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ESSENTIALLY RATIONALIZED TOEPLITZ HANKEL OPERATORS

RUCHIKA BATRA

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Abstract. In this paper, the notion of Essentially Rationalized Toeplitz Hankel Operators on the space L^2 is introduced and some properties about the set of all such kind of operators are discussed, Conditions under which the product of two such operators commute are also discussed. We also discuss essential spectrum of such operators. We define the set ERTHO (L^2) as the set of all Essentially Rationalized Toeplitz Hankel operators. Precisely, we attempt to investigate the properties of ERTHO (L^2).

1. Introduction

The study of Toeplitz and Hankel operators and their generalizations have been a subject of investigation by many researchers around the globe. Toeplitz in the year 1911 introduced the notion of Toeplitz operators and subsequently many researchers like Devinatz [8], Barria and Halmos [7] give the generalization of these operators. Along with Toeplitz operators, Hankel operators were also introduced and become a subject of investigation for the researchers. Hankel operators are the formal companions of Toeplitz operators which have occured in realization problem for certain discrete time linear systems and in determining which systems are exactly controllable. Motivated by these, Ho in the year 1995 [10] introduced Slant Toeplitz operators on the space $L^2(\mathbb{T})$, $\mathbb{T}$ being the unit circle in the complex plane. We introduced [2] the notions of Slant Hankel operators and also generalized the notion of Slant Toeplitz operators [2] of k^{th} order. After that a lot of mathematicians have worked on these kind of operators and also give many kinds of generalizations. The notions of Essentially Toeplitz operators, Essentially Slant Toeplitz operators were also introduced and studied by different mathematicians around the globe.

Ever since the introduction of the class of slant Toeplitz operators,the study has gained importance due to its applications. Villemoes [12] associated the Besov regularity

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of solutions of the refinement equation with the spectral radius of the associated Slant Toeplitz operator.

Motivated by these, in [5], the notions of all kinds of Toeplitz, Hankel, Slant Toeplitz, Slant Hankel are further generalized and a rationalization of these operators is introduced as the Rationalized Toeplitz Hankel operator on the space $L^2(\mathbb{T})$. Precisely we define [5] for $\varphi \in L^\infty$, a Rationalized Toeplitz Hankel operator on the space L^2 of order (k_1, k_2) as

$$R_\varphi : L^2 \to L^2,\ R_\varphi(f) = W_{k_1} M_\varphi W_{k_2}^*(f)\ \forall\ f \in L^2,$$

where k_1 and k_2 are non zero integers and $W_k(z^i) = \begin{cases} z^{i|k}, i \text{ is divisible by k} \\ 0, \text{otherwise} \end{cases}$.

It is also proved in [5] that if k_1 and k_2 are relatively prime then a bounded linear operator R on L^2 is a Rationalized Toeplitz Hankel operator if and only if $M_{z^{k_2}} R = RM_{z^{k_1}}$.

Further if k_1 and k_2 are not relatively prime then if $d = g.c.d(k_1, k_2)$. i.e, let $k_1 = dn$ and $k_2 = dm$ then a bounded operator R on L^2 is Rationalized Toeplitz Hankel operator if and only if

$$R|\tilde{N}_i = W_m M_{\tilde{\varphi}_i} W_n^* |\tilde{N}_i$$

for some $\tilde{\varphi}_i$ in L^∞ and for $i = 1, 2, \cdots d-1$.

$$L^2 = \widetilde{N}_0 \oplus \widetilde{N}_1 \oplus \cdots \oplus \widetilde{N}_{d-1}$$

where

$$\begin{aligned} \widetilde{N}_0 &= N_0 \oplus N_1 \oplus \cdots \oplus N_{m-1} \\ \widetilde{N}_1 &= N_m \oplus N_{m+1} \oplus \cdots \oplus N_{2m-1} \\ &\vdots \qquad\qquad \vdots \\ \widetilde{N}_{d-1} &= N_{(d-1)m} \oplus N_{(d-1)m+1} \oplus \cdots \oplus N_{(dm-1)}, \end{aligned}$$

where N_i denotes the closed linear span of $\{z^{k_1 t+1}\ t \in \mathbb{Z}\}$.

Here in this paper we introduce a new class of operators as the Essentially Rationalized Toeplitz Hankel operators which is the Rationalization of all kinds of Essentially Toeplitz and Essentially Hankel operators.

