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ON COMPLEX BIPOLAR FUZZY WEIGHTED AGGREGATION OPERATORS

MABRUKA ALI AND ADEM KILICMAN†

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Abstract. This study is aim to elucidate some series weighted averaging operators for aggregating the dissimilar complex bipolar fuzzy (CBF) sets by utilizing t-norm operations. The unpredictability in the data was controlled using membership degrees that are subset of real numbers, which can reduce some useful information and therefore impact decision-making results. As a refinement to these, the complex bipolar fuzzy set controls the unpredictability with the degrees whose range is extended from the real subset to the complex with the unit disk. Hence, it handles the two-dimensional information in a single set. Finally, some new averaging aggregation operators were developed, namely; complex bipolar weighted averaging, CBF ordered weighted average, and CBF hybrid averaging. Some numerical examples are also given to illustrate the extended operators.

1. Introduction

Aggregation plays a major role in many of the technical tasks we are faced with. The ordered weighted averaging operators, a parameterized family of aggregation operators that embody many of the well-known operators like the ordered weighted aggregation (OWA) operator, was proposed by Yager [30] to supply for aggregation lying between the Min and Max operators. Xu and Yager [27] extended some new aggregation operators. Xu [28] proposed a weighted averaging operator for aggregation of the different

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†*Corresponding author*

intuitionistic fuzzy numbers(IFN_S) while Xu and Yager [29] proposed fuzzy Bonferroni means aggregation operators. Wang and Liu [24] developed some interactive averaging aggregation operators to solve the multiple attribute decision making $MCDM$ problems. Kaur and Gary [14] developed some aggregation operators for the cubic intuitionistic fuzzy (IF) set. Ye [31] introduced some hybrid averaging and geometric aggregation operators with IFS environment to solve the $MCDM$ problems. Wei et al.[25] proposed bipolar fuzzy Hamacher weighted average operator while Wei et al. [26] developed some aggregation operators for aggregating hesitant bipolar fuzzy information. Lu et al. [16] investigate the $MADM$ problem with bipolar 2-tuple linguistic information. In [22], Tamir at al. extended a generalized improved score function to rank the $IVIFS$. Khameneh and Kılıçman [33] proposed a new aggregation method for processing the $MAGDM$ problems with M-polar fuzzy soft information. Gary and Pani [13] presented some weighted averaging aggregation operators for aggregating the different $CIFS$ utilizing t-norm operations.

Bipolar fuzzy sets and relations (BFS) are introduced in [34] and BFS is a generalization of fuzzy sets in [32] whose membership degree range in $[-1, 1]$. In a bipolar fuzzy set, the value 0 of membership degree of an element means that the element is irrelative to the identical property, and any value lies in the interval $(0, 1]$ of membership degree of an element indicates that the element satisfies the property (Positive information). In contrast to the membership degree that lies in the interval $[-1, 0)$ indicates that the element satisfies the implicit counter-property (negative information). Bipolar fuzzy information is two sides in coordination and decision making. For example, profit and loss, hostility and friendship, competition and cooperation, feedback and feedforward, and so on. Therefore, bipolar fuzzy set theory has been applied and studied speedily and increasingly. Thus, bipolar fuzzy sets not only have applications in mathematical theories but also in real-world problems [1], [18], and [19]. In the decision-making field, bipolarity is also used to select the best optimal solution according to positive and negative evaluations of their attributes or criteria [12] and [4]. Also, bipolarity has been used to express negative and positive information or goals as in [3], [5], [6], and [7].

Fuzzy complex numbers (FCN) are closed under the basic arithmetic operations namely; division, multiplication, addition, and subtraction ($\div, \times, +$ and $-$). Also, introducing a metric space of FCN allocated a good place to discuss the continuity and differentiability of complex fuzzy functions [5]. The work in [17] is considered to be the nearest approach to the work of [5]. They employed complex numbers to represent truth values and False values by expanding the range of truth and false values from the interval $[0, 1]$ to the complex plane. The complex-valued degrees of truth and false have an ability to be utilized in representing inconsistencies or paradoxes. This is the motivation to introduce in [20] approach. This approach emphasizes problems encountered in humanities, especially psychology and philosophy.

Complex fuzzy set (in short, CFS), [17] is a fuzzy set described by a complex-valued membership function, which the range lies in the unit disk in C:

$$\mu(x) : U \to \{a | a \in C, |a| \leq 1\}.$$

The innovation of CFS may appear in the further dimensional membership. The phase of grade of membership, denoted by $w(x)$ That is, without the phase membership the CFS reduces to traditional fuzzy set. Some advantages of CFS were summarized in [17]. The idea of generalizing the range of fuzzy set to a wider range of complex fuzzy set lies in its ability to represent semantics of uncertainty and periodicity information simultaneously. They were identifying the phase of degrees to distinguish the different meaning in different phases or levels for similar data/degrees. The uncertainty and periodicity semantics can be represented on the form of amplitude and phas term $r(x)$ and $w(x)$,respectively, of the complex numbers $r(x)e^{iw(x)}$. Besides that both values of $r(x)$ and $w(x)$ have the ability to be fuzzy set [32]. The complex fuzzy set is closed under the basic set theoretic operations: complements, union, intersection.

A novel interpretation of complex fuzzy membership grade and the concept of pure complex fuzzy classes were obtained in [22], in which the complex fuzzy membership grade can be represented in polar form and Cartesian form with two fuzzy components. Tamir and Kandel [23] developed an axiomatic for propositional complex fuzzy logic, which resolves most of the limitations of the complex fuzzy logic theory. Some constraints and limitation in the concept of CFS and its resolution to develop an axiomatic for propositional complex fuzzy logic are explained by Tamir et al. [21], [22], Tamir and Kandel [23].

Recently, Singh, [20] proposed the bipolar complex fuzzy concept lattice with its application. He studied the problem of calculating the periodic variation in bipolar information at the given phase of time. He proposed three methods and illustrative examples for the suitable representation of bipolar complex data sets. Singh represented the phase term in the complex fuzzy set as a real-valued component.

This paper is organized as the following. In the Section 2, we discuss some basic concepts related to CFS and $CBFS$. In section 3, we give some basic generalized operational laws of $CBFNs$ and their basic properties. In section 4, some weighted averaging aggregation operators, namely $CBFWA$, $CBFOWA$ and $CBFHA$ were developed.

2. Preliminaries

In this section, we introduce some necessary definitions for the complex bipolar fuzzy sets. Our construction will be based on the Zadeh [32] definition of fuzzy set as follows:

Definition 2.1. [32] A fuzzy set A in a universe of discourse U is characterised by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.

Further, Zadeh [32] has defined the following basic operators: Let A and B be defined as $A = (x, \mu_A(x)), x \in U$, $B = (x, \mu_B(x)), x \in U$. Then:

(1) Complement of A is shown as, $A^c = \{(x, 1 - \mu_A) : x, \in U\}$.
(2) Union between two sets is shown as, $A \cup B = \{(x, \mu_A \diamond \mu_B) : x \in U\}$.
(3) Intersection between two sets is shown as, $A \cap B = \{(x, \mu_A * \mu_B) : x \in U\}$.

Here, $\diamond$ and $*$ denote the s-norm and t-norm operators,respectively. Ramot in [17] extended the fuzzy set idea which was introduced by Zadeh in [32] and defined complex fuzzy set as follows: Let $CFS(U)$ be a set of all complex fuzzy sets in U. Then we have;

Definition 2.2. [17] A complex fuzzy set CFS A, defined on a universe of discourse U, is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in U$, a complex grade of membership is A.

Note that, by definition, the values $\mu_A(x)$ may receive all lying within the unit circle in the complex plane, and thus of the form $\mu_A(x) = r_A(x)e^{iw_A(x)}$, where $i = \sqrt{-1}$, all of $r_A(x)$ and $w_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$. The CFS may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in U\} = \{x, r_A(x)e^{iw_A(x)} : x \in U\}.$$

Definition 2.3. [35] Let A and B be two CFS on U, with complex-valued membership function $\mu_A = r_A(x)e^{jw_A(x)}$ and $\mu_B(x) = r_B(x)e^{jw_B(x)}$, $A \subseteq B$ if and only if $r_A \leq r_B$ and $w_A \leq w_B$.

Definition 2.4. [17] The complement of A is denoted by $A^c = \{(x, r^c_A(x)e^{jw^c_A(x)} : x \in U\}$ where $r^c_A(x) = 1 - r_A(x), w^c(x) = 2\pi - w_s(x)$.

