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ON N-POWER HYPONORMAL OPERATORS IN MINKOWSKI SPACE

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Abstract. In this paper, we extend the concept of n-power hyponormal operators in Minkowski space, which is weaker than the case of normal operators. Furthermore, we give some basic properties of these operators.

1. Introduction

Minkowski space was initially developed by German mathematician Hermann Minkowski as a four dimensional space, with fourth dimension as time, for Maxwell's equations of electromagnetism. The mathematical structure of Minkowski spacetime was shown to be an immediate consequence of the postulates of special relativity. Minkowski space mathematically forms an indefinite inner product space with a particular type of metric matrix, known as Minkowski metric matrix. Minkowski space is considered to be the geometrically most suitable space for the study of Einstein's Theory of Special Relativity. The concept of hyponormal operators were introduced by Stampfli [8] in 1962. In 1990, Aluthge [1] extended the concept of p-hyponormal operators. Alzuraigi et al. [2] studied the n-normal operators in 2010. Guesba et al. [4] developed the concept of n-power-hyponormal operators in 2016. Here $B(H)$ denotes the set of all bounded linear operator in Minkowski space $\mathcal{M}$.

2. Preliminaries

Definition 2.1. An operator N is called n-EP in $\mathcal{M}$, if $N^n N^{\oplus} = N^{\oplus} N^n$

Definition 2.2. An operator N is called normal in $\mathcal{M}$, if $NN^{\sim} = N^{\sim}N$

Definition 2.3. An operator N is called skew-EP in $\mathcal{M}$, if $N^2 = -N^{\oplus 2}$

Definition 2.4. An operator N is called projection in $\mathcal{M}$, if $N^2 = N = N^{\sim}$.

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Definition 2.5. An operator N is called unitary in $\mathcal{M}$, if $NN^{\sim} = N^{\sim}N = I$.

Definition 2.6. An operator N is called hypo-EP in $\mathcal{M}$, if $NN^{\oplus} \leq N^{\oplus}N$.

We refer various properties and advantages of this product in ([6], [7],[5]).

3. Main Results

Definition 3.1. For an operator $N \in B(H)$, if $N^nN^{\sim} \leq N^{\sim}N^n$ then N is called n-power hyponormal operator in $\mathcal{M}$.

Proposition 3.2. If $S, N \in B(H)$ are unitarily equivalent and if N is n-power hyponormal operator in $\mathcal{M}$, then so is S.

Proof. Let N be n-power hyponormal operator in $\mathcal{M}$ and S be unitary equivalent of N. Then there exists unitary operator U such that $S = UNU^{\sim}$ so $S^n = UN^nU^{\sim}$.
We have,
$$\begin{aligned} S^nS^{\sim} &= UN^nU^{\sim}(UNU^{\sim})^{\sim} \\ &= UN^nU^{\sim}UN^{\sim}U^{\sim} \\ &= UN^nN^{\sim}U^{\sim} \\ &\leq UN^{\sim}N^nU^{\sim} \text{ (Since N is a n-power hyponormal in } \mathcal{M}.) \\ &= S^{\sim}S^n. \end{aligned}$$

Thus, $S^nS^{\sim} \leq S^{\sim}S^n$. Therefore S is a n-power hyponormal operator in $\mathcal{M}$. □

Proposition 3.3. Let $N \in B(H)$ be n-power hyponormal operator in $\mathcal{M}$. Then $N^{\sim}$ is a n-power hyponormal operator in $\mathcal{M}$.

Proof. Since, N is a n-power hyponormal operator in $\mathcal{M}$. We have

$$\begin{aligned} N^nN^{\sim} \leq N^{\sim}N^n &\Rightarrow (N^nN^{\sim})^{\sim} \leq (N^{\sim}N^n)^{\sim} \\ &\Rightarrow (N^{\sim})^{\sim}(N^n)^{\sim} \leq (N^n)^{\sim}(N^{\sim})^{\sim} \\ &\Rightarrow N(N^{\sim})^n \leq (N^{\sim})^nN \\ &\Rightarrow (N^{\sim})^nN \geq N(N^{\sim})^n. \end{aligned}$$
Thus $N^{\sim}$ is a n-power hyponormal operator in $\mathcal{M}$. □

Corollary 3.4. If N and $N^{\sim}$ are two n-power hyponormal operators, then N is n-normal operator in $\mathcal{M}$.

