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A GENERALIZED LOCAL FUNCTION IN IDEAL TOPOLOGICAL SPACES

JOSÉ TORMET, RAFAEL LOZADA-YAVINA†, AND MARGOT SALAS-BROWN

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Abstract. In this paper we introduce a generalization of the notion of local function using operators associated to the topology. Similarly, a notion of compatibility of the ideal with the topology is introduced utilizing this new local function and the classical concept of star closure is extended.

1. Introduction

The notion of closure of a set has been fundamental in the development and expansion of the general topology, to such an extent that it is rare that in some topic of this area a characterization or a result that depends on it is not studied.

In 1979, S. Kasahara [8] introduced the notion of operators associated with a topology; based on this notion D. Jankovic [6] defined the notion of γ-closure, generalizing the notions of θ-closure and δ-closure introduced by N. Veličko [17]. On the other hand, the concept of ideals in topological spaces is treated in the classic text by K. Kuratowski [10]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following properties: (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal topological space (or simply a space) is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset A of X, $A^\star(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ [10]. We simply write $A^\star$ in case there is no chance confusion. According to D. Janković and T. Hamlett [7], $Cl^\star(A) = A \cup A^\star$ defines a Kuratowski closure for a topology $\tau^\star(\mathcal{I})$, finer than τ. When there is not chance for confusion, we will simply write $\tau^\star$ for $\tau^\star(\mathcal{I})$.

In addition, for a ideal $\mathcal{I}$ on X, the collection $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau \text{ and } J \in \mathcal{I}\}$ is a basis for the topology $\tau^\star$. Investigating generalizations of the local function in terms of weak notions of open sets has been the subject of study in many works (see [2, 3, 4, 5, 9, 13, 14, 15]). In this paper, we carry out a detailed study of K. Kuratowski

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† *Corresponding author*

[10], D. Janković and T. Hamlett [7], S. Kasahara [8], H. Ogata [12] and A. Al-Omari and T. Noiri [1], thanks to this it is defined in terms of operators, a new application that generalizes not only the concepts of local function and closure, as well as other variants of closure existing in the literature such as δ-closure, θ-closure, γ-closure and the local closure function $\Gamma(\cdot)(\mathcal{I},\tau)$. In addition, a detailed study of the properties that this new application satisfies is also carried out.

2. Preliminaries

Throughout this paper, (X,τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of X, we denote the closure of A, the interior of A and the boundary of A by $Cl(A)$, $Int(A)$ and $Bd(A)$, respectively.

A point $x \in X$ is called a *δ-cluster* (resp. *θ-cluster*) *point* of A if $Int(Cl(U)) \cap A \neq \emptyset$ (resp. $Cl(U) \cap A \neq \emptyset$) for each open set U containing x (see [17]). The set of all δ-cluster (resp. θ-cluster) points of A is called the *δ-closure* (resp. *θ-closure*) of A and is denoted by $\delta Cl(A)$ (resp. $Cl_\theta(A)$). A subset A of X is said to be *δ-closed* (resp. *θ-closed*) if $A = \delta Cl(A)$ (resp. $A = Cl_\theta(A)$). The complement of a δ-closed (resp. θ-closed) set is said to be a *δ-open* (resp. *θ-open*) set. A subset A is said to be δ-open (resp. θ-open) if $A = \delta\, Int(A)$ (resp. $A = Int_\theta(A)$). It follows from [17] that the collection of all δ-open sets in a topological space (X,τ) forms a topology on X which is denoted by τ_δ. From the definitions it follows that $\tau_\delta \subset \tau$.

Let (X,τ) be a topological space. A operator associated with the topology τ is a function $\gamma : \tau \to \mathcal{P}(X)$ such that $U \subset \gamma(U)$ for all $U \in \tau$ (see [8]). The operator γ is said to be *monotonous* if for each pair of open subsets A, B such that $A \subset B$, is satisfied that $\gamma(A) \subset \gamma(B)$. The operator γ is said to be *regular* if for each pair of open subsets U, V , $x \in U \cap V$ there exist a open set W such that $x \in W$ and $\gamma(W) \subset \gamma(U) \cap \gamma(V)$ (see [8]).