Definition 2.5. [17] The complex fuzzy union of A and B, are denoted by $A \cup B$, with complex valued membership function $\mu_A(x)$ and $\mu_B(x), respectively$ is specified by a function:

$$u : \{a|a \in C, |a| \leq 1\} \diamond \{b|b \in C, |b| \leq 1\} \to \{d|d \in C, |d| \leq 1\}$$

where u assigns a complex value, $u(\mu_A(x), \mu_B(x)) = \mu_{(A\cup B)}(x)$ to all $x \in U$. The complex fuzzy union function u must satisfy at least the following axiomatic requirements, for any $a, b, c, b \in \{x|x \in C, |x| \leq 1\}$;

1 -$u(a, 0) = a$ (boundary conditions)
2 -if $|b| \leq |d|$ then, $|u(a, b)| \leq |u(a, d)|$ (monotonicity)
3 -$u(a, b) = u(b, a)$ (commutativity)
4 -$u(a, u(b, d)) = u(u(a, b), d).(associativity)$
5 -u is continuous function (continuity)
6 -$|u(a, a)| > |a|$(super idempotency)
7 -if $|a| \leq |b|$ and $|b| \leq |d|$ then $|u(a, b)| \leq |u(c, d)|$.

Definition 2.6. [17] The complex fuzzy intersection of A and B, are denoted by $A \cap B$, with complex valued membership function $\mu_A(x)$ and $\mu_B(x)$, respectively and specified by a function:

$$i : \{a|a \in C, |a| \leq 1\} * \{b|b \in C, |b| \leq 1\} \to \{d|d \in C, |d| \leq 1\}$$

i assigns a complex value, $i(\mu_A(x), \mu_B(x)) = \mu_{(A\cap B)}(x)$ to all $x \in U$. The complex fuzzy intersection function i must satisfy at least the following axiomatic requirements, for any $a, b, c, b \in \{x|x \in C, |x| \leq 1\}$

1 $|b| = 1$, if $i(a, b) = |a|$ (boundary conditions)
2 if $|b| \leq |d|$ then, $|i(a, b)| \leq |i(a, d)|$ (monotonicity)
3 $i(a, b) = i(b, a)$ (commutativity)
4 $i(a, i(b, d)) = i(i(a, b), d)$ (associativity)

5 i is continuous function (continuity)
6 $|i(a,a)| < |a|$(super idempotency)
7 if $|a| \leq |c|$ and $|b| \leq |d|$ then $|i(a,b)| \leq |i(c,d)|$.

Definition 2.7. [15] A function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called t-norm if it satisfies the boundary condition, monotonicity, commutativity and associativity.conversely,a function F defined by $F(a,b) = 1 - T(1-a, 1-b)$ for all $a, b \in [0,1]$ is called t-conorm.

Definition 2.8. [15] An Archimedean t-conorm (t-norm) function, T(or F), is a continuous norm(t-norm)satisfying the condition $T(a,a) < a$(or $F(a,a) > a$) for $a \in (0,1)$. A strict Archimedean t-conorm(t-norm)is strictly increasing t-conorm(t-norm).

Strict Archimedean t-norm T and t-conorm F can be expressed using continuous functions $f, g : [0,1] \to [0,\infty)$, respectively, as

$$T(a,b) = f^{-1}(f(a) + f(b)) \text{ and } F(a,b) = g^{-1}(g(a) + g(b))$$

where f is decreasing function with $f(1) = 0$, and g is an increasing function with $g(0) = 0$ and $g(a) = f(1-a)$. The concept of $CBFS$ was defined by Singh (2019) which is the extension of bipolar fuzzy set Zhang (1994)as follows:

Definition 2.9. [20] A complex Bipolar fuzzy set A, is characterised by positive $\mu_A^+(x)$ and negative $\mu_A^-(x)$ membership functions, that assign to any element x on a universe of discourse U. A complex-valued grade of both positive and negative memberships in A. By definition, the values of $\mu_A^+(x)$ and $\mu_A^-(x)$ may receive all values that lies within the unit disks in the complex plane, and are of the form $\mu{+}_A(x) = r^+(x)e^{iw_{r^+}(x)}$ for positive membership function in A and $\mu{-}_A(x) = r^-(x)e^{iw_{r^-}(x)}$ for negative membership function in A, where $i = \sqrt{-1}$, each of $r_A^+(x)$ is belong to $[0,1]$, $r_A^-(x)$ is belong to $[-1,0]$ and w_{μ^+}, $w_{\mu^+} \in [0,2\pi]$.
We represent the $CBFS$ A as

$$\mu_A^+(x) : U \to r^+(x)e^{iw_{r^+}(x)}$$

and

$$\mu_A^-(x) : U \to r^-(x)e^{iw_{r^-}(x)}$$

where

$$A = \{(x, \mu_A^+(x), \mu_A^-(x)) : x \in U\}$$

and w_{μ^+} and w_{μ^+} are real-valued interval in $(0,2\pi]$.

Definition 2.10. [20] Suppose A and B are two CBFS expressed as

$$A = \{x, \mu_A^+(x), \mu_A^-(x) : x \in U\},$$

and

$$B = \{x, \mu_B^+(x), \mu_B^-(x) : x \in U\},$$

the set operations union, intersection and complement of CBFS were defined as follows:

$$\begin{aligned}
A \cup B &= \{x, \mu^+_{A\cup B}(x), \mu^-_{A\cup B}(x) : x \in U\} \\
&= \{(\max(\mu^+_A(x), \mu^+_B(x), \min(\mu^-_A(x), \mu^-_B(x))\}, \\
A \cap B &= \{x, \mu^+_{A\cap B}(x), \mu^-_{A\cap B}(x) : x \in U\} \\
&= \{(\min(\mu^+_A(x), \mu^+_B(x), \max(\mu^-_A(x), \mu^-_B(x))\}, \\
A^c &= \{x, \mu^+_{A^c}(x), \mu^-_{A^c}(x) : x \in U\} \\
&= \{x, 1 - \mu^+_A(x), 1 - \mu^-_A(x)\}.
\end{aligned}$$

3. Operational laws in complex bipolar fuzzy numbers

In the following, some operational laws of the $CBFNs$ are defined.

3.1. Score and Accuracy Functions.

3.1. Score and Accuracy Functions. In subsection, we define the score function following as

Definition 3.1. The score function S of $c = \langle \mu^+ e^{iw_{\mu^+}}, \mu^- e^{iw_{\mu^-}} \rangle$ is shown as

$$S(c) = \frac{1}{2}\left(1 + \mu^+ + \mu^-\right) + \frac{1}{2\pi}\left(w_{\mu^+} + w_{\mu^-}\right),\ S(c) \in [0, 1].$$

Definition 3.2. The accuracy function H of

$$c = \langle \mu^+ e^{iw_{\mu^+}}, \mu^- e^{iw_{\mu^-}} \rangle$$

is shown as

$$H(c) = (\mu^+ + \mu^-) + \frac{1}{2\pi}(w_{\mu^+} + w_{\mu^-}),\ H(c) \in [0, 1].$$

Now, we define an order relation between two $CBFNs$ and it is shown as below

(1) If $S(c_1) < S(c_2)$ then $c_1 < c_2$,
(2) If $S(c_1) = S(c_2)$ then $c_1 = c_2$

and

(i) If $H(c_1) > H(c_2)$ then $c_1 > c_2$,
(ii) If $H(c_1) = H(c_2)$ then $c_1 = c_2$

where

$$c_1 = \langle \mu_1^+ e^{iw_{\mu_1^+}}, \mu_1^- e^{iw_{\mu_1^-}} \rangle \text{ and } c_2 = \langle \mu_2^+ e^{iw_{\mu_2^+}}, \mu_2^- e^{iw_{\mu_2^-}} \rangle.$$

3.2. Some basic operational Laws of CBFNs. In subsection, the various the basic operational laws of $CBFNs$ were shown as