2. Essentially Rationalized Toeplitz Hankel Operators

It is known [5] that a bounded linear operator R on the space L^2 is the Rationalized Toeplitz Hankel operator of order (k_1, k_2) if and only if $M_{z^{k_2}} R = RM_{z^{k_1}}$, we introduce the following.

Definition 2.1. A bounded linear operator R on the space L^2 is said to be a Essentially Rationalized Toeplitz Hankel operator of order (k_1, k_2) if $M_{z^{k_2}} R - RM_{z^{k_1}} = K$, for some compact operator K on L^2.

The class of all Essentially Rationalized Toeplitz Hankel operators on L^2 of order (k_1, k_2) is denoted by ERTHO (L^2)

Example 2.2. Let $k_1 = 2$ and $k_2 = 1$. Define an operator S on L^2 as $S = W_2$, where

$$W_2(z^n) = \begin{cases} z^{n|2} \; if \; n \; is \; even \\ 0, otherwise \end{cases}$$

where $e_n(z) = z^n \; \forall \; n \in \mathbb{Z}$. Then

$$(M_{z^{k_2}} S - S M_{z^{k_1}}) e_n(z) = (M_z S - S M_{z^2}) e_n(z) \; .$$

For $n = 2k$, we have $(M_z S - S M_{z^2}) e_{2k}(z) = e_{k+1}(z) - e_{k+1}(z) = 0$ and for $n = 2k+1$, we have $(M_z S - S M_{z^2}) e_{2k+1}(z) = 0$. Hence S is an Essentially Rationalized Toeplitz Hankel operator.

In fact, more general example would be if $k_2 = 1$ and $k_1 = k$, a fixed integer $k \geq 2$. Then $S = W_k$ is an Essentially Rationalized Toeplitz Hankel operator.

If $\mathcal{K}_o$ is the ideal of all compact operator on L^2 then from above definition we can say that every compact operator on L^2 is in ERTHO (L^2) and therefore ERTHO$(L^2) \cap \mathcal{K}_0 = \mathcal{K}_0$.

In [6], we have proved that the only compact Rationalized Toeplitz Hankel operator is the zero operator. So if RTHO(L^2) is the set of all Rationalized Toeplitz Hankel operator then

$$RTHO(L^2) \cap \mathcal{K}_0 = \{0\}.$$

Remark 2.3. If R is a Rationalized Toeplitz Hankel operator of order (k_1, k_2) then $M_{z^{k_2}} R = R M_{z^{k_1}}$ and therefore $M_{z^{k_2}} R - R M_{z^{k_1}} = 0$. As zero operator on L^2 is a compact operator, therefore R in an Essentially Rationalized Toeplitz Hankel operator. That is, every Rationalized Toeplitz Hankel operator is member of the set ERTHO (L^2).

Infact, if T is any compact perturbation of a Rationalized Toeplitz Hankel operator on L^2 then $T \in ERTHO(L^2)$.

By above Remark, every Rationalized Toeplitz Hankel operator is a member of ERTHO(L^2). However we have the following example to show that the converse need not be true.

Example 2.4. Let k_1 and k_2 are relatively prime integers and Let $\mathcal{K}$ on L^2 be defined as

$$\mathcal{K} e_n = \begin{cases} e_1, & if, \; n = 0 \\ 0, & otherwise \end{cases}$$

where $e_n(z) = z^n \;\; \forall \;\; n \in \mathbb{Z}$. Define an operator S on L^2 as $S = W_{k_1} M_{z^{k_2}} + \mathcal{K}$, where for non zero integer k, $W_k(z^n) = \begin{cases} z^{n|k}, & if \; n \; is \; divisible \; by \; k \\ 0, & otherwise \end{cases}$.

Then $M_{z^{k_2}} S - S M_{z^{k_1}} = M_{z^{k_2}} W_{k_1} M_{z^{k_2}} - W_{k_1} M_{z^{k_2}} M_{z^{k_1}} + K'$ (where $K' \in \mathcal{K}_0$ is a compact operator). So, $M_{z^{k_2}} S - S M_{z^{k_1}} = 0 + K' \in \mathcal{K}_0$. Hence S is an Essentially Rationalized Toeplitz Hankel operator. However it is in not Rationalized Toeplitz Hankel operator. We begin with the following result.

