ^*-closed, then it is σ-$\mathcal{H}_\sigma$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and μ^*-closed. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma(L) \subseteq c^*_\mu(L) \subseteq L$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu(L) = c^*_\sigma i_\mu(L)$. Hence L is σ-$\mathcal{H}_\sigma$-open. □

Theorem 2.11. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and $\sigma\mu^*$-closed, then it is σ-$\mathcal{H}_\sigma$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and $\sigma\mu^*$-closed. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma(L) \subseteq A$, which implies $i_\mu c^*_\sigma(L) \subseteq i_\mu L$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu(L) = c^*_\sigma i_\mu(L)$. Hence σ-$\mathcal{H}_\sigma$-open. □

Theorem 2.12. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and μ-σ-closed, then it is σ-$\mathcal{H}_\sigma$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and μ-σ-closed. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma(L) \subseteq A$ by Proposition 2.9 of [16]. Which implies $i_\mu c^*_\sigma(L) \subseteq i_\mu(L)$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu(L) = c^*_\sigma i_\mu(L)$. Hence σ-$\mathcal{H}_\sigma$-open. □

Theorem 2.13. *If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open such that $i_\mu(L) = \emptyset$, then it is π-$\mathcal{H}_\sigma$-open.*

Proof. Let L be a b-$\mathcal{H}_\sigma$-open and $i_\mu(L) = \emptyset$. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) = i_\mu c^*_\sigma(L)$. Hence L is π-$\mathcal{H}_\sigma$-open. □

Theorem 2.14. *If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open, then it is strong β-$\mathcal{H}$-open.*

Proof. Let L be a b-$\mathcal{H}_\sigma$-open. Then L is b-$\mathcal{H}$-open by Proposition 2.5. Hence L is strong β-$\mathcal{H}$-open by Proposition of 2.26 of [17]. □

Theorem 2.15. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and δ-$\mathcal{H}$-open, then it is σ-$\mathcal{H}$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and δ-$\mathcal{H}$-open. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $i_\mu c^*_\mu(L) \subseteq c^*_\mu i_\mu(L)$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\mu(L) \cup c^*_\mu i_\mu(L) \subseteq c^*_\mu i_\mu(L)$. Hence L is σ-$\mathcal{H}$-open. □

Theorem 2.16. *If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open and $L \in \mathcal{H}$, then it is σ-$\mathcal{H}_\sigma$-open.*

Proof. Let L is b-$\mathcal{H}_\sigma$-open and $L \in \mathcal{H}$. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma(L) = A$ by Remark 2.10 of [16]. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) = i_\mu(L) \cup c^*_\sigma i_\mu(L) = c^*_\sigma i_\mu(L)$. Hence L is σ-$\mathcal{H}_\sigma$-open. □

Theorem 2.17. *If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open and $\mathcal{H} = P(Z)$ then it is σ-$\mathcal{H}_\sigma$-open.*

Proof. Let L is b-$\mathcal{H}_\sigma$-open and $L \in \mathcal{H}$. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma(L) = A$ by Remark 2.10 of [16]. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) = i_\mu(L) \cup c^*_\sigma i_\mu(L) = c^*_\sigma i_\mu(L)$. Hence L is σ-$\mathcal{H}_\sigma$-open. □

Theorem 2.18. *If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open and $L \subset L^*_\sigma$, then it is μ-β-open.*

Proof. Let L is b-$\mathcal{H}_\sigma$-open and $L \subset L^*_\sigma$. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma i_\mu(L) \subset c^*_\sigma i_\mu c^*_\sigma(L)$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu c^*_\sigma(L) \subseteq c^*_\sigma i_\mu c^*_\sigma(L) \subseteq c^*_\mu i_\mu c^*_\mu(L) \subseteq c_\mu i_\mu c_\mu(L)$. Hence L is μ-β-open. □

Remark 2.19. If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open and $L \subset L^*_\sigma$, then it is strong β-$\mathcal{H}$-open.

Remark 2.20. If $L \subset Z$ is b-$\mathcal{H}_\sigma$-open and $L \subset L^*_\sigma$, then it is β-$\mathcal{H}$-open.