Theorem 3.5. *If S, N are commuting n-power-hyponormal operators and $SN^{\sim} = N^{\sim}S$ then SN is an n-power hyponormal operator in $\mathcal{M}$.*

Proof. Since $SN = NS$, so $S^nN^n = (SN)^n$ and $SN^{\sim} = N^{\sim}S$, so $S^nN^{\sim} = N^{\sim}S^n$. Now,
$$\begin{aligned} SN^{\sim} &= N^{\sim}S \\ \Rightarrow NS^{\sim} &= S^{\sim}N \\ \Rightarrow N^n\, S^{\sim} &= S^{\sim}N^n. \end{aligned}$$
Similarly $S^nN^{\sim} = N^{\sim}S^n$ We have,
$$\begin{aligned} (SN)^n(SN)^{\sim} &= S^nN^n\, N^{\sim}S^{\sim} \\ &\leq S^nN^{\sim}N^nS^{\sim} \text{ (since N is a n-power hyponormal)} \\ &= N^{\sim}S^nS^{\sim}N^n \\ &\leq N^{\sim}S^{\sim}S^nN^n \text{ (since } S \text{ is a n-power hyponormal)} \end{aligned}$$

Hence, $(SN)^n(SN)^\sim \leq (SN)^\sim(SN)^n$.

Therefore SN is a n-power hyponormal operator in $\mathcal{M}$. □

Proposition 3.6. Let S, N are commuting n-power hyponormal operators, such that $NS^\sim = S^\sim N$ and $(S+N)^\sim$ is commutes with $\sum_{k=1}^{n-1} C_n^k\ S^{n-k}N^k$. Then $(S+N)$ a n-power hyponormal operator in $\mathcal{M}$.

Proof.

$$(S+N)^n(S+N)^\sim = \left(\sum_{k=0}^{n-1} C_n^k\ S^{n-k}N^k\right)(S^\sim N^\sim)$$

$= S^nN^\sim + \sum_{k=0}^{n-1} C_n^k S^{n-k}N^k(S+N)^\sim + N^nS^\sim$
$+ S^nN^\sim + N^nN^\sim$
and since $NS^\sim = S^\sim N \Rightarrow N^nS^\sim = S^\sim N^n$.
Now,

$$\begin{aligned} NS^\sim &= S^\sim N \Rightarrow SN^\sim \\ &= N^\sim S \Rightarrow S^nN^\sim \\ &= N^\sim S^n.\ (Since, (S+N)^\sim\, commutewith \sum_{k=1}^{n-1} C_n^k\ S^{n-k}N^k.) \end{aligned}$$

Hence,

$$\begin{aligned} (S+N)^n\,(S+N)^\sim &= S^nS^\sim + (S+N)^\sim \sum_{k=1}^{n-1} C_n^k S^{n-k}N^k + S^\sim N^n + N^\sim S^n + N^nN^\sim \\ &\leq S^\sim S^n + (S+N)^\sim \sum_{k=1}^{n-1} C_n^k\ S^{n-k}N^k\ + S^\sim N^n + N^\sim N^n \\ &= (S+N)^\sim \left(\sum_{k=0}^{n-1} C_n^k\ S^{n-k}N^k\right) \\ &= (S+N)^\sim\,(S+N)^n. \end{aligned}$$

□

Proposition 3.7. If $S, N \in B(H)$ are 2 -power-hyponormal operators, such that $NS^\sim = S^\sim N$ and $SN + NS = 0$, then $S+N$ and SN are 2-power-hyponormal operators in $\mathcal{M}$.

Proof. Since $SN + NS = 0$, hence $S^2N^2 = N^2S^2$, so$(S+N)^2 = S^2 + N^2$.

$$\begin{aligned} (S+N)^2(S+N)^\sim &= \left(S^2+N^2\right)(S^\sim + N^\sim) \\ &= S^2S^\sim + S^2N^\sim + N^2S^\sim + N^2N^\sim \\ &= S^2S^\sim + N^\sim S^2 + S^\sim N^2 + N^2N^\sim \quad \text{(since } NS^\sim = S^\sim N\text{)} \\ &\leq S^\sim S^2 + N^\sim S^2 + S^\sim N^2 + N^\sim N^2 \end{aligned}$$

$$= (S+N)^{\sim} (S+N)^2.$$

Now, $(\mathrm{SN})^2 (\mathrm{SN})^{\sim} = S^2N^2N^{\sim}S^{\sim}$
$$\leq S^2N^{\sim}N^2S^{\sim}$$
$$= N^{\sim}S^2S^{\sim}N^2$$
$$\leq N^{\sim}S^{\sim}S^2N^{\sim}$$
$$= (SN)^{\sim}(SN)^2.$$
□