Definition 3.3. Let $c_1 = \langle \mu_1^+ e^{iw_{\mu_1^+}}, -\mu_1^- e^{iw_{\mu_1^-}} \rangle$ and $c_2 = \langle \mu_2^+ e^{iw_{\mu_2^+}}, -\mu_2^- e^{iw_{\mu_2^-}} \rangle$ be two CBFNs, $\lambda > 0$. Then,

$$\begin{aligned}
c_1 \oplus c_2 &= \langle (f^{-1}(f(\mu_1^+ + f(\mu_2^+)))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_1^+}}{2\pi}+f(\frac{w_{\mu_2^+}}{2\pi}))),} \\
&\qquad -(g^{-1}(g(|\mu_1^-|) + g(|\mu_2^-|))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_1^-}}{2\pi}+g(\frac{w_{\mu_2^-}}{2\pi}))))\rangle} \\
c_1 \otimes c_2 &= \langle (g^{-1}(g(\mu_1^+ + g(\mu_2^+)))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_1^+}}{2\pi}+t(\frac{w_{\mu_2^+}}{2\pi}))),} \\
&\qquad -(f^{-1}(f(|\mu_1^-|) + f(|\mu_2^-|))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_1^-}}{2\pi}+f(\frac{w_{\mu_2^-}}{2\pi}))))\rangle} \\
\lambda c_1 &= \langle (f^{-1}(\lambda f(\mu_1^+)))e^{i2\pi(f^{-1}(\lambda f(\frac{w_{\mu_1^+}}{2\pi})}), -\langle (g^{-1}(\lambda g(|\mu_1^-)|))e^{i2\pi(g^{-1}(\lambda g(\frac{w_{\mu_1^-}}{2\pi}))}\rangle \\
(c_1)^\lambda &= \langle (g^{-1}(\lambda g(\mu_1^+)))e^{i2\pi(g^{-1}(\lambda g(\frac{w_{\mu_1^+}}{2\pi})}), -(f^{-1}(\lambda f(|\mu_1^-|)))e^{i2\pi(f^{-1}(\lambda f(\frac{w_{\mu_1^-}}{2\pi}))}\rangle
\end{aligned}$$

where f, g two functions defined in the Definition 2.8

Theorem 3.4. *If c_1 and c_2 be any two CBFNs and $\lambda > 0$, then $c_1 \oplus c_2$, λc_1, $c_1 \otimes c_2$ and c_1^λ are also CBFNs.*

Proof. We only prove part $c_1 \oplus c_2$. The other parts follow a similar. By Definition 3.3, and $c_1 = \langle \mu_1^+ e^{iw_{\mu_1^+}},\ \mu_1^- e^{iw_{\mu_1^-}} \rangle$ and $c_2 = \langle \mu_2^+ e^{iw_{\mu_2^+}},\ \mu_2^- e^{iw_{\mu_2^-}} \rangle$ we have

$$\begin{aligned}
c_1 \oplus c_2 &= \langle (f^{-1}(f(\mu_1^+ + f(\mu_2^+)))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_1^+}}{2\pi}+f(\frac{w_{\mu_2^+}}{2\pi}))),} \\
&\qquad -(g^{-1}(g(|\mu_1^-|) + g(|\mu_2^-|))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_1^-}}{2\pi}+g(\frac{w_{\mu_2^-}}{2\pi}))))\rangle} \\
c_1 \oplus c_2 &= \langle (f^{-1})(f)[\mu_1^+ + \mu_2^+]e^{i2\pi(f^{-1})f[\frac{w_{\mu_1^+}}{2\pi}+\frac{w_{\mu_2^+}}{2\pi}]}, \\
&\qquad -[(g^{-1})g[|\mu_1^-| + |\mu_2^-|)]e^{i2\pi(g^{-1})g[\frac{w_{\mu_1^-}}{2\pi}+\frac{w_{\mu_2^-}}{2\pi}]}]\rangle \\
&= (\mu_1^+ + \mu_2^+)e^{i[w_{\mu_1^+}+w_{\mu_2^+}]}, -[|\mu_1^-| + |\mu_2^-|]e^{i[w_{\mu_1^-}+w_{\mu_2^-}]}\rangle \\
&= (\mu^+ e^{iw_{\mu^+}}, -\mu^- e^{iw_{\mu^-}})
\end{aligned}$$

where

$$\mu^+ = \mu_1^+ + \mu_2^+,\ w_{\mu^+} = w_{\mu_1^+} + w_{\mu_2^+},\ \mu^- = |\mu_1^-| + |\mu_2^-| \text{ and } w_{\mu^-} = [w_{\mu_1^-} + w_{\mu_2^-}].$$

So, $c_1 \oplus c_2$ is CBFNs. □

Theorem 3.5. *Let c_1, c_2 be two CBFNs and let $\lambda_1, \lambda_2, \lambda_3 > 0$ be three real numbers. Then, we have*

(1) $c_1 \oplus c_2 = c_2 \oplus c_1$;
(2) $c_1 \otimes c_2 = c_2 \otimes c_1$;
(3) $\lambda_1(c_1 \oplus c_2) = \lambda_1 c_1 \oplus \lambda_1 c_2$;
(4) $(c_1 \otimes c_2)^{\lambda_1} = c_1^{\lambda_1} \otimes c_2^{\lambda_1}$;
(5) $\lambda_1 c_1 \oplus \lambda_2 c_2 \lambda = (\lambda_1 + \lambda_2)c_1$;
(6) $c_1^{\lambda_1} \otimes c_1^{\lambda_2} = c_1^{\lambda_1+\lambda_2}$.

Proof. (1) By Definition (3.2), we get

$$\begin{aligned}
c_1 \oplus c_2 &= \langle (f^{-1}(f(\mu_1^+ + f(\mu_2^+)))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_1^+}}{2\pi}+f(\frac{w_{\mu_2^+}}{2\pi})))}, \\
&\qquad -(g^{-1}(g(|\mu_1^-|)+g(|\mu_2^-|))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_1^-}}{2\pi}+g(\frac{w_{\mu_2^-}}{2\pi}))))}\rangle \\
c_1 \oplus c_2 &= \langle (f^{-1}(f(\mu_1^+ + f(\mu_2^+)))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_1^+}}{2\pi}+f(\frac{w_{\mu_2^+}}{2\pi})))}, \\
&\qquad -(g^{-1}(g(|\mu_1^-|)+g(|\mu_2^-|))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_1^-}}{2\pi}+g(\frac{w_{\mu_2^-}}{2\pi}))))}\rangle \\
&= \langle (f^{-1}(f(\mu_2^+ + f(\mu_1^+)))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_2^+}}{2\pi}+f(\frac{w_{\mu_1^+}}{2\pi})))}, \\
&\qquad -(g^{-1}(g(|\mu_2^-|)+g(|\mu_1^-|))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_2^-}}{2\pi}+g(\frac{w_{\mu_2^-}}{2\pi}))))}\rangle \\
&= c_2 \oplus c_1.
\end{aligned}$$

□

Theorem 3.6. *Let c_1, c_2, be two CBFNs and $\lambda > 0$. Then, we have*

(1) $(c_1^c)^\lambda = (\lambda c_1)^c$;
(2) $\lambda(c_1)^c = (c_1^\lambda)^c$;
(3) $(c_1 \oplus_2)^c = c_1^c \otimes c_2^c$;
(4) $(c_1 \otimes c_2)^c = c_1^c \oplus c_2^c$.

Proof. It obvious on using definitions. □

Now in the next, some new averaging aggregation operators of complex bipolar fuzzy environment including CBFWA, CBFOWA and CBFHA are discussed.

4. Complex Bipolar Fuzzy Aggregation Operators

4.1. Complex bipolar Weighted Averaging CBFWA Operators. Now, we present the following definition.

Definition 4.1. Let $c_j = \langle \mu_j^+ e^{iw_{\mu_j^+}}, \mu_j^- e^{iw_{\mu_j^-}} \rangle (j = 1, 2, \ldots, n)$ be a collection of all CBFNs with corresponding weights $w = (w_1, w_2, \ldots, w_n)^T$ such as $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$ and let $CBFWA : \Psi^n \to \Psi$, if

$$CBFWA(c_1, c_2, \ldots, c_n) = w_1 c_1 \oplus w_2 c_2 \oplus \ldots \oplus w_n c_n \tag{4.1}$$

then, CBFWA is called CBFWA operator.