Let (X,τ) be a topological space. If γ is a monotonous operator, then γ is regular operator [8]. A point $x \in X$ is called a *γ-cluster* point of A if $\gamma(U) \cap A \neq \emptyset$ for each open set U containing x (see [6]). The set of all *γ-cluster* points of A is called the *γ-closure* of A and is denoted by $Cl_\gamma(A)$. A subset A of X is called *γ-closed* if $A = Cl_\gamma(A)$. A subset A of X is said to be *γ-open* if for each $x \in A$ there exist an open set U such that $x \in U$ and $\gamma(U) \subset A$ (see [12]).

Definition 2.1. [11] Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A a subset of X. The ideal $\mathcal{I}$ is said to be:

(1) Weakly compatible with τ, if $A^\star = \emptyset$ implies that $A \in \mathcal{I}$.
(2) Compatible with τ, if $A^\star \cap A = \emptyset$ implies that $A \in \mathcal{I}$.

Definition 2.2. [1] Let $(X,\tau,\mathcal{I})$ be an ideal topological space. For a subset A of X, is defined the following operator: $\Gamma(A)(\mathcal{I},\tau) = \{x \in X : Cl(U) \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. In case there is no confusion $\Gamma(A)(\mathcal{I},\tau)$ is briefly denoted by $\Gamma(A)$ and is called the local closure function of A with respect to $\mathcal{I}$ and τ.

Definition 2.3. [1] Let $(X,\tau,\mathcal{I})$ be an ideal topological space. A topology τ is said to be closure compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_\Gamma \mathcal{I}$, if for every subset A of X and $x \in A$, there exist an open set U containing the point x such that $Cl(U) \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 2.4. [12] *Let (X, τ) be a topological space and γ an operator associated with the topology τ. If γ is a regular operator, then τ_γ is a topology on X.*

Theorem 2.5. [11] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A a subset of X. If $\mathcal{I}$ is an ideal compatible with τ, then $\mathcal{I}$ is an ideal weakly compatible with τ.*

Theorem 2.6. [1] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is a subset of X, then*

$$A^\star(\mathcal{I}, \tau) \subseteq \Gamma(A)(\mathcal{I}, \tau).$$

Theorem 2.7. [1] *If τ is compatible with $\mathcal{I}$, then τ is closure compatible with $\mathcal{I}$.*

Theorem 2.8. [1] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A a subset of X. If τ is closure compatible with $\mathcal{I}$, then $A \cap \Gamma(A) = \emptyset$ implies that $\Gamma(A) = \emptyset$.*

3. Generalized local function

In this section, introduces the generalized local function in terms of operators associated with a topology. Also, some properties determined by this new generalized local function are studied, as well as the relationship it has with other types of local function that appear in the literature.

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ be an operator associated with τ and A a subset of X. We define the generalized local function of A with respect to an operator γ as follows: $A^\star_\gamma(\mathcal{I}, \tau) = \{x \in X : \gamma(U) \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$. In case there is no confusion $A^\star_\gamma(\mathcal{I}, \tau)$ is briefly denoted by $A^\star_\gamma$.

Remark 3.2. Using different ideals $\mathcal{I}$ and the γ operators, some concepts known in the literature are obtained.

(1) $A^\star_{id}(\{\emptyset\}, \tau) = Cl(A)$.
(2) $A^\star_\gamma(\mathcal{P}(X), \tau) = \emptyset$.
(3) $A^\star_{id}(\mathcal{I}, \tau) = A^\star$ [10].
(4) $A^\star_{id}(\mathcal{N}, \tau) = Cl(Int\, Cl(A)))$ [16].
(5) $A^\star_{Cl}(\{\emptyset\}, \tau) = Cl_\theta(A)$ [17].
(6) $A^\star_{Int\, Cl}(\{\emptyset\}, \tau) = \delta Cl(A)$ [17].
(7) $A^\star_\gamma(\{\emptyset\}, \tau) = Cl_\gamma(A)$ [6].
(8) $A^\star_{Cl}(\mathcal{I}, \tau) = \Gamma(A)$ [1].