Theorem 2.21. *Let $(Z, \mu, \mathcal{H})$ be a strong H.G.T.S., where Z is C_0-space and μ-extremally disconnected space, $L \subset Z$. Then the following conditions are equivalent.*

(1) *L is μ-open,*
(2) *L is b-$\mathcal{H}_\sigma$-open and μ-locally closed set.*

Proof. (1) ⇒ (2) This is obvious from definitions.

(2) ⇒ (1) Let L is b-$\mathcal{H}_\sigma$-open and μ-locally closed set. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq i_\mu c_\mu(L) \cup c_\mu i_\mu(L)$ and $L = U \cap c_\mu(L)$. Now

$$\begin{aligned}
L &\subset U \cap c_\mu(L).\\
&\subset U \cap [i_\mu c_\mu(L) \cup c_\mu i_\mu(L)]\\
&\subset [U \cap i_\mu c_\mu(L)] \cup [U \cap c_\mu i_\mu(L)]\\
&\subset [i_\mu(U) \cap i_\mu c_\mu(L)] \cup [i_\mu(U) \cap c_\mu i_\mu(L)]\\
&\subset [i_\mu(U) \cap i_\mu c_\mu(L)] \cup [i_\mu(U) \cap i_\mu c_\mu(L)]\\
&\subset [i_\mu(U \cap c_\mu(L))] \cup [i_\mu(U \cap c_\mu(L))]\\
&= [i_\mu(L)] \cup [i_\mu(L)]\\
&= i_\mu(L).
\end{aligned}$$

Hence L is μ-open. □

Definition 2.22. For $L \subset Z$, $i_{b\mathcal{H}_\sigma}(L)$ is the largest b-$\mathcal{H}_\sigma$-open open set contained in L.

Definition 2.23. A subset L of H.G.T.S $(Z, \mu, \mathcal{H})$ is called $D_b(c, \mathcal{H}_\sigma)$-set, if $i_\mu(L) = i_{b\mathcal{H}_\sigma}(L)$.

Theorem 2.24. *For a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$, the following conditions are equivalent.*

(1) *L is μ-open,*
(2) *L is b-$\mathcal{H}_\sigma$-open and $D_b(c, \mathcal{H}_\sigma)$-set.*

Proof. (1) $\Rightarrow$ (2) Let L is μ-open. Then L is b-$\mathcal{H}_\sigma$-open. So $L = i_\mu(L)$ and $L = i_{b\mathcal{H}_\sigma}(L)$. Therefore $i_\mu(L) = i_{b\mathcal{H}_\sigma}(L)$. Hence L is $D_b(c, \mathcal{H}_\sigma)$-set.

(2) $\Rightarrow$ (1) Let L is b-$\mathcal{H}_\sigma$-open and $D_b(c, \mathcal{H}_\sigma)$-set. Then $L = i_{b\mathcal{H}_\sigma}(L)$ and $i_\mu(L) = i_{b\mathcal{H}_\sigma}(L)$. Therefore $i_\mu(L) = L$. Hence L is μ-open. □

Remark 2.25. The notions of L is b-$\mathcal{H}_\sigma$-open and μ-locally closed set(resp. $D_b(c, \mathcal{H}_\sigma)$-set) are independent.

Example 2.26. Let $Z = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $L = \{a, d\}$ is μ-locally closed set (resp. $D_b(c, \mathcal{H}_\sigma)$-set) but not b-$\mathcal{H}_\sigma$-open.

Example 2.27. Let $Z = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, b, c\}, \{c, d\}, Z\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then $L = \{a, c, d\}$ is b-$\mathcal{H}_\sigma$-open but neither μ-locally closed set and nor $D_b(c, \mathcal{H}_\sigma)$-set.

3. π^*-B-$\mathcal{H}_\sigma$-sets, α^*-B-$\mathcal{H}_\sigma$-sets and σ^*-B-$\mathcal{H}_\sigma$-sets

Definition 3.1. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is called

(1) π^*-$\mathcal{H}_\sigma$-set, if $i_\mu c^*_\sigma(L) = i_\mu(L)$.
(2) α^*-$\mathcal{H}_\sigma$-set, $i_\mu c^*_\sigma i_\mu(L) = i_\mu(L)$.
(3) σ^*-$\mathcal{H}_\sigma$-set, if $c^*_\sigma i_\mu(L) = i_\mu(L)$.