Theorem 4.2. *Let*

$$CBFNsc_j = \langle \mu_j^+ e^{iw_{\mu_j^+}}, \mu_j^- e^{iw_{\mu_j^-}} \rangle (j = 1, 2, \ldots, n)$$

be a collection of $CBFN$ then by utilizing CBFWA operator is also CBFN and

$$\begin{aligned}
CBFWA(c_1, c_2, \ldots, c_n) &= \langle (f^{-1}(\sum_{j=1}^{n} w_j f(\mu_j^+))))e^{i2\pi(f^{-1}(\sum_{j=1}^{n} w_j f(\frac{w_{\mu_j^+}}{2\pi})))}, \\
&\qquad -(g^{-1}(\sum_{j=1}^{n} w_j g(|\mu_j^-|))))e^{i2\pi(g^{-1}(\sum_{j=1}^{n} w_j g(\frac{w_{\mu_j^-}}{2\pi})))}\rangle.
\end{aligned}$$

Proof. The first result we get from Def (4.1); Now, by utilizing mathematical induction we get:

Step 1: For $n = 2$, since $c_1 = \langle \mu_1^+ e^{iw_{\mu_1^+}}, \mu_1^- e^{iw_{\mu_1^-}} \rangle$ and $c_2 = \langle \mu_2^+ e^{iw_{\mu_2^+}}, \mu_2^- e^{iw_{\mu_2^-}} \rangle$. Thus, we have

$$\begin{aligned} w_1 c_1 &= \langle (f^{-1}(w_1 f(\mu_1^+))) e^{i2\pi(f^{-1} w_1 f(\frac{w_{\mu_1^+}}{2\pi})))}, \\ &\qquad -(g^{-1}(w_1 g(|\mu_1^-|))) e^{i2\pi(g^{-1}(w_1 g(\frac{w_{\mu_1^-}}{2\pi})))} \rangle \end{aligned}$$

and

$$\begin{aligned} w_2 c_2 &= (g^{-1}(g(g^{-1}(w_1 g(\mu_1^-)))) + g(g^{-1}(w_2 f(\mu_2^-))))), \\ &\qquad -(g^{-1}(w_2 g(|\mu_2^-|))) e^{i2\pi(g^{-1}(w_2 g(\frac{w_{\mu_2^-}}{2\pi})))} \rangle. \end{aligned}$$

Hence, we get

$$\begin{aligned} CBFWA(c_1, c_2) &= w_1 c_1 \oplus w_2 c_2 \\ &= \langle (f^{-1}(f(f^{-1}(w_1 f(\mu_1^+))) + f(f^{-1}(w_2 f(\mu_2^+))))) \\ &\times e^{i2\pi(f^{-1}(f(f^{-1}(w_1 f(\frac{w_{\mu_1^+}}{2\pi}))) + f(f^{-1}(w_2 f(\frac{w_{\mu_2^+}}{2\pi})))}, \\ &- (g^{-1}(g(g^{-1}(w_1 f(|\mu_1^-|))) + g(g^{-1}(w_2 g(|\mu_2^-|))))) \\ &\times e^{i2\pi(g^{-1}(g(g^{-1}(w_1 g(\frac{w_{\mu_1^-}}{2\pi}))) + g(g^{-1}(w_2 g(\frac{w_{\mu_2^-}}{2\pi})))} \rangle \\ &= \langle (f^{-1}(\sum_{j=1}^{2} w_j f(\mu_j^+))) e^{i2\pi(f^{-1}(\sum_{j=1}^{2} w_j f(\frac{w_{\mu_j^+}}{2\pi})))}, \\ &- (g^{-1}(\sum_{j=1}^{2} w_j g(|\mu_j^-|))) e^{i2\pi(g^{-1}(\sum_{j=1}^{2} w_j g(\frac{w_{\mu_j^-}}{2\pi})))} \rangle. \end{aligned}$$

Thus, holds for $n = 2$.

Step 2: If Eq(4.1) holds for $n = k$, then

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_k = \langle (f^{-1}(\sum_{j=1}^{m} w_j f(\mu_j^+))) e^{i2\pi(f^{-1}(\sum_{j=1}^{m} w_j f(\frac{w_{\mu_j^+}}{2\pi})))}, \\ - (g^{-1}(\sum_{j=1}^{m} w_j g(|\mu_j^-|))) e^{i2\pi(g^{-1}(\sum_{j=1}^{m} w_j g(\frac{w_{\mu_j^-}}{2\pi})))} \rangle \end{aligned}$$

then for $n = k+1$, we get

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_{k+1}) &= CBFWA(c_1, c_2, \ldots {}_k) \oplus w_{k+1} c_{k+1} \\ &= \langle (f^{-1}(\sum_{j=1}^{m} w_j f(\mu_j^+)))e^{i2\pi(f^{-1}(\sum_{j=1}^{m} w_j f(\frac{w_{\mu_j^+}}{2\pi})))}, \\ &- (g^{-1}(\sum_{j=1}^{m} w_j g(|\mu_j^-|))) e^{i2\pi(g^{-1}(\sum_{j=1}^{m} w_j g(\frac{w_{\mu_j^-}}{2\pi})))} \rangle \\ &\oplus \langle (f^{-1}(w_{k+1} f(\mu_{k+1}^+)))e^{i2\pi(f^{-1}(w_{k+1} f(\frac{w_{\mu_{k+1}^+}}{2\pi})))}, \\ &- (g^{-1}(w_{k+1} g(|\mu_{k+1}^-|))) e^{i2\pi(g^{-1}(w_{k+1} g(\frac{w_{\mu_{k+1}^-}}{2\pi})))} \rangle \\ &= \langle (f^{-1}(\sum_{j=1}^{k+1} w_j f(\mu_j^+)))e^{i2\pi(f^{-1}(\sum_{j=1}^{k+1} w_j f(\frac{w_{\mu_j^+}}{2\pi})))}, \\ &- (g^{-1}(\sum_{j=1}^{k+1} w_j g(|\mu_j^-|))) e^{i2\pi(g^{-1}(\sum_{j=1}^{k+1} w_j g(\frac{w_{\mu_j^-}}{2\pi})))} \rangle. \end{aligned}$$

Thus, when $n = k+1$, is also true and Eq (4.1) holds for each n. □

Proposition 4.3. If $w_{\mu_j^+}$, $w_{\mu_j^-} = 0$ then, the CBFWA operator convert to BFWA operator in CBS environment.

Proof Since $w_{\mu_j^+}, w_{\mu_j^-} = 0$. Therefore, Eq.(4.1) converts to:

$$CBFWA(C_1, C_2, \ldots, C_n) = \langle f^{-1}(\sum_{j=1}^{n} w_j f(\mu^+)), g^{-1}(\sum_{j=1}^{n} w_j g(\mu_j)) \rangle.$$

Proposition 4.4. If $w_{\mu_j^+}$, $w_{\mu_j^-} = 0$ all j then, the CBFWA operator become a weighted operator in F environment.

Proof is similar to the Proposition (4.1)

Example 4.5. Let

$$\begin{aligned} c_1 &= \langle 0.3e^{i2\pi(0.2)}, -0.1e^{i2\pi(0.5)} \rangle, \; c_2 = \langle 0.4e^{i2\pi(0.8)}, -0.4e^{i2\pi(0.7)} \rangle, \\ c_3 &= \langle 0.6e^{i2\pi(0.5)}, -0.2e^{i2\pi(0.3)} \rangle, \; c_4 = \langle 0.1e^{i2\pi(0.6)}, -0.3e^{i2\pi(0.2)} \rangle \end{aligned}$$

be four $CBFNs$ and $w = (0.15, 0.3, 0.1, 0.45)^T$ be the corresponding weight vector of $c_j (j = 1, 2, 3, 4)$. Then, using Eq.(4.1) becomes

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_n) \quad = \quad & \langle (1 - \prod_{j=1}^{n} (1 - \mu_j^+)^{w_j}) e^{i2\pi(1 - \prod_{j=1}^{n} (1 - \frac{w_{\mu_j^+}}{2\pi})^{w_j})} \\ & - \prod_{j=1}^{n} (|\mu_j^-|)^{w_j} e^{i2\pi \prod_{j=1}^{n} (\frac{w_{\mu_j^-}}{)} 2\pi)^{w_j}} \rangle. \end{aligned}$$

Now, we have

$$\begin{aligned}
\prod_{j=1}^{4}\left(1-\mu_j^{+}\right)^{w_j} &= (1-0.3)^{0.15}\times(1-0.4)^{0.3}\times(1-0.6)^{0.1}\times(1-0.1)^{0.45}\\
&= 0.9479\times 0.8579\times 0.9124\times 0.9537 = 0.7076\\
\prod_{j=1}^{4}\left(|\mu_j^{-}|\right)^{w_j} &= (|-0.1|)^{0.15}\times(|-0.4|)^{0.3}\times(|-0.2|)^{0.1}\times(|-0.3|)^{0.45}\\
&= 0.7079\times 0.7597\times 0.8513\times 0.5817 = 0.2663\\
\prod_{j=1}^{4}\left(1-\frac{w_{\mu_j^{+}}}{2\pi}\right)^{w_j} &= (1-0.2)^{0.15}\times(1-0.8)^{0.3}\times(1-0.5)^{0.1}\times(1-0.6)^{0.45}\\
&= 0.9671\times 0.6170\times 0.9330\times 0.6622 = 0.3687\\
\prod_{j=1}^{4}\left(\frac{w_{\mu_j^{-}}}{2\pi}\right)^{w_j} &= (0.5)^{0.15}\times(0.7)^{0.3}\times(0.3)^{0.1}\times(0.2)^{0.45}\\
&= 0.9013\times 0.8985\times 0.8866\times 0.4847 = 0.3480\\
CBFWA(c_1,c_2,c_3,c_4) &= \langle 1-0.7076)e^{i2\pi(1-0.3687)}, -0.2663e^{i2\pi(0.3480)}\rangle\\
&= \langle 0.2924e^{i2\pi(0.6313)}, -0.2663e^{i2\pi(0.3480)}\rangle.
\end{aligned}$$