The following example shows that in general $X^\star_\gamma$ is a subset of its own but not equal to X.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Using the operator $\gamma = Int(Cl)$, we obtain $X^\star_\gamma = \{b, c\} \subset \{a, b, c\} = X$.

The following theorem shows the relationship between the local function and the generalized local function.

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If γ is an operator associated with τ and A is a subset of X, then $A^\star(\mathcal{I}, \tau) \subset A^\star_\gamma(\mathcal{I}, \tau)$.*

Proof. Let $x \in A^\star(\mathcal{I}, \tau)$ and U be an open subset of X containing the x. Then $U \cap A \notin \mathcal{I}$. Since γ is an operator associated with τ, we obtain

$$A \cap U \subset A \cap \gamma(U).$$

Hence, $A \cap \gamma(U) \notin \mathcal{I}$, otherwise $U \cap A \in \mathcal{I}$, which is not true. Therefore, $A \cap \gamma(U) \notin \mathcal{I}$ and $x \in A^\star_\gamma(\mathcal{I},\tau)$. This shows that $A^\star(\mathcal{I},\tau) \subset A^\star_\gamma(\mathcal{I},\tau)$. □

The following two examples shows that the inclusion of Theorem 3.4 in general is proper.

Example 3.5. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$ with operator $\gamma = Int(Cl)$. If defined $A = \{b,c\}$, we conclude that $A^\star_\gamma = \{a,b,c\}$ and $A^\star = \{b\}$. Therefore, $A^\star \subset A^\star_\gamma$.

Example 3.6. Let $(\mathbb{R},\tau)$ be a topological space with the discrete topology, the ideal $\mathcal{F}$ of the finite subsets of $\mathbb{R}$ and the operator $\gamma = \mathbb{R} - Bd$. Consider $A = [1,2]$ and $U \in \tau$. Note that $(\mathbb{R} - Bd(U)) \cap A = \mathbb{R} \cap A = A$ for all open set U and since A is a infinite set, we have to $(\mathbb{R} - Bd(U)) \cap A \notin \mathcal{F}$. This shows that $A^\star_\gamma = \mathbb{R}$.

On the other hand, we have $A^\star = \emptyset$. Indeed, suppose that $x \in A^\star$, then for all open set U containing the point x is true that $U \cap A \notin \mathcal{F}$. In addition, $U = \{x\}$ is a open set containing the point x, given that U is finite its intersection with any set will be finite, therefore $U \cap A \in \mathcal{F}$, which is a contradiction. This shows that $A^\star \subset A^\star_\gamma$.

Now, some basic properties of the generalized local function are presented.

Theorem 3.7. *Let (X,τ) be a topological space, $\mathcal{I},\mathcal{J}$ be ideals on X, γ be an operator associated with τ and A,B be subsets of X. Then, the following properties hold.*

(1) *If $A \subset B$, then $A^\star_\gamma \subset B^\star_\gamma$.*
(2) *$A^\star_\gamma = Cl(A^\star_\gamma) \subset Cl_\gamma(A)$.*
(3) *$\emptyset^\star_\gamma = \emptyset$.*
(4) *If $A \in \mathcal{I}$, then $A^\star_\gamma = \emptyset$.*
(5) *If $\mathcal{I} \subset \mathcal{J}$, then $A^\star_\gamma(\mathcal{J},\tau) \subset A^\star_\gamma(\mathcal{I},\tau)$.*

Proof. (1) Suppose that $x \notin B^\star_\gamma$, then there exists an open set U containing x such that $B \cap \gamma(U) \in \mathcal{I}$. Since $A \subset B$, we have $A \cap \gamma(U) \subset B \cap \gamma(U)$ and $A \cap \gamma(U) \in \mathcal{I}$. Therefore, $x \notin A^\star_\gamma$.

(2) Let $x \in Cl(A^\star_\gamma)$ and U be an open set containing x. Then, $U \cap A^\star_\gamma \neq \emptyset$ and there exists $y \in U \cap A^\star_\gamma$ such that U containing y and $y \in A^\star_\gamma$. Therefore, $\gamma(U) \cap A \notin \mathcal{I}$ and $x \in A^\star_\gamma$. Since always it is true that $A^\star_\gamma \subset Cl(A^\star_\gamma)$ we obtain $A^\star_\gamma = Cl(A^\star_\gamma)$.