Remark 3.2. The notions of π^*-$\mathcal{H}_\sigma$-set (resp. α^*-$\mathcal{H}_\sigma$-set, σ^*-$\mathcal{H}_\sigma$-set) and π-$\mathcal{H}_\sigma$-open (resp. α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open) are independent.

Example 3.3. Let $Z = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $A = \{a, d\}$ is π^*-$\mathcal{H}_\sigma$-set (resp. α^*-$\mathcal{H}_\sigma$-set, σ^*-$\mathcal{H}_\sigma$-set) but not π-$\mathcal{H}_\sigma$-open (resp. α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open).

Example 3.4. let $Z = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, Z\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Then $A = \{a, b, c\}$ is π-$\mathcal{H}_\sigma$-open (resp. α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open) but not π^*-$\mathcal{H}_\sigma$-set (resp. α^*-$\mathcal{H}_\sigma$-set, σ^*-$\mathcal{H}_\sigma$-set).

Definition 3.5. A subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is called

(1) π^*-B-$\mathcal{H}_\sigma$-set, if $L = M \cap N$, where M is μ-open and N is π^*-$\mathcal{H}_\sigma$-set.
(2) α^*-B-$\mathcal{H}_\sigma$-set, if $L = M \cap N$, where M is μ-open and N is α^*-$\mathcal{H}_\sigma$-set.
(3) σ^*-B-$\mathcal{H}_\sigma$-set, if $L = M \cap N$, where M is μ-open and N is σ^*-$\mathcal{H}_\sigma$-set

Theorem 3.6. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and π^*-$\mathcal{H}_\sigma$-set, then it is σ-$\mathcal{H}_\sigma$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and π^*-$\mathcal{H}_\sigma$-set. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $i_\mu c^*_\sigma(L) = i_\mu(L)$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L) \cup i_\mu(L) = c^*_\sigma i_\mu(L)$. Hence L is σ-$\mathcal{H}_\sigma$-open. □

Theorem 3.7. *If $L \subset Z$ is both b-$\mathcal{H}_\sigma$-open and σ^*-$\mathcal{H}_\sigma$-set, then it is π-$\mathcal{H}_\sigma$-open.*

Proof. Let L is both b-$\mathcal{H}_\sigma$-open and σ^*-$\mathcal{H}_\sigma$-set. Then $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L)$ and $c^*_\sigma i_\mu(L) = i_\mu(L)$. Now $L \subseteq i_\mu c^*_\sigma(L) \cup c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\sigma(L) \cup i_\mu(L) = i_\mu c^*_\sigma(L)$. Hence L is π-$\mathcal{H}_\sigma$-open. □

Theorem 3.8. *If $L \subset Z$ is both π-$\mathcal{H}_\sigma$-open and μ^*-closed, then it is π^*-$\mathcal{H}_\sigma$-set.*

Proof. Let L is both π-$\mathcal{H}_\sigma$-open and μ^*-closed. Then $L \subseteq i_\mu c^*_\sigma(L)$ and $c^*_\mu(L) \subset L$. Now $i_\mu c^*_\sigma(L) \subset c^*_\sigma(L) \subset c^*_\mu(L) \subset L$. So, $L = i_\mu c^*_\sigma(L)$. Thus $i_\mu(L) = i_\mu c^*_\sigma(L)$. Hence L is π^*-$\mathcal{H}_\sigma$-set. □

Theorem 3.9. *If $L \subset Z$ is both σ-$\mathcal{H}_\sigma$-open and μ^*-closed, then it is α^*-$\mathcal{H}_\sigma$-set.*

Proof. Let L is both σ-$\mathcal{H}_\sigma$-open and μ^*-closed. Then $L \subseteq c^*_\sigma i_\mu(L)$ and $c^*_\mu(L) \subset L$. Now $c^*_\sigma i_\mu(L) \subset c^*_\sigma(L) \subset c^*_\mu(L) \subset L$. So, $L = c^*_\sigma i_\mu(L)$. Thus $i_\mu(L) = i_\mu c^*_\sigma i_\mu(L)$. Hence L is α^*-$\mathcal{H}_\sigma$-set. □