Proposition 4.6. (Idempotency) Let C be $CBFN$ and if $c_j = c$ for all $j = 1, 2, \dots, n$, then,

$$CBFWA(c_1, c_2, \dots, c_n) = c.$$

Proof. Let $c = \langle \mu^+ e^{iw_{\mu^+}}, \mu^- e^{iw_{\mu^-}}\rangle$ and $c_j = \langle \mu_j^+ e^{iw_{\mu_j^+}}, \mu_j^- e^{iw_{\mu_j^-}}\rangle (j = 1, 2, \dots, n)$ be two $CBFNs$ such as $c_j = c$ which implies that $\mu_j^+ = \mu^+, \mu_j^- = \mu^-$, $w_{\mu_j^+} = w_{\mu^+}$ and $w_{\mu_j^-} = w_{\mu^-}$ for all j. Also, we have $w_j > 0$ such that $\sum_{j=1}^{n} w_j = 1$. Then,

$$\begin{aligned}
CBFWA(c_1, c_2, \dots,_n) &= \langle (f^{-1}\left(\sum_{j=1}^{n} w_j f(\mu^+))\right)) e^{i2\pi(f^{-1}\left(\sum_{j=1}^{n} w_j f(\frac{w_{\mu^+}}{2\pi})\right))},\\
&\quad -(s^{-1}\left(\sum_{j=1}^{n} w_j s(|\mu^-|))\right)) e^{i2\pi(s^{-1}(\sum_{j=1}^{n} w_j s(\frac{w_{\mu^-}}{2\pi})))}\rangle\\
&= \langle (f^{-1}(f(\mu^+)))) e^{i2\pi(f^{-1}(f(\frac{w_{\mu^+}}{2\pi})))},\\
&\quad -(g^{-1}(g(|\mu^-|)))) e^{i2\pi(g^{-1}(g(\frac{w_{\mu^-}}{2\pi})))}\rangle\\
&= \langle \mu^+ e^{iw_{\mu^+}}, -|\mu^-| e^{iw_{\mu^-}}\rangle = c.
\end{aligned}$$

Hence, $CBFWA(c_1, c_2, \dots,_n) = c$. □

Proposition 4.7. (Monotonicity) Let

$$c_j = \langle \mu_{c_j}^+ e^{iw_{\mu_{c_j}^+}}, \mu_{c_j}^- e^{iw_{\mu_{c_j}^-}}\rangle$$

and

$$\beta_j = \langle \mu_{\beta_j}^+ e^{iw_{\mu_{\beta_j}^+}}, \mu_{\beta_j}^- e^{iw_{\mu_{\beta_j}^-}}\rangle$$

be two collections of $CBFN$ satisfying

$$\mu^+_{c_j} \le \mu^+_{\beta_j},\ \mu^-_{c_j} \le \mu^-_{\beta_j},\ w_{\mu^+_{c_j}} \le w_{\mu^+_{\beta_j}}$$

and $w_{\mu^-_{c_j}} \le w_{\mu^-_{\beta_j}}$ for all j then

$$CBFWA(c_1, c_2, \ldots, c_n) \le CBFWA(\beta_1, \beta_2, \ldots, \beta_n).$$

Proof. For two CBFN we have

$$c_j = \langle \mu^+_{c_j} e^{iw_{\mu^+_{c_j}}}, \mu^-_{c_j} e^{iw_{\mu^-_{c_j}}} \rangle$$

and

$$\beta_j = \langle \mu^+_{\beta_j} e^{iw_{\mu^+_{\beta_j}}}, \mu^-_{\beta_j} e^{iw_{\mu^-_{c^\star_j}}} \rangle$$

such that $\mu^+_{c_j} \le \mu^+_{\beta_j}$, $\mu^-_{c_j} \le \mu^-_{\beta_j}$, $w_{\mu^+_{c_j}} \le w_{\mu^+_{\beta_j}}$ and $w_{\mu^-_{c_j}} \le w_{\mu^-_{\beta_j}}$ □

$$f^{-1}\left(\sum_{j=1}^{n} w_j f(\mu^+_{c_j})\right) \le f^{-1}\left(\sum_{j=1}^{n} w_j f(\mu^+_{\beta_j})\right)$$

$$g^{-1}\left(\sum_{j=1}^{n} w_j g(\mu^-_{c_j})\right) \le g^{-1}\left(\sum_{j=1}^{n} w_j g(|\mu^-_{\beta_j})|\right)$$

$$f^{-1}\left(\sum_{j=1}^{n} w_j f(\frac{w_{\mu^+_{c_j}}}{2\pi})\right) \le f^{-}\left(\sum_{j=1}^{n} w_j f(\frac{w_{\mu^+_{\beta_j}}}{2\pi})\right)$$

and

$$g^{-1}\left(\sum_{j=1}^{n} w_j g(\frac{w_{\mu^-_{c_j}}}{2\pi})\right) \le g^{-}\left(\sum_{j=1}^{n} w_j g(\frac{w_{\mu^-_{\beta_j}}}{2\pi})\right)$$

by utilizing the score function, we have

$$\begin{aligned} S(CBFWA(c_1, c_2, \ldots, c_n)) &= \frac{1}{2}(1 + f^{-1}\left(\sum_{j=1}^{n} w_j f(\mu^+_{c_j})\right) + g^{-1}\left(\sum_{j=1}^{n} w_j g(|\mu^-_{c_j}|)\right) \\ &\quad + \frac{1}{2\pi}(2\pi f^{-1}\left(\sum_{j=1}^{n} w_j f(\frac{w_{\mu^+_{c_j}}}{2\pi})\right) \\ +(2\pi g^{-1}\left(\sum_{j=1}^{n} w_j g(\frac{w_{\mu^-_{c_j}}}{2\pi})\right) &\le \frac{1}{2}(1 + f^{-1}\left(\sum_{j=1}^{n} w_j f(\mu^+_{\beta_j})\right) + g^{-1}\left(\sum_{j=1}^{n} w_j g(|\mu^-_{\beta_j}|)\right) \\ &\quad + \frac{1}{2\pi}(2\pi f^{-1}\left(\sum_{j=1}^{n} w_j f(\frac{w_{\mu^+_{\beta_j}}}{2\pi})\right) - (2\pi g^{-1}\left(\sum_{j=1}^{n} w_j g(\frac{w_{\mu^-_{\beta_j}}}{2\pi})\right) \\ &= S(CBFWA(\beta_1, \beta_2, \ldots, \beta_n)). \end{aligned}$$

Hence, we get

$$CBFWA(c_1, c_2, \ldots, c_n) \le (CBFWA(\beta_1, \beta_2, \ldots, \beta_n).$$

Proposition 4.8. (Boundedness.) Let

$$c_j = \langle \mu_j^+ e^{iw_{c_j}}, \mu_j^- e^{iw_{\mu_j^-}} \rangle (j = 1, 2, \dots, n),$$

be a collection of $CBFNs$ and let $[c]^+ = \max_j c_j$ where $[\mu^+]^+ = \max_j(\mu_j^+)$, $w_{\mu^+}^+$ and let $[c]^- = \min_j c_j$ where $[\mu^+]^- = \min(\mu_j^+)$, $w_{\mu^+}^- = \min w_{\mu_j^+}$, $[\mu^-]^+ = \max_j(\mu_j^-)$ and $w_{\mu^-}^+ = \max_j((w_{\mu^-}))$, $[\mu^-]^- = \min_j(\mu_j^-)$ and $w_{\mu^-}^- = \min_j(w_{\mu_{\mu_j^-}})$. Then, we have

$$c^- \leq CBFWA(c_1, c_2, \dots, c_n) \leq c^+.$$

Proof. Let $CBFWA(c_1, c_2, \dots, c_n) = \{\mu_s^+ e^{iw_{\mu_s^+}}, \mu_s^- e^{iw_{\mu_s^-}}\}$. For a $CBFNC_j$, we have

$$\min_j\{\mu_j^+\} \leq \mu_j^+ \leq \max_j\{\mu_j^+\} \text{ and } \min_j\{\mu_j^-\} \leq \mu_j^- \leq \max_j\{\mu_j^-\}.$$