On the other hand, let $x \in Cl(A^\star_\gamma)$ and U an open set containing x, then $U \cap A^\star_\gamma \neq \emptyset$. Therefore, there exists $y \in U \cap A^\star_\gamma$ with $\gamma(U) \cap A \notin \mathcal{I}$, hence $\gamma(U) \cap A \neq \emptyset$. This show that $x \in Cl_\gamma(A)$ and $Cl(A^\star_\gamma) \subset Cl_\gamma(A)$.

(3) Suppose that $\emptyset^\star_\gamma \neq \emptyset$. Then, there exists $x \in \emptyset^\star_\gamma$ and an open set U containing x, implies that $\gamma(U) \cap \emptyset \notin \mathcal{I}$. Hence, $\emptyset \notin \mathcal{I}$ which is a contradiction. This show that $\emptyset^\star_\gamma = \emptyset$

(4) Suppose that $A \in \mathcal{I}$ and $A^\star_\gamma \neq \emptyset$. Then, there exists $x \in A^\star_\gamma$ and for some open set U containing x, we have $\gamma(U) \cap A \notin \mathcal{I}$. Since $A \in \mathcal{I}$ we obtain $\gamma(U) \cap A \in \mathcal{I}$, which is a contradiction. This show that $A^\star_\gamma = \emptyset$.

(5) If $x \notin A^\star_\gamma(\mathcal{I},\tau)$, then there exists an open set U containing x such that $A \cap \gamma(U) \in \mathcal{I}$. Since $\mathcal{I} \subset \mathcal{J}$ we have $A \cap \gamma(U) \in \mathcal{J}$ and hence $x \notin A^\star_\gamma(\mathcal{J},\tau)$. Therefore, $A^\star_\gamma(\mathcal{J},\tau) \subset A^\star_\gamma(\mathcal{I},\tau)$. □

Note that the generalized local function with respect to an operator can be seen as a monotonous application of sets of X.

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ an monotone operator associated with τ. If A an subset of X, then*

$$U \cap A^\star_\gamma = U \cap (U \cap A)^\star_\gamma \subset (U \cap A)^\star_\gamma,$$

for all subset γ-open U of X.

Proof. Let $x \in U \cap A^\star_\gamma$. Taking V an open set containing x. Since $x \in U$ and U is a γ-open set, there exists an open set W such that $x \in W \subset \gamma(W) \subset U$. Note that $V \cap W$ is an open set and $x \in V \cap W$. Since $x \in A^\star_\gamma$, we obtain $\gamma(V \cap W) \cap A \notin \mathcal{I}$. Since γ is monotone, $\gamma(V \cap W) \subset \gamma(V) \cap \gamma(W)$, but $\gamma(W) \subset (U)$.

In this way $\gamma(V \cap W) \cap A \subset \gamma(V) \cap (U \cap A)$ and since $\gamma(V \cap W) \cap A \notin \mathcal{I}$, we obtain $\gamma(V) \cap (U \cap A) \notin \mathcal{I}$. Therefore, $x \in (U \cap A)^\star_\gamma$.

Given that $U \cap A^\star_\gamma \subset (U \cap A)^\star_\gamma$, we have $U \cap (U \cap A^\star_\gamma) \subset U \cap (U \cap A)^\star_\gamma$, what it means to

$$U \cap A^\star_\gamma \subset U \cap (U \cap A)^\star_\gamma.$$

Finally, since $U \cap A \subset A$ we obtain $(U \cap A)^\star_\gamma \subset A^\star_\gamma$ and hence, $U \cap (U \cap A)^\star_\gamma \subset U \cap A^\star_\gamma$. This shows that $x \in (U \cap A)^\star_\gamma$. □

Theorem 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ an operator associated with τ. If A, B are subsets of X, then $A^\star_\gamma \cup B^\star_\gamma \subset (A \cup B)^\star_\gamma$. The equality is obtained if γ is monotone.*

Proof. By Theorem 3.7, it is satisfied that $A^\star_\gamma \subset (A \cup B)^\star_\gamma$ and $B^\star_\gamma \subset (A \cup B)^\star_\gamma$. Thus, $A^\star_\gamma \cup B^\star_\gamma \subset (A \cup B)^\star_\gamma$.