Theorem 2.5. *The space ERTHO(L^2) is a closed vector space of $B(L^2)$ under the *-strong operator topology, where $B(L^2)$ is the set of all bounded linear operators on the space L^2.*

Proof. Let S_1, S_2 be in ERTHO (L^2) and let α, β are complex numbers. Then consider $\alpha S_1 + \beta S_2$, we get

$M_{z^{k_2}}(\alpha S_1+\beta S_2)-(\alpha S_1+\beta S_2)M_{z^{k_1}} = \alpha(M_{z^{k_2}}S_1-S_1M_{Z^{k_1}})+\beta(M_{z^{k_2}}S_2-S_2M_{z^{k_1}}) \in \mathcal{K}_0$
$\implies \alpha S_1 + \beta S_2 \in \mathcal{K}_0$.

Now for each n, consider the sequence of operators (S_n) and (S_n^*)in ERTHO(L^2) such that $S_n \to S$ and $S_n^* \to S^*$ uniformly in $B(L^2)$. So for each n, we have $M_{z^{k_2}}S_n - S_nM_{z^{k_1}}$ $=K_n^{'}$ and $M_{z^{k_2}}S_n^* - S_n^*M_{z^{k_1}} = K_n^{"}$, where $K_n^{'}$ and $K_n^{"}$ are compact operators on L^2.This implies that

$$\begin{aligned} \|M_{z^{k_2}}S - SM_{z^{k_1}} - K_n^{'}\| &= \|M_{z^{k_2}}S - SM_{z^{k_1}} - M_{z^{k_2}}S_n + S_nM_{z^{k_2}}\| \\ &\leq (\|M_{z^{k_2}}\| + \|M_{z^{k_1}}\|)\|\|S_n - S\| \to 0 \ as \ n \to \infty. \end{aligned}$$

Thus the sequence of compact operators $K_n^{'} \to M_{z^{k_2}}S - SM_{z^{k_1}}$ uniformly. Since $(K_n^{'})$ is uniformly closed,therefore $M_{z^{k_2}}S - SM_{z^{k_1}}$ is compact and hence $S \in ERTHO(L^2)$. Similarly, we can show that$S^* \in ERTHO(L^2)$.

Thus the space ERTHO(L^2) is a closed vector space of $B(L^2)$ under the *-strong operator topology. □

In the following example we can see that the product of two Essentially Rationalized Toeplitz Hankel operators in not an Essentially Rationalized Toeplitz Hankel operators, and so ERTHO (L^2) is not an algebra of operators on L^2 and also it is not a self adjoint set.

Example 2.6. Let $S_1 = S_2 = W_{k_1}M_{z^{k_2}} + \mathcal{K}$ as defined in Example 4. Then we have seen that $S_1, S_2 \in ERTHO(L^2)$. But the product $S = S_1S_2 \notin ERTHO(L^2)$ as

$$\begin{aligned} M_{z^{k_2}}S - SM_{z^{k_1}} &= M_{z^{k_2}}S_1S_2 - S_1S_2M_{z^{k_1}} \\ &= M_{z^{k_2}}(W_{k_1}M_{z^{k_2}})^2 - (W_{k_1}M_{z_2^k})^2M_{k_1}(mod\mathcal{K}_0). \end{aligned}$$

Thus $M_{z^{k_2}}S - SM_{z^{k_1}} \in \mathcal{K}_0$ if and only if $M_{z^{k_2}}(W_{k_1}M_{z^{k_2}})^2 - (W_{k_1}M_{z^{k_2}})^2M_{z^{k_1}} \in \mathcal{K}_0$ which is not true. Hence the product $S_1S_2 \notin ERTHO(L^2)$.

However for the product we have the following theorem.

Theorem 2.7. *If $S_1, S_2 \in ERTHO(L^2)$ then the product $S_1S_2 \in ERTHO(L^2)$ if and only if*

$$S_1M_{z^{k_2}}S_2 = S_1M_{z^{k_1}}S_2(mod\mathcal{K}_0).$$

Proof. Let $S_1, S_2 \in ERTHO(L^2)$. Then Consider

$$\begin{aligned} M_{z^{k_2}}S_1S_2 - S_1S_2M_{z^{k_1}} &= S_1M_{z^{k_1}}S_2 - S_1S_2M_{z^{k_1}}(mod\mathcal{K}_0) \\ &= S_1M_{z^{k_1}}S_2 - S_1M_{z^{k_2}}S_2(mod\mathcal{K}_0). \end{aligned}$$

Hence $S_1S_2 \in ERTHO(L^2)$ if and only if $S_1M_{z^{k_1}}S_2 = S_1M_{z^{k_2}}S_2(mod\mathcal{K}_0)$. □

In general, we can prove the following result.