Therefore,

$$f^{-1}\left(\sum_{j=1}^n w_j f(\min_j\{\mu_j^+\})\right) \leq f^{-1}\left(\sum_{j=1}^n w_j f(\mu_j^+)\right) \leq f^{-1}\left(\sum_{j=1}^n w_j f(\max_j\{\mu_j^+\})\right)$$

and

$$g^{-1}\left(\sum_{j=1}^n w_j g(\min_j\{\mu_j^-\})\right) \leq g^{-1}\left(\sum_{j=1}^n w_j g(\mu_j^-)\right) \leq g^{-1}\left(\sum_{j=1}^n w_j g(\max_j\{\mu_j^-\})\right)$$

which

$$\min_j\{\mu_j^+\} \leq \mu_s^+ \leq \max_j\{\mu_j^+\} \text{ and } \min_j\{\mu_j^-\} \leq \mu_s^- \leq \max_j\{\mu_j^-\},$$

i.e,

$$(\mu^-)^- \leq \mu_s^+ \leq (\mu^+)^+ \text{ and } (\mu^-)^- \leq \mu_s^- \leq (\mu^-)^+.$$

Similarly, we get

$$w_{\mu^+}^- \leq w_{\mu_s^+} \leq w_{\mu^+}^+ \text{ and } w_{\mu^+}^- \leq w_{\mu_s^-} \leq w_{\mu^-}^+.$$

Hence, we have

$$\begin{aligned} s(c)^- &= \frac{1}{2}(1 + (\mu^+)^- + (\mu^-)^+) + \frac{1}{2\pi}(w_{\mu^+}^- + w_{\mu^-}^+) \\ &\leq \frac{1}{2}(1 + \mu_s^+ + \mu_s^-) + \frac{1}{2\pi}(w_{mu_s^+} + w_{\mu_s^-}) = S(CBFWA(c_1, c_2, \dots, c_n)) \end{aligned}$$

and

$$\begin{aligned} s(c)^+ &= \frac{1}{2}(1 + (\mu^+)^+ + (\mu^-)^-) + \frac{1}{2\pi}(w_{\mu^+}^+ + w_{\mu^-}^-) \\ &\geq \frac{1}{2}(\mu_s^+ - \mu_s^-) + \frac{1}{2\pi}(w_{mu_s^+} - w_{\mu_s^-}) = S(CBFWA(c_1, c_2, \dots, c_n)). \end{aligned}$$

Therefore, by utilizing Definition 4.2, we have

$$c^- \leq CBFWA(c_1, c_2, \dots, c_n) \leq c^+. \quad \square$$

Proposition 4.9. (Shift Invariance) Let $c_j (j = 1, 2, \dots, n)$ be the collection of CBFNs and CBFN$c^\star$, we have

$$CBFWA(c_1 \oplus c^\star, c_2 \oplus c^\star, \dots, c_n \oplus c^\star) = CBFWA(c_1, c_2, \dots, c_n) \oplus c^\star.$$

Proof. Let $c_j = \langle \mu_j^+ e^{iw_{\mu_j^+}}, \mu_j^- e^{iw_{\mu_j^-}} \rangle$ and $c^\star = \langle \mu_{c^\star}^+ e^{iw_{\mu_{c^\star}^+}}, \mu_{c^\star}^- e^{iw_{\mu_{c^\star}^-}} \rangle$ be CBFNs. Then, by utilizing Def (3.2) for CBFNs for each j, we have

$$\begin{aligned} c_j \oplus c^\star &= \langle (f^{-1}(f(\mu_j^+) + f(\mu_{c^\star}^+)))e^{i2\pi(f^{-1}(f(\frac{w_{\mu_j^+}}{2\pi})+f(\frac{w_{\mu_{c^\star}^+}}{2\pi})))} \\ &\quad -(g^{-1}(g(|\mu_j^-|) + g(|\mu_{c^\star}^-|)))e^{i2\pi(g^{-1}(g(\frac{w_{\mu_j^-}}{2\pi})+g(\frac{w_{\mu_{c^\star}^-}}{2\pi})))} \rangle. \end{aligned}$$

Now, utilizing Eq.(4.1), we get

$$\begin{aligned} CBFWA(c_1 \oplus c^\star, c_2 \oplus c^\star, \ldots, c_n \oplus c^\star) &= \langle ((f^{-1}(\sum_{j=1}^n w_j f(f^{-1}(f(\mu_j^+) + f(\mu_{c^\star}^+))) \\ &\quad e^{i2\pi(f^{-1}(\sum_{j=1}^n w_j f(f^{-1}(f(\frac{w_{\mu_j^+}}{2\pi})+f(\frac{w_{\mu_{c^\star}^+}}{2\pi)})))}, \\ &\quad -(g^{-1}(\sum_{j=1}^n w_j g(g^{-1}(g(|\mu_j^-|) + g(|\mu_{c^\star}^-|))) \\ &\quad e^{i2\pi(g^{-1}(\sum_{j=1}^n w_j g(g^{-1}(g(\frac{w_{\mu_j^-}}{2\pi})+g(\frac{w_{\mu_{c^\star}^-}}{2\pi)})))} \\ &= \langle ((g^{-1}(\sum_{j=1}^n w_j (f(\mu_j^+) + f(\mu_{c^\star}^+))) \\ &\quad e^{i2\pi(f^{-1}(\sum_{j=1}^n w_j (f(\frac{w_{\mu_j^+}}{2\pi})+f(\frac{w_{\mu_{c^\star}^+}}{2\pi)})))}, \\ &\quad -(g^{-1}(\sum_{j=1}^n w_j (g|(\mu_j^-)| + g(|\mu_{c^\star}^-|))) \\ &\quad e^{i2\pi(g^{-1}(\sum_{j=1}^n w_j (g(\frac{w_{\mu_j^-}}{2\pi})+g(\frac{w_{\mu_{c^\star}^-}}{2\pi)})))} \\ &= \langle ((f^{-1}[(\sum_{j=1}^n w_j (f(\mu_j^+)] + f(\mu_{c^\star}^+))) \\ &\quad e^{i2\pi(f^{-1}[(\sum_{j=1}^n w_j (f(\frac{w_{\mu_j^+}}{2\pi})]+f(\frac{w_{\mu_{c^\star}^+}}{2\pi)})))}, \\ &\quad -(g^{-1}[(\sum_{j=1}^n w_j (g(|\mu_j^-)|] + g(|\mu_{c^\star}^-|))) \\ &\quad e^{i2\pi(g^{-1}[(\sum_{j=1}^n w_j (g(\frac{w_{\mu_j^-}}{2\pi})]+g(\frac{w_{\mu_{c^\star}^-}}{2\pi)})))} \\ &= \langle ((f^{-1}(f(f^{-1}[(\sum_{j=1}^n w_j f(\mu_j^+)]) + f(\mu_{c^\star}^+)) \\ &\quad e^{i2\pi(f^{-1}(f(f^{-1}[(\sum_{j=1}^n w_j (f(\frac{w_{\mu_j^+}}{2\pi})]+f(\frac{w_{\mu_{c^\star}^+}}{2\pi)})))}, \\ &\quad -(g^{-1}(g(g*-1[(\sum_{j=1}^n w_j (g(|\mu_j^-)|])) + g(|\mu_{c^\star}^-|))) \\ &\quad e^{i2\pi(g^{-1}(g(g^{-1}[(\sum_{j=1}^n w_j (g(\frac{w_{\mu_j^-}}{2\pi})])+g(\frac{w_{\mu_{c^\star}^-}}{2\pi)})))} \end{aligned}$$

$$= \quad CBFWA(C_1, C_2, \ldots, C_n) \oplus C^{\star}. \quad \square$$

Proposition 4.10. (Homogeneity) Let $c_j (j = 1, 2, \ldots, n)$ a collection of CBFNs and any real number $\lambda > 0$, then

$$CBFWA(\lambda c_2, \lambda c_2, \ldots, \lambda c_n) = \lambda CBFWA(c_1, c_2, \ldots, c_n).$$

Proof. Let $CBFNs = \langle \mu_j^+ e^{iw_{\mu_j^+}}, \mu_j^- e^{iw_{\mu_j^-}} \rangle (j = 1, 2, \ldots, n)$ and $\lambda > 0$, we get

$$\lambda c_j = \langle (f^{-1}(\lambda f(\mu_j^+)))e^{i2\pi(f^{-1}(\lambda f(\frac{w_{\mu_j^+}}{2\pi})))}, -(g^{-1}(\lambda g(|\mu_j^-|)))e^{i2\pi(g^{-1}(\lambda g(\frac{w_{\mu_j^-}}{2\pi})))}.$$