Theorem 3.8. *Let $N_1, N_2, \cdots, N_m$ be n-power hyponormal operators in B(H). Then $(N_1 \oplus N_2 \oplus ... \oplus N_n)$ and $(N_1 \otimes N_2 \otimes \cdots N_n)$ are the n-power hyponormal operators in $\mathcal{M}$.*

Proof. Since $(N_1 \oplus N_2 \oplus \cdots \oplus N_n)^n (N_1 \oplus N_2 \oplus \cdots \oplus N_n)^{\sim}$
$$= (N_1^n \oplus N_2^n \oplus \cdots \oplus N_n^n)(N_1^{\sim} \oplus N_2^{\sim} \oplus \cdots \oplus N_n^{\sim})$$
$$= N_1^nN_1^{\sim} \oplus N_2^nN_2^{\sim} \oplus \cdots \oplus N_m^nN_m^{\sim}$$
$$\leq N_1^{\sim}N_1^n \oplus N_2^{\sim}N_2^n \oplus \cdots \oplus N_m^{\sim}N_m^n$$
$$= (N_1^{\sim} \oplus N_2^{\sim} \oplus \ldots \oplus N_m^{\sim})(N_1^n \oplus N_2^n \oplus \cdots \oplus N_m^n)$$
$$= (N_1 \oplus N_2 \oplus \cdots \oplus N_m)^{\sim}(N_1 \oplus N_2 \oplus \cdots \oplus N_m)^n.$$

Then $(N_1 \oplus N_2 \oplus \cdots \oplus N_m)$ is a n-power hyponormal operator in $\mathcal{M}$.
Now, for $x_1, \ldots, x_m \in H$
$$(N_1 \otimes N_2 \otimes \cdots \otimes N_m)^n (N_1 \otimes N_2 \otimes \cdots \otimes N_m)^{\sim}(x_1 \otimes \cdots \otimes x_m)$$
$$= (N_1^n \otimes N_2^n \otimes \cdots \otimes N_m^n)(N_1^{\sim} \otimes N_2^{\sim} \otimes \cdots \otimes N_m^{\sim})\ (x_1 \otimes \cdots \otimes x_m)$$
$$= N_1^nN_1^{\sim}x_1 \otimes \cdots \otimes N_m^nN_m^{\sim}x_m$$
$$\leq N_1^{\sim}N_1^nx_1 \otimes \cdots \otimes N_m^{\sim}N_m^nx_m \text{ (since } N \text{ is an n-power hyponormal operator)}$$
$$= (N_1^{\sim} \otimes N_2^{\sim} \otimes \ldots \otimes N_m^{\sim})(N_1^n \otimes N_2^n \otimes \cdots \otimes N_m^n)(x_1 \otimes \ldots, \otimes x_m)$$
$$= (N_1 \otimes N_2 \otimes \cdots \otimes N_m)^{\sim}(N_1 \otimes N_2 \otimes \cdots \otimes N_m)^n(x_1 \otimes \cdots \otimes x_m)$$

So $(N_1 \otimes N_2 \otimes \cdots \otimes N_m)^n (N_1 \otimes N_2 \otimes \cdots \otimes N_m)^{\sim}$
$$\leq (N_1 \otimes N_2 \otimes \cdots \otimes N_m)^{\sim}(N_1 \otimes N_2 \otimes \cdots \otimes N_m)^n.$$

Hence $(N_1 \otimes N_2 \otimes \cdots \otimes N_m)$ is a n-power hyponormal operator in $\mathcal{M}$. □

Proposition 3.9. If N is 3-power-hyponormal and $N^2 = -N^{\sim^2}$. Then N is 3-normal operator in $\mathcal{M}$.

Proof. Since $N^3N^{\sim} = NN^2N^{\sim} = -NN^{\sim^3}$ and $N^{\sim}N^3 = N^{\sim}N^2N = -N^{\sim^3}N$ N is a 3-power-hyponormal in $\mathcal{M}$,
then $N^3N^{\sim} \leq N^{\sim}N^3 \Rightarrow -NN^{\sim^3} \leq -N^{\sim^3}N$
$$\Rightarrow NN^{\sim^3} \geq N^{\sim^3}N$$
$$\Rightarrow \left(NN^{\sim^3}\right)^{\sim} \geq (N^{\sim^3}N)^{\sim}$$
$$\Rightarrow N^3N^{\sim} \geq N^{\sim}N^3.$$

Thus $N^3N^{\sim} \geq N^{\sim}N^3$. □

Proposition 3.10. If N is 4-power hyponormal and N is skew-normal operator in $\mathcal{M}$, then N is 4-normal operator in $\mathcal{M}$.