Theorem 2.8. *Let $S \in ERTHO(L^2)$ and $r \in \mathbb{N}$, $r > 1$. If $n(r)$ denotes the number of partition of r as the sum of two natural numbers $r = s_i + t_i(s_i, t_i \in \mathbb{N}, i = 1, 2, \ldots n(r))$. $S^{s_i}, S^{t_i} \in ERTHO(L^2)$. Then the following are equivalent.*

(1) $S^r \in ERTHO(L^2)$.
(2) $S^{s_i} M_{z^{k_2}} S^{t_i} = S^{s_i} M_{z^{k_1}} S^{t_i} (mod \mathcal{K}_0) i = 1, 2, ...n(r)$.
(3) $S^{t_i} M_{z^{k_2}} S^{s_i} = S^{t_i} M_{z^{k_1}} S^{s_i} (mod \mathcal{K}_0) i = 1, 2, ...n(r)$.

We can see that the product is the Essentially Rationalized Toeplitz Hankel operator under the following condition.

Theorem 2.9. *(i) If $S_1, S_2 \in ERTHO(L^2)$ such that S_1 commutes essentially with $M_{z^{k_2}}$ OR S_2 commutes essentially with $M_{z^{k_1}}$ then the product $S_1 S_2 \in ERTHO(L^2)$*
(ii) In particular, if S_1 commutes essentially with $M_{z^{k_2}}$ and $S_2 \in ERTHO(L^2)$ then $S_1 S_2 \in ERTHO(L^2)$. Similarly if $S_1 \in ERTHO(L^2)$ and S_2 commutes essentially with $M_{z^{k_1}}$ then $S_1 S_2 \in ERTHO(L^2)$.

Proof. Let $S_1, S_2 \in ERTHO(L^2)$.
Case-I Let S_1 commutes essentially with $M_{z^{k_2}}$ then

$$S_1 M_{z^{k_2}} = M_{z^{k_2}} S_1 (mod \mathcal{K}_0).$$

So,

$$\begin{aligned} M_{z^{k_2}} S_1 S_2 - S_1 S_2 M_{z^{k_1}} &= M_{z^{k_2}} S_1 S_2 - S_1 M_{z^{k_2}} S_2 (mod \mathcal{K}_0) \\ &= S_1 M_{z^{k_2}} S_2 - S_1 M_{z^{k_2}} S_2 (mod \mathcal{K}_0) \\ &= 0 (mod \mathcal{K}_0). \end{aligned}$$

Case-II Let S_2 commutes essentially with $M_{z^{k_1}}$ then

$$S_2 M_{z^{k_1}} = M_{z^{k_1}} S_2 (mod \mathcal{K}_0).$$

So,

$$\begin{aligned} M_{z^{k_2}} S_1 S_2 - S_1 S_2 M_{z^{k_1}} &= S_1 M_{z^{k_1}} S_2 - S_1 S_2 M_{z^{k_1}} (mod \mathcal{K}_0) \\ &= S_1 S_2 M_{z^{k_1}} - S_1 S_2 M_{z^{k_1}} (mod \mathcal{K}_0) \\ &= 0 (mod \mathcal{K}_0). \end{aligned}$$

Therefore $S_1 S_2 \in ERTHO(L^2)$.
In particular part (ii) is the trivial consequence of part (i). □

Corollary 2.10. For φ in L^∞, if M_φ is the Multiplication operator on L^2 induced by φ and $S \in ERTHO(L^2)$ then both SM_φ and $M_\varphi S$ are in ERTHO(L^2).

We can observe in the following example that the class ERTHO(L^2) is not closed with respect to adjoint.

Example 2.11. Let $k_1 = k$, ($k \geq 2$ be fixed) and $k_2 = 1$. Define an operator S on L^2 as $S = W_k$, where

$$W_k(z^n) = \begin{cases} z^{n|k}, & if\ n\ is\ divisible\ by\ k \\ 0, & otherwise \end{cases}$$

where $e_n(z) = z^n \quad \forall \quad n \in \mathbb{Z}$. Then

$$(M_{z^{k_2}} S - S M_{z^{k_1}}) e_n(z) = (M_z S - S M_{z^k}) e_n(z).$$

For $n = ki$, we have

$$(M_z S - S M_{z^k}) e_{ki}(z) = e_{i+1}(z) - e_{i+1}(z) = 0.$$

If n is not a multiple of k, then

$$(M_z S - S M_{z^k}) e_n(z) = 0.$$

Hence S is an Essentially Rationalized Toeplitz Hankel operator. But we can see that its adjoint S^* is not Essentially Rationalized Toeplitz Hankel operator as $(M_z S^* - S^* M_{z^k})e_n(z) \notin \mathcal{K}_0$. So $S^* \notin ERTHO(L^2)$.