Theorem 3.10. *If $L \subset Z$ is both π-$\mathcal{H}_\sigma$-open and $\sigma\mu^*$-closed, then it is π^*-$\mathcal{H}_\sigma$-set.*

Proof. Let L is both π-$\mathcal{H}_\sigma$-open and $\sigma\mu^*$-closed. Then $L \subseteq i_\mu c^*_\sigma(L)$ and $c^*_\sigma(L) \subset L$. Now $i_\mu c^*_\sigma(L) \subset c^*_\sigma(L) \subset L$. So, $L = i_\mu c^*_\sigma(L)$. Thus $i_\mu(L) = i_\mu c^*_\sigma(L)$. Hence L is π^*-$\mathcal{H}_\sigma$-set. □

Theorem 3.11. *If $L \subset Z$ is both σ-$\mathcal{H}_\sigma$-open and $\sigma\mu^*$-closed, then it is α^*-$\mathcal{H}_\sigma$-set.*

Proof. Let L is both σ-$\mathcal{H}_\sigma$-open and $\sigma\mu^*$-closed. Then $L \subseteq c^*_\sigma i_\mu(L)$ and $c^*_\sigma(L) \subset L$. Now $c^*_\sigma i_\mu(L) \subset c^*_\sigma(L) \subset L$. So, $L = c^*_\sigma i_\mu(L)$. Thus $i_\mu(L) = i_\mu c^*_\sigma i_\mu(L)$. Hence L is α^*-$\mathcal{H}_\sigma$-set. □

Proposition 3.12. Let $(Z, \mu, \mathcal{H})$ be a strong H.G.T.S. and $L \subset Z$. Then the following holds:

(1) If L is π^*-$\mathcal{H}_\sigma$-set, then L is π^*-B-$\mathcal{H}_\sigma$-set,
(2) If L is α^*-$\mathcal{H}_\sigma$-set, then L is α^*-B-$\mathcal{H}_\sigma$-set,
(3) If L is σ^*-$\mathcal{H}_\sigma$-set, then L is σ^*-B-$\mathcal{H}_\sigma$-set.

Proof. (1) Let L be a π^*-$\mathcal{H}_\sigma$-set. If we take $M = Z \in \mu$, then $L = M \cap L$ and hence L is a π^*-B-$\mathcal{H}_\sigma$-set.
(2) This is obvious.
(3) Trivial.

□

Theorem 3.13. *Let $(Z, \mu, \mathcal{H})$ be a strong H.G.T.S., where Z is C_0-space and $L \subset Z$. Then the following conditions are equivalent.*

(1) L is μ-open,
(2) L is π-$\mathcal{H}_\sigma$-open and π^*-B-$\mathcal{H}_\sigma$-set,
(3) α-$\mathcal{H}_\sigma$-open and α^*-B-$\mathcal{H}_\sigma$-set,
(4) σ-$\mathcal{H}_\sigma$-open and σ^*-B-$\mathcal{H}_\sigma$-set.

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), are obvious.

(2) $\Rightarrow$ (1). Let L is both π-$\mathcal{H}_\sigma$-open and π^*-B-$\mathcal{H}_\sigma$-set. Then we have $L \subseteq i_\mu c^*_\sigma(L) = i_\mu c^*_\sigma(M \cap N)$, where $M \in \mu$ and N is π^*-$\mathcal{H}_\sigma$-set. Hence $L \subseteq i_\mu c^*_\sigma(M) \cap i_\mu c^*_\sigma(N)$. Now $L \subseteq M \cap L \subseteq M \cap [i_\mu c^*_\sigma(M) \cap i_\mu(N)] = M \cap i_\mu(N) = i_\mu(L)$. Hence L is μ-open.

(3) $\Rightarrow$ (1). Let L is both α-$\mathcal{H}_\sigma$-open and α^*-B-$\mathcal{H}_\sigma$-set. Then we have $L \subseteq i_\mu c^*_\sigma i_\mu(L) = i_\mu c^*_\sigma i_\mu(M \cap N)$, where $M \in \mu$ and N is t^*-$\mathcal{H}_\sigma$-set. Hence $L \subseteq i_\mu c^*_\sigma i_\mu(M) \cap i_\mu c^*_\sigma i_\mu(N)$. Now $L \subseteq M \cap L \subseteq M \cap [i_\mu c^*_\sigma i_\mu(M) \cap i_\mu(N)] = M \cap i_\mu(N) = i_\mu(L)$. Hence L is μ-open.