Conversely, let $x \notin A^\star_\gamma \cup B^\star_\gamma$. Then, there exist two open sets U and V containing x such that $\gamma(U) \cap A \in \mathcal{I}$ and $\gamma(V) \cap B \in \mathcal{I}$. Since γ is monotone, we have $\gamma(U \cap V) \cap A \subset \gamma(U) \cap A$ and so, $\gamma(U \cap V) \cap A \in \mathcal{I}$. Analogously, $\gamma(U \cap V) \cap B \in \mathcal{I}$ and as $\mathcal{I}$ is an ideal we conclude $[\gamma(U \cap V) \cap A] \cup [\gamma(U \cap V) \cap B] \in \mathcal{I}$, but $[\gamma(U \cap V) \cap A] \cup [\gamma(U \cap V) \cap B] = (\gamma(U \cap V)) \cap (A \cup B)$. This shows that $(\gamma(U \cap V)) \cap (A \cup B) \in \mathcal{I}$ and $x \notin (A \cup B)^\star_\gamma$. Therefore, $(A \cup B)^\star_\gamma \subset A^\star_\gamma \cup B^\star_\gamma$ and the equality is obtained. □

The following example shows that the inclusion of Theorem 3.9 is proper if the operator is not monotone.

Example 3.10. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$ with $\mathcal{I} = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$. We define the operator γ as:

$$\gamma(U) = \begin{cases} Cl(U), & \text{for} \quad a \notin U \\ X - Bd(U), & \text{for} \quad a \in U \end{cases}$$

Choosing $\{b\} \subset \{a, b\}$, we have $\gamma(\{b\}) = Cl(\{b\}) = \{b, c, d\}$ and $\gamma(\{a, b\}) = X - Bd(\{a, b\}) = \{a, b\}$. Therefore, $\gamma(\{b\}) \nsubseteq \gamma(\{a, b\})$. This show that γ is not a monotone operator.

Now, let $A = \{a, b\}$ and $B = \{c, d\}$ be two subsets of X. We conclude that $A^\star_\gamma = \{a, c, d\}$ and $B^\star_\gamma = \{c\}$. Therefore, $A^\star_\gamma \cup B^\star_\gamma = \{a, c, d\} \cup \{c\} = \{a, c, d\}$. On the other hand, $(A \cup B)^\star_\gamma = X^\star_\gamma = X$. In consequence, $A^\star_\gamma \cup B^\star_\gamma = \{a, c, d\} \subset X = (A \cup B)^\star_\gamma$.

Theorem 3.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ be a monotone operator associated with τ. If A, B are subsets of X, then*

$$\begin{aligned} A^\star_\gamma - B^\star_\gamma &= (A-B)^\star_\gamma - B^\star_\gamma \\ &\subset (A-B)^\star_\gamma. \end{aligned}$$

Proof. Using Theorem 3.9, note that:

$$\begin{aligned} A^\star_\gamma &= [(A-B) \cup (A \cap B)]^\star_\gamma \\ &= (A-B)^\star_\gamma \cup (A \cap B)^\star_\gamma \\ &\subset (A-B)^\star_\gamma \cup B^\star_\gamma. \end{aligned}$$

Hence, we obtain:

$$\begin{aligned} A^\star_\gamma - B^\star_\gamma &\subset [(A-B)^\star_\gamma \cup B^\star_\gamma] - B^\star_\gamma \\ &= [(A-B)^\star_\gamma - (B)^\star_\gamma] \cup [B^\star_\gamma - B^\star_\gamma] \\ &= (A-B)^\star_\gamma - (B)^\star_\gamma. \end{aligned}$$

On the other hand, $(A-B)^\star_\gamma \subset A^\star_\gamma$ and $(A-B)^\star_\gamma - B^\star_\gamma \subset A^\star_\gamma - B^\star_\gamma$. Therefore,

$$A^\star_\gamma - B^\star_\gamma = (A-B)^\star_\gamma - B^\star_\gamma \subset (A-B)^\star_\gamma.$$