Proof. N is skew-normal operator in $\mathcal{M}$ then $N^2 = -N^{\sim^2}$.
Since $N^4N^{\sim} = N^2N^2N^{\sim} = N^{\sim^5}$ and
$N^{\sim}N^4 = N^{\sim}N^2N^2 = N^{\sim^5}$.
Thus $N^4N^{\sim} = N^{\sim}N^4$. □

Proposition 3.11. If N is 2-power hyponormal operator in $\mathcal{M}$ and N is idempotent. Then N is hyponormal operator in $\mathcal{M}$.

Proof. Since N is 2-power hyponormal operator in $\mathcal{M}$, then
$N^2 N^{\sim} \leq N^{\sim} N^2$.
Since N is idempotent $N^2 = N$, which implies $NN^{\sim} \leq N^{\sim}N$.
Hence, N is a hyponormal operator in $\mathcal{M}$. □

Proposition 3.12. If N is 3-power hyponormal operator in $\mathcal{M}$ and N is idempotent. Then N is 2-power hyponormal operator in $\mathcal{M}$.

Proof. Since N is 3-power hyponormal operator in $\mathcal{M}$,
then $N^3 N^{\sim} \leq N^{\sim} N^3$. Since N is idempotent which implies $N^2 N^{\sim} \leq N^{\sim} N^2$.
Hence N is 2-power hyponormal operator in $\mathcal{M}$. □

4. n-power hypo-EP operator in Minkowski space $\mathcal{M}$

In this section, we state the generalization of the results in Section.2 to n-power hypo-EP operators in $\mathcal{M}$.

Definition 4.1. *For $N \in B(H)$ is called n-power hypo-EP in $\mathcal{M}$ operator, if $N^n N^{\oplus} \leq N^{\oplus} N^n$.*

Proposition 4.2. If $S, N \in B(H)$ are unitarily equivalent and N is n-power hypo-EP operators in $\mathcal{M}$, then so is S.

Proposition 4.3. Let $N \in B(H)$ be an n-power hypo-EP operator in $\mathcal{M}$. Then $N^{\oplus}$ is n-power hypo-EP operator in $\mathcal{M}$.

Corollary 4.4. If N and $N^{\oplus}$ are two n-power hypo-EP operators in $\mathcal{M}$, then N is n-EP operator in $\mathcal{M}$.

Theorem 4.5. *If S, N are commuting n-power-hypo-EP operators in $\mathcal{M}$ and $SN^{\oplus} = N^{\oplus}S$, then SN is an n-power hypo-EP operator in $\mathcal{M}$.*

Proposition 4.6. Let S, N are commuting n-power hypo-EP operators in $\mathcal{M}$, such that $NS^{\oplus} = S^{\oplus}N$ and $(S+N)^{\oplus}$ is commutes with $\sum\limits_{k=1}^{n-1} C_n^k \, S^{n-k}N^k$. Then $(S+N)$ is an n-power hypo-EP operator in $\mathcal{M}$.

Proposition 4.7. If $S, N \in B(H)$ are 2 -power-hypo-EP operators in $\mathcal{M}$, such that $NS^{\oplus} = S^{\oplus}N$ and $SN + NS = 0$, then $S + N$ and SN are 2-power-hypo-EP operators in $\mathcal{M}$.

Theorem 4.8. *Let $N_1, N_2, \cdots, N_m$ be n-power hypo-EP operators in $\mathcal{M}$. Then $(N_1 \oplus N_2 \oplus ... \oplus N_n)$ and $(N_1 \otimes N_2 \otimes \cdots N_n)$ are the n-power hypo-EP operators in $\mathcal{M}$.*

Proposition 4.9. If N is 3-power hypo-EP and $N^2 = -N^{\oplus^2}$ in $\mathcal{M}$. Then N is 3-EP operator in $\mathcal{M}$.

Proposition 4.10. If N is 4-power hypo-EP and N is skew-EP operator in $\mathcal{M}$, then N is 4-EP operator in $\mathcal{M}$.

Proposition 4.11. If N is 2-power hypo-EP operator and N is idempotent in $\mathcal{M}$. Then N is hypo-EP operator in $\mathcal{M}$.

Proposition 4.12. If N is 3-power hypo-EP operator and N is idempotent in $\mathcal{M}$. Then N is 2-power hypo-EP operator in $\mathcal{M}$.

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