However for the adjoint, we have the following theorem.

Theorem 2.12. *If S and its adjoint S^* both are Essentially Rationalized Toeplitz Hankel operators then $TS^* = S^*T^*(mod\mathcal{K}_0)$, where $T = M_{z^{k_2}} + M_{\bar{z}^{k_1}}$.*

Proof. It is given that both S and S* are members of ERTHO(L^2), so by definition

$$M_{z^{k_2}} S - S M_{z^{k_1}} = \mathcal{K}_1 \tag{2.1}$$

$$M_{z^{k_2}} S^* - S^* M_{z^{k_2}} = \mathcal{K}_2, \tag{2.2}$$

where K_1 and K_2 are compact operators $(ie., K_1,\ K_2 \in \mathcal{K}_0)$
From equation (2.1) and (2.2), we get on taking adjoints of both sides of (2.1) and then subtracting (2.2), we get

$$(M_{\bar{z}^{k_1}} + M_{z^{k_2}})S^* - S^*(M_{\bar{z}^{k_2}} + M_{z^{k_1}}) = \mathcal{K}^{'},$$

for some $\mathcal{K}^{'} \in \mathcal{K}_0$. So if $T = M_{z^{k_2}} + M_{\bar{z}^{k_1}}$
Then we get

$$TS^* - S^*T^*(mod\mathcal{K}_0)$$

This completes the proof. □

Now,we can deduce the following Corollary.

Corollary 2.13. A necessary condition for any operator $S \in ERTHO(L^2)$ to be self adjoint is that TS in essentially self adjoint, where

$$T = M_{z^{k_2}} + M_{\bar{z}^{k_1}}.$$

Foe essentially nornal operators we have the following.

Theorem 2.14. *If $S \in ERTHO(L^2)$ be an essentially normal operator. Then the operators SS^* and S^*S are both in $ERTHO(L^2)$ if and only if S^* commutes essentially with $SM_{z^{k_1}}$.*

Proof. Let $S \in ERTHO(L^2)$ be an essentially normal operator. So $SS^* = S^*S(mod\mathcal{K}_0)$
Then we get

$$\begin{aligned} M_{z^{k_2}} S^*S - S^*SM_{z^{k_1}} &= M_{z^{k_2}} SS^* - S^*SM_{z^{k_1}}(mod\mathcal{K}_0) \\ &= (SM_{z^{k_1}})S^* - S^*(SM_{z^{k_1}})(mod\mathcal{K}_0) \end{aligned}$$

Therefore, $S^*S \in ERTHO(L^2)$ if and only if S^*commutes essentially with $SM_{z^{k_1}}$.
Similarly,

$$\begin{aligned} M_{z^{k_2}} SS^* - SS^*M_{z^{k_1}} &= SM_{z^{k_1}} S^* - SS^*M_{z^{k_1}}(mod\mathcal{K}_0) \\ &= (SM_{z^{k_1}})S^* - S^*(SM_{z^{k_1}})(mod\mathcal{K}_0) \end{aligned}.$$

Therefore, $SS^* \in ERTHO(L^2)$ if and only if S^*commutes essentially with $SM_{z^{k_1}}$.
This completes the proof. □

The concluding result of this section proves that the essential spectrum of an Essentially Rationalized Toeplitz Hankel operator contains zero.

Theorem 2.15. *If $S \in ERTHO(L^2)$. Then, $0 \in \sigma_e(S)$,the essential spectrum of S .*

Proof. Let $S \in ERTHO(L^2)$ and let if possible $0 \notin \sigma_e(S)$ .This gives us that the operator S is essentially invertible in L^2.Since $S \in ERTHO(L^2)$, therefore,we have $M_{z^{k_2}}S - SM_{z^{k_1}} \in \mathcal{K}_0$ and therefore it follows that $M_{z^{k_2}} - SM_{z^{k_1}}S^{-1} \in \mathcal{K}_0$. This gives that $M_{z^{k_2}}$ and $M_{z^{k_1}}$ are essentially similar. But this is not possible as the Fredholm indexes are different.Hence our assumption was wrong and so $0 \in \sigma_e(S)$. This completes the proof. □

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(R. Batra) Ramjas College, University of Delhi, Delhi-110007, India

E-mail address: ruchikaverma@ramjas.du.ac.in