(3) $\Rightarrow$ (1). Let L is both σ-$\mathcal{H}_\sigma$-open and σ^*-B-$\mathcal{H}_\sigma$-set. Then we have $L \subseteq c^*_\sigma i_\mu(L) = c^*_\sigma i_\mu(M \cap N)$, where $M \in \mu$ and N is σ^*-B-$\mathcal{H}_\sigma$-set. Hence $L \subseteq c^*_\sigma i_\mu(M) \cap c^*_\sigma i_\mu(N)$. Now $L \subseteq M \cap L \subseteq M \cap [c^*_\sigma i_\mu(M) \cap i_\mu(N)] = M \cap i_\mu(N) = i_\mu(L)$. Hence L is μ-open. □

Remark 3.14. The notions of π-$\mathcal{H}_\sigma$-open and π^*-B-$\mathcal{H}_\sigma$-set are independent.

Remark 3.15. The notions of α-$\mathcal{H}_\sigma$-open and α^*-B-$\mathcal{H}_\sigma$-set are independent.

Remark 3.16. The notions of σ-$\mathcal{H}_\sigma$-open and σ^*-B-$\mathcal{H}_\sigma$-set are independent.

Example 3.17. Let $Z = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $A = \{c, d\}$ is π^*-B-$\mathcal{H}_\sigma$-set (resp. α^*-B-$\mathcal{H}_\sigma$-set, σ^*-B-$\mathcal{H}_\sigma$-set) but not π-$\mathcal{H}_\sigma$-open (resp. α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open).

Example 3.18. let $Z = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, Z\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Then $A = \{a, b, c\}$ is π-$\mathcal{H}_\sigma$-open (resp. α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open) but not π^*-B-$\mathcal{H}_\sigma$-set (resp. α^*-B-$\mathcal{H}_\sigma$-set, σ^*-B-$\mathcal{H}_\sigma$-set).

4. Decomposition of (μ, λ)-continuity

Definition 4.1. A function $f : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ is said to be $(\pi^*$-B-$\mathcal{H}_\sigma, \lambda)$-continuous (resp.$(\alpha^*$-$B$-$\mathcal{H}_\sigma, \lambda)$-continuous, $(\sigma^*$-B-$\mathcal{H}_\sigma, \lambda)$-continuous, $(D_b(c, \mathcal{H}_\sigma), \lambda)$-continuous,) if $f^{-1}(N)$ is π^*-B-$\mathcal{H}_\sigma$-set (resp. α^*-B-$\mathcal{H}_\sigma$-set, σ^*-B-$\mathcal{H}_\sigma$-set, $D_b(c, \mathcal{H}_\sigma)$-set) for each λ-open set N in (W, λ).

Definition 4.2. A function $f : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ is said to be $(\mu L, \lambda)$-continuous, if $f^{-1}(N)$ is μ-locally closed set for each λ-closed set N in (W, λ).

Theorem 4.3. *Let $(Z, \mu, \mathcal{H})$ be a strong H.G.T.S. where Z is C_0-space and μ-extremally disconnected space, for a function $f : (Z, \mu, \mathcal{H}) \to (W, \lambda)$. Then the following conditions are equivalent.*

(1) f is (μ, λ)-continuous,
(2) f is $(b$-$\mathcal{H}_\sigma, \lambda)$-continuous and $(\mu L, \lambda)$-continuous.

Proof. Proof is trivial from Theorem 2.21. □

Theorem 4.4. *Let $(Z, \mu, \mathcal{H})$ be a H.G.T.S. for a function $f : (Z, \mu, \mathcal{H}) \to (W, \lambda)$. Then the following conditions are equivalent.*

(1) f is (μ, λ)-continuous,

(2) f is $(b\text{-}\mathcal{H}_\sigma, \lambda)$-continuous and $(D_b(c, \mathcal{H}_\sigma), \lambda)$-continuous.