Now, utilizing Eq.(4.1), we get

$$\begin{aligned} CBFWA(\lambda c_1, \lambda c_2, \ldots, \lambda c_n) &= \langle (f^{-1}(\sum_{j=1}^{n} w_j f(f^{-1}(\lambda f(\mu_j^+)))) \\ &\quad e^{i2\pi(f^{-1}(\sum_{j=1}^{n} w_j f(f^{-1}(\lambda f(\frac{w_{\mu_j^+}}{2\pi})))))}, \\ &\quad -(g^{-1}(\sum_{j=1}^{n} w_j g(g^{-1}(\lambda g(|\mu_j^-|)))) \\ &\quad e^{i2\pi(g^{-1}(\sum_{j=1}^{n} w_j g(g^{-1}(\lambda g(\frac{w_{\mu_j^-}}{2\pi})))))} \rangle \\ &= \langle (f^{-1}(\sum_{j=1}^{n} w_j \lambda f(\mu_j^+))))e^{i2\pi(f^{-1}(\sum_{j=1}^{n} w_j \lambda f(\frac{w_{\mu_j^+}}{2\pi}))))}, \\ &\quad -(g^{-1}(\sum_{j=1}^{n} w_j \lambda g(|\mu_j^-|))))e^{i2\pi(g^{-1}(\sum_{j=1}^{n} w_j (\lambda g(\frac{w_{\mu_j^-}}{2\pi})))))} \rangle \\ &= \langle (f^{-1}(\lambda f(f^{-1}(\sum_{j=1}^{n} w_j f(\mu_j^+)))) \\ &\quad e^{i2\pi(f^{-1}(\lambda f(f^{-1}(\sum_{j=1}^{n} w_j f(\frac{w_{\mu_j^+}}{2\pi})))))}, \\ &\quad -(g^{-1}(\lambda g(g^{-1}(\sum_{j=1}^{n} w_j g(|\mu_j^-|)))) \\ &\quad e^{i2\pi(g^{-1}(\lambda g(g^{-1}(\sum_{j=1}^{n} w_j (g(\frac{w_{\mu_j^-}}{2\pi})))))} \rangle \\ &= \lambda CBFWA(c_1, c_2, \ldots, c_n). \end{aligned}$$

$\square$

4.2. Complex Bipolar Fuzzy Ordered Weighted Averaging operator. Now, we present the following defintion

Definition 4.11. Let Ψ be a collection of CBFNs. A map $CBFOWA : \Psi^n \to \Psi$ defined by

$$CBFOWA(c_1, c_2, \ldots, c_n) = w_1 c_{\varrho(1)} \oplus w_2 c_{\varrho(2)}, \ldots, w_n c_{\varrho(n)}$$

for all $c_j \in \Psi$ where $(\varrho(1), \varrho(2), \ldots, \varrho(n))$ is called a permutation of $(1, 2, \ldots, n)$ such that $c_{\varrho_{j-1}} \geq c_{\varrho}(j)$ for $j = 2, 3, .., n$ and further $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of c_j with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then, the CBFOWA is called complex bipolar fuzzy ordered weighted averaging operator.

Theorem 4.12. *The aggregated number by utilizing CBFOWA operator for a collection of*

$$CBFNsc_j = \langle \mu_j^+ e^{iw_{\mu^+}},\ \mu_j^- e^{iw_{\mu^-}})(j = 1, 2, \ldots, n)$$

is also a CBFN and is given by

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_n) \quad &= \quad \langle (f^{-1}(\sum_{j=1}^{n} w_j f(\mu^+_{\varrho(j)}))))e^{i2\pi(f^{-1}(\sum_{j=1}^{n} w_j f(\frac{w_{\mu^+_{\varrho(j)}}}{2\pi})))}, \\ &\quad -(g^{-1}(\sum_{j=1}^{n} w_j g(|\mu^-_{\varrho(j)}|)))e^{i2\pi(g^{-1}(\sum_{j=1}^{n} w_j g(\frac{w_{\mu^-\varrho(j)}}{2\pi})))} \rangle. \end{aligned}$$

Proof is similar to the theorem 4.1 The following example shows the theorem above

Example 4.13. Let

$$\begin{aligned} c_1 \quad &= \quad \langle 0.5e^{i2\pi(0.6)}, -0.3e^{i2\pi(0.5)}\rangle,\ c_2 = \langle 0.4e^{i2\pi(0.8)}, -0.1e^{i2\pi(0.2)}\rangle, \\ c_3 \quad &= \quad \langle 0.6e^{i2\pi(0.5)}, -0.4e^{i2\pi(0.3)}\rangle,\ c_4 = \langle 0.7e^{i2\pi(0.9)}, -0.2e^{i2\pi(0.7)}\rangle \end{aligned}$$

be four CBFNs and $w = (0.2, 0.3, 0.15, 0.35)^T$ be the associated weight vector. Then, by Eq (3.1) we get

$$s(c_1) = 1.7,\ s(c_2) = 1.65,\ s(c_3) = 1.4,\ s(c_4) = 2.35$$

since

$$s(c_4) > s(c_1) > s(c_2) > s(c_3)$$

and we have

$$\begin{aligned} c_{\varrho(1)} \quad &= \quad \langle 0.7e^{i2\pi(0.9)}, -0.2e^{i2\pi(0.7)}\rangle,\ c_{\varrho(2)} = \langle 0.5e^{i2\pi(0.6)}, -0.3e^{i2\pi(0.5)}\rangle, \\ c_{\varrho(3)} \quad &= \quad \langle 0.4e^{i2\pi(0.8)}, -0.1e^{i2\pi(0.2)}\rangle,\ c_{\varrho(4)} = \langle 0.6e^{i2\pi(0.5)}, -0.4e^{i2\pi(0.3)}\rangle. \end{aligned}$$

Then, we get

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_n) \quad &= \quad \langle (1 - \prod_{j=1}^{n}(1 - \mu^+_{\varrho(j)})^{w_j})e^{i2\pi(1-\prod_{j=1}^{n}(1-\frac{w_{\mu^+_{\varrho(j)}}}{2\pi})^{w_j})} \\ &\quad - \prod_{j=1}^{n}(|\mu^-_{\varrho(j)}|)^{w_j} e^{i2\pi \prod_{j=1}^{n}(\frac{w_{\mu^-_{\varrho(j)}}}{)}2\pi)^{w_j}} \rangle. \end{aligned}$$

Further, we obtain

$$\begin{aligned}
\prod_{j=1}^{4}\left(1-\mu^{+}_{\varrho(j)}\right)^{w_j} &= (1-0.7)^{0.2}\times(1-0.5)^{0.3}\times(1-0.4)^{0.15}\times(1-0.6)^{0.35}\\
&= 0.7860\times 0.8123\times 0.9262\times 0.7256 = 0.4291\\
\prod_{j=1}^{4}\left(|\mu^{-}_{\varrho(j)|}\right)^{w_j} &= (|-0.2|)^{0.2}\times(|-0.3|)^{0.3}\times(|-0.1|)^{0.15}\times(|-0.4|)^{0.35}\\
&= 0.7248\times 0.6968\times 0.7079\times 0.7256 = 0.2594\\
\prod_{j=1}^{4}\left(1-\frac{w_{\mu^{+}_{\varrho(j)}}}{2\pi}\right)^{w_j} &= (1-0.9)^{0.2}\times(1-0.6)^{0.3}\times(1-0.8)^{0.15}\times(1-0.5)^{0.35}\\
&= 0.6310\times 0.7597\times 0.7855\times 0.7846 = 0.2954\\
\prod_{j=1}^{4}\left(\frac{w_{\mu^{-}_{\varrho(j)}}}{2\pi}\right)^{w_j} &= (0.7)^{0.2}\times(0.5)^{0.3}\times(0.2)^{0.15}\times(0.3)^{0.35}\\
&= 0.9311\times 0.8123\times 0.7855\times 0.6561 = 0.3898\\
CBFWA(c_1,c_2,c_3,c_4) &= \langle(1-0.4291)e^{i2\pi(1-0.2954)}, -0.2594e^{i2\pi(0.3898)}\rangle\\
&= \langle 0.5709e^{i2\pi(0.7046)}, -0.2594e^{i2\pi(0.3898)}\rangle.
\end{aligned}$$

4.3. Complex Bipolar Fuzzy Hybrid Averaging Operator. In this section, the HA aggregation operator of $CBFNs$ were discussed.