□

Corollary 3.12. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ be a monotone operator associated with τ and A, B two subsets of X. If $B \in \mathcal{I}$, then

$$(A \cup B)^\star_\gamma = A^\star_\gamma = (A-B)^\star_\gamma.$$

Proof. Suppose that $B \in \mathcal{I}$. By Theorem 3.7, we have $B^\star_\gamma = \emptyset$. In addition, by Theorem 3.11, we conclude $A^\star_\gamma = (A-B)^\star_\gamma$. On the other hand,

$$(A \cup B)^\star_\gamma = A^\star_\gamma \cup B^\star_\gamma = A^\star_\gamma \cup \emptyset = A^\star_\gamma.$$

□

The example presented below shows that $(A^\star_\gamma)^\star_\gamma$ is not a subset of $A^\star_\gamma$ in general.

Example 3.13. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ with $\mathcal{I} = \{\emptyset, \{a\}\}$. We define an operator γ as:

$$\gamma(U) = \begin{cases} Cl(U), & \text{for} \quad a \in U \\ X - Bd(U), & \text{for} \quad a \notin U \end{cases}$$

Let $A = \{a, c\}$. Note that $A^\star_\gamma = \{b, c\}$ and $(A^\star_\gamma)^\star_\gamma = \{a, b, c\}$. That is, $(A^\star_\gamma)^\star_\gamma$ is not a subset of $A^\star_\gamma$.

4. Compatible ideals in terms of an operator

In this section introduces the notion of compatibility of an ideal $\mathcal{I}$ with a topology τ in terms of the generalized local function given in Definition 3.1, the relationship between this concept and the concept of compatibility of an ideal $\mathcal{I}$ with τ due to O. Njåstad [11]. In addition, a characterization and some properties are studied using this notion.

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ an operator associated with τ. We say that τ is γ-compatible with $\mathcal{I}$, denoted $\tau \sim_\gamma \mathcal{I}$, if for every subset A of X containing x, there exists an open set U containing x such that $\gamma(U) \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Remark 4.2. Note that the fact that τ is γ-compatible with $\mathcal{I}$, means that for each subset A of X if $A \cap A^\star_\gamma = \emptyset$, then $A \in \mathcal{I}$.

Theorem 4.3. *If τ is compatible with $\mathcal{I}$, then τ is γ-compatible with $\mathcal{I}$.*

Proof. Let A be a subset of X containing x and suppose that there exists an open set U containing x such that $\gamma(U) \cap A \in \mathcal{I}$. Since $U \cap A \subset \gamma(U) \cap A$, we have $U \cap A \in \mathcal{I}$. Since, τ is compatible with $\mathcal{I}$ and hence $A \in \mathcal{I}$. Therefore τ is γ-compatible with $\mathcal{I}$. □

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ a monotone operator associated with τ. If A is a subset of X, then the following properties are equivalent:*

(1) $\tau \sim_\gamma \mathcal{I}$.
(2) If there exist an open covering $\{U_\lambda : \lambda \in \Lambda\}$ of A such that $\gamma(U_\lambda) \cap A \in \mathcal{I}$ for every $\lambda \in \Lambda$, then $A \in \mathcal{I}$.
(3) If $A \cap A^\star_\gamma = \emptyset$, then $A \in \mathcal{I}$.
(4) $A - A^\star_\gamma \in \mathcal{I}$.
(5) If A contains a nonempty subset B with $B \subset B^\star_\gamma$, then $A \in \mathcal{I}$.

Proof. (1) ⇒ (2) Let $\{U_\lambda : \lambda \in \Lambda\}$ an open covering of A such that $\gamma(U_\lambda) \cap A \in \mathcal{I}$ for all $\lambda \in \Lambda$. If $x \in A$, then exists a set $U_\lambda \in \tau$ such that $x \in U_\lambda$ and $\gamma(U_\lambda) \cap A \in \mathcal{I}$. Since, $\tau \sim_\gamma \mathcal{I}$ we obtain $A \in \mathcal{I}$.