Proof. Proof is trivial from Theorem 2.24. □

Theorem 4.5. *Let $(Z, \mu, \mathcal{H})$ be a strong H.G.T.S., for a function $f : (Z, \mu, \mathcal{H}) \to (W, \lambda)$. Then the following conditions are equivalent.*

(1) f is (μ, λ)-continuous,
(2) f is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuous and $(\pi^*\text{-}B\text{-}\mathcal{H}_\sigma, \lambda)$-continuous,
(3) f is $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuous and $(\alpha^*\text{-}B\text{-}\mathcal{H}_\sigma, \lambda)$-continuous,
(4) f is $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuous and $(\sigma^*\text{-}B\text{-}\mathcal{H}_\sigma, \lambda)$-continuous.

Proof. Proof is trivial from Theorem 3.13. □

Acknowledgment

The authors are highly grateful to editor and referees for their valuable comments and suggestions for improving the paper.

References

[1] A. Al-Omari and T. Noiri, *On operators in ideal minimal spaces*, Mathematica, **58(81)**(2016), no.1-2, 3–13.
[2] A. Al-Omari and T. Noiri, *Operators in minimal spaces with hereditary classes*, Mathematica, **61(84)**(2019), no. 2, 101-110.
[3] A. Al-Omari and T. Noiri, *Generalizations of Lindelöf spaces via hereditary classes*, Acta Univ. Sapientie Math., **13**(2021), no. 2, 281-291.
[4] A. Al-Omari and T. Noiri, *Generalizations of regular and normal spaces II*, Mathematica, **63(86)**(2021), no. 1, 3-12.
[5] Ahmad Al-Omari and Mohd. Salmi Md. Noorani, *Decomposition of continuity via b-open set*, Bol. Soc. Paran. Mat., **26**(2008), no. 1-2, 53-64.
[6] A. Csaszar, *Generalized topology, generalized continuity*, Acta Math. Hungar., **96**(2002), 351-357.
[7] A. Csaszar, *Extremally disconneted genealized topologies*, Annales Univ. Sci. Budapest., **47**(2004), 91-96.
[8] A. Csaszar, *Generalized open sets in generalized topologies*, Acta Mathematica Hungarica., **106**(2005), 53-56.
[9] A. Csaszar, *Modification of generalized topologies via hereditary classes*, Acta Mathematica Hungarica., **115**(2007), 29-36.
[10] E. Ekici, *On e-open sets, $\mathcal{DP}^*$-sets and $\mathcal{DP}\epsilon^*$-sets and decompositions of continuity*, Arabian Journal for Science and Engineering., **33**(2008), no. 2A, 269-282.
[11] E. Ekici, *Some generalizations of almost contra-super-continuity*, Filomat., **21**(2007), no. 2, 31-44.
[12] E. Ekici, *New forms of contra-continuity*, Carpathian Journal of Mathematis., **24**(2008), no. 1, 37-45.
[13] E. Ekici, *Generalized submaximal spaces*, Acta Math. Hungar., **134**(2012), no. 1-2, 132-138.
[14] K. Karuppayi, *A note on δ-$\mathcal{H}$-sets in GTS with hereditary classes*, Intern. J. Math. Anal., **5**(2014), 226-229.

[15] K. Karuppayi, *On decompositions of continuity and complete continuity with hereditary class*, J. Adv. Stud. Topol., **5**(2014), 18-26.

[16] M. Rajamani, V. Inthumathi and R. Ramesh, *Some new generalized topologies via hereditary classes*, Bol. Soc. Paran. Mat., **30**(2012), no. 2, 71-77.

[17] R. Ramesh and R. Mariappan, *Generalized open sets in hereditary generalized topological spaces*, J. Math. Comput. Sci., **5(2)**(2015), 149-159.

[18] R. Ramesh and Ahmad Al-Omari, *Decomposition of* $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$*-continuity*, Communicated.

[19] M. S. Sarsak, *On some properties of Generalized open sets in Generalized topological spaces*, Demonstratio Math., **46**(2013), no. 2,.

[20] X. Ge and Y. Ge, μ*-Separations in generalized topological spaces*, Appl. Math. J. Chinese Univ., **25**(2010), no. 2, 243-252.

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