Definition 4.14. let Ψ be the collection of $CBFNs$. A map $CBFHA : \Psi^n \to \Psi$, getting associated weight vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ with $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$, defined by

$$CBFHA(c_1, c_2, \ldots, c_n) = \xi_1 c^{\bullet}_{\varrho(1)} \oplus \xi_2 c^{\bullet}_{\varrho(2)} \oplus \ldots \oplus \xi_n c^{\bullet}_{\varrho(n)}$$

where $c^{\bullet}_j = nw_j c_j, j = 1, 2, \ldots, n$ and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of c_j with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$,

Theorem 4.15. *Let* $C_j = \langle \mu^{+}e^{iw_{\mu^{+}_j}}, \mu^{-}e^{iw_{\mu^{-}_j}}\rangle, (j = 1, 2, \ldots, n)$ *be the collection of* $CBFNs$*. Then by utilizing* $CBFHA$ *operator is a* $CBFN$ *and*

$$\begin{aligned}
CBFWA(c_1, c_2, \ldots, c_n) = \ & \langle (f^{-1}(\textstyle\sum_{j=1}^{n} \xi_j f(\mu^{+}_{\varrho(j)}))))e^{i2\pi(f^{-1}(\sum_{j=1}^{n}\xi_j f(\frac{w_{\mu^{+}_{\varrho(j)}}}{2\pi})))},\\
& (g^{-1}(\textstyle\sum_{j=1}^{n} \xi_j g(\mu^{-}_{\varrho(j)})))e^{i2\pi(g^{-1}(\sum_{j=1}^{n}\xi_j g(\frac{w_{\mu^{-}\varrho(j)}}{2\pi})))}\rangle.
\end{aligned}$$

Proof is similar to the theorem 4.1.

Example 4.16. Let

$$\begin{aligned}
c_1 &= \langle 0.5e^{i2\pi(0.4)}, -0.3e^{i2\pi(0.2)}\rangle,\ c_2 = \langle 0.8e^{i2\pi(0.6)}, -0.4e^{i2\pi(0.5)}\rangle\\
c_3 &= \langle 0.7e^{i2\pi(0.5)}, -0.4e^{i2\pi(0.4)}\rangle,\ c_4 = \langle 0.8e^{i2\pi(0.7)}, -0.5e^{i2\pi(0.2)}\rangle
\end{aligned}$$

be four $CBFNs$ and $w = (0.2, 0.35, 0.3, 0.15)^T$ be the corresponding weight vector of c_j for each $j = 1, 2, 3, 4$ and $\xi = (0.1, 0.3, 0.4, 0.2)^T$ be the weight vector associated with $CBFHA$. Then, utilizing Def(3.2), we get

$$\lambda c_1 = \langle (1-(1-\mu_1^+)^\lambda e^{i2\pi(1-(1-\frac{w_{\mu_1^+}}{2\pi})^\lambda)}, -|\mu_1^-|^\lambda e^{i2\pi(\frac{w_{\mu_1^-}}{2\pi})^\lambda} \rangle$$

then

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_n) &= \langle (1-\prod_{j=1}^{n}\left((1-\mu^+)^\bullet_{\varrho(j)}\right)^{\xi_j}) e^{i2\pi(1-\prod_{j=1}^n(1-\frac{w_{\mu^+_{\varrho(j)}}}{2\pi})^{w_j})}, \\ & \qquad -\prod_{j=1}^{n}(|\mu^-_{\varrho(j)}|)^{w_j} e^{i2\pi\prod_{j=1}^n(\frac{w_{\mu^-_{\varrho(j)}}}{2\pi})^{w_j}} \rangle. \end{aligned}$$

Now, by using $w = (0.2, 0.35, 0.3, 0.15)^T$ and $CBFNs$ $c_j (j = 1, 2, 3, 4)$, we get

$$\begin{aligned} c^\bullet_{\varrho(1)} &= \langle (1-(1-0.5)^{4\times0.2} e^{i2\pi(1-(1-0.4)^{4\times0.2}}, -(|0.3|)^{4\times0.2} e^{i2\pi(0.2)^{4\times0.2}} \rangle \\ &= \langle (1-(1-0.5)^{0.8} e^{i2\pi(1-(1-0.4)^{0.8}}, -(|0.3|)^{0.8} e^{i2\pi(0.2)^{0.8}} \rangle \\ &= \langle 0.4257 e^{i2\pi(0.3356)}, -0.3817 e^{i2\pi(0.2760)} \rangle \\ c^\bullet_{\varrho(2)} &= \langle (1-(1-0.8)^{4\times0.35} e^{i2\pi(1-(1-0.6)^{4\times0.35}}, -|0.4|^{4\times0.35} e^{i2\pi(0.5)^{4\times0.35}} \rangle \\ &= \langle 0.8949 e^{i2\pi(0.5109)}, -0.2773 e^{i2\pi(0.3789)} \rangle \\ c^\bullet_{\varrho(3)} &= \langle (1-(1-0.7)^{4\times0.3} e^{i2\pi(1-(1-0.5)^{4\times0.3}}, -|0.4|^{4\times0.3} e^{i2\pi(0.4)^{4\times0.3}} \rangle \\ &= \langle 0.7642 e^{i2\pi(0.5647)}, -0.3330 e^{i2\pi(0.3330)} \rangle \\ c^\bullet_{\varrho(4)} &= \langle (1-(1-0.8)^{4\times0.15} e^{i2\pi(1-(1-0.7)^{4\times0.15}}, -|0.5|^{4\times0.15} e^{i2\pi(0.2)^{4\times0.15}} \rangle \\ &= \langle 0.6193 e^{i2\pi(0.5144)}, -0.6598 e^{i2\pi(0.3807)} \rangle. \end{aligned}$$

Thus, by score function of each $CBFN$ we get

$$s(c_1^\bullet) = 1.1334,\ s(c_2^\bullet) = 1.6986,\ s(c_3^\bullet) = 1.6133,\ s(c_4^\bullet) = 1.37485.$$

Note that

$$s(c_2^\bullet) > s(c_3^\bullet) > s(c_4^\bullet) > s(c_1^\bullet).$$

Therefore

$$\begin{aligned} c^\bullet_{\varrho(1)} &= \langle 0.8949 e^{i2\pi(0.5104)}, -0.2773 e^{i2\pi(0.3789)} \rangle, \\ c^\bullet_{\varrho(2)} &= \langle 0.7642 e^{i2\pi(0.5647)}, -0.3330 e^{i2\pi(0.3330)} \rangle, \\ c^\bullet_{\varrho(3)} &= \langle 0.6193 e^{i2\pi(0.5144)}, -0.6598 e^{i2\pi(0.3807)} \rangle, \\ c^\bullet_{\varrho(4)} &= \langle 0.4257 e^{i2\pi(0.3355)}, -0.3817 e^{i2\pi(0.2759)} \rangle. \end{aligned}$$

now that yields

$$\begin{aligned} CBFWA(c_1, c_2, \ldots, c_n) = & \quad \langle 1 - 0.0024 e^{1-0.00030}, -0.3955 e^{0.3214} \rangle \\ = & \qquad \langle 0.9976 e^{0.9997}, -0.4417 e^{0.0133} \rangle. \end{aligned}$$

Theorem 4.17. *If* $\xi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ *then, CBFHA operator becomes a CBFWA operator.*

Proof. By definition $c^{\bullet}_{\varrho(j)} = nw_j c_j, j = 1, 2, \ldots, n$, since, $\xi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ which implies that $\xi_j c^{\bullet}_{\varrho(j)} = w_j c_j$. Then,

$$\begin{aligned} CBFHA(c_1, c_2, \ldots, c_n) &= \xi_1 c^{\bullet}_{\varrho(1)} \oplus \xi_2 c^{\bullet}_{\varrho(2)} \oplus \ldots \oplus \xi_n c^{\bullet}_{\varrho(n)} \\ &= w_1 c_1 \oplus w_2 c_2 \oplus \ldots \oplus w_n c_n \\ &= CBFWA(c_1, c_2, \ldots, c_n). \end{aligned}$$

□

Theorem 4.18. *If* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ *then the CBFHA operator becomes a CBFOWA operator.*

Proof is similar to the theorem 4.4

Conclusion

In this work, some laws of complex bipolar fuzzy numbers were discussed. Some new aggregation operators including $CBFW$ operator, $CBFOW$ operator and $CBFH$ operator were developed which extend the BFW and $BFOW$ operators. In addition, many further properties of CBF aggregation operator such as including idempotency, monotonicity, boundedness, shift invariance and homogeneity were presented. Some numerical examples were provided to illustrate the extended operators.

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(Mabruka Ali) Department of Mathematics, 43400 UPM, Serdang, Selangor, Malaysia
E-mail address: altwer2016@gmail.com

(Adem Kilicman) Department of Mathematics and Statistics, University Putra Malaysia(UPM), 43400 Serdang, Selangor, Malaysia
E-mail address: akilic@upm.edu.my