(2) ⇒ (3) Suppose that $A \cap A^\star_\gamma = \emptyset$. Let $x \in A$, then $x \notin A^\star_\gamma$ and hence there exists an open set U_x containing x such that $\gamma(U_x) \cap A \in \mathcal{I}$. Now, note that the collection $\{U_x : x \in A\}$, is an open covering of A that satisfies $\gamma(U_x) \cap A \in \mathcal{I}$ for every $x \in A$. Therefore, we obtain $A \in \mathcal{I}$.

(3) ⇒ (4) Note that $A - A^\star_\gamma \subset A$ and hence $(A - A^\star_\gamma) \cap (A - A^\star_\gamma)^\star_\gamma \subset (A - A^\star_\gamma) \cap A^\star_\gamma = \emptyset$. In addition, $(A - A^\star_\gamma) \cap (A - A^\star_\gamma)^\star_\gamma = \emptyset$, by hypothesis, we obtain $A - A^\star_\gamma \in \mathcal{I}$.

(4) ⇒ (5) Let $A - A^\star_\gamma \in \mathcal{I}$. Choosing $J = A - A^\star_\gamma$, we have $J = A - A^\star_\gamma \in \mathcal{I}$. Note that $A = J \cup (A \cap A^\star_\gamma)$, of the fact that γ is a monotone operator and from Theorem 3.9, we obtain $A^\star_\gamma = J^\star_\gamma \cup (A \cap A^\star_\gamma)^\star_\gamma$.

In addition, $J \in \mathcal{I}$ and by Theorem 3.7 we conclude that $J^\star_\gamma = \emptyset$. Therefore, $A^\star_\gamma = (A \cap A^\star_\gamma)^\star_\gamma$ and $A \cap A^\star_\gamma = A \cap (A \cap A^\star_\gamma)^\star_\gamma \subset (A \cap A^\star_\gamma)^\star_\gamma$. Since $(A \cap A^\star_\gamma) \subset A$, we have $A \cap A^\star_\gamma = \emptyset$. In consequence, $A = J \cup (A \cap A^\star_\gamma) = J \cup \emptyset = J \in \mathcal{I}$.

(5) ⇒ (1) Let A be an open subset in X. Suppose for all $x \in A$, there exists an open set U containing x such that $\gamma(U) \cap A \in \mathcal{I}$. Then, $A \cap A^\star_\gamma = \emptyset$.

Let B be a subset in A such that $B \subset B^\star_\gamma$. Note that $B = \emptyset$, indeed $B = B \cap B^\star_\gamma \subset A \cap A^\star_\gamma = \emptyset$ and by hypothesis, we have $A \in \mathcal{I}$. □

Note that the condition that γ is a monotone operator is only used in the proof of (4) ⇒ (5).

Theorem 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ an monotone operator associated with τ and A an subset of X. If τ is γ-compatible with $\mathcal{I}$, then $A \cap A^\star_\gamma = \emptyset$ implies that $A^\star_\gamma = \emptyset$.*

Proof. Suppose that $A \cap A^\star_\gamma = \emptyset$. From Theorem 4.4, we have $A \in \mathcal{I}$ and by Theorem 3.7, we conclude $A^\star_\gamma = \emptyset$. $\square$

Theorem 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ be a monotone operator associated with τ. If A is a subset of X, then the following properties are equivalent:*

(1) If $A \cap A^\star_\gamma = \emptyset$, then $A^\star_\gamma = \emptyset$.
(2) $(A - A^\star_\gamma)^\star_\gamma = \emptyset$.
(3) $(A \cap A^\star_\gamma)^\star_\gamma = A^\star_\gamma$.

Proof. (1) $\Rightarrow$ (2) Let $B = A - A^\star_\gamma$. Then

$$\begin{aligned} B \cap B^\star_\gamma &= (A - A^\star_\gamma) \cap (A - A^\star_\gamma)^\star_\gamma \\ &= (A \cap (X - A^\star_\gamma)) \cap (A \cap (X - A^\star_\gamma))^\star_\gamma, \\ &\subset (A \cap (X - A^\star_\gamma)) \cap [A^\star_\gamma \cap (X - A^\star_\gamma)^\star_\gamma] \\ &= [A^\star_\gamma \cap (X - A^\star_\gamma)] \cap [A \cap (X - A^\star_\gamma)^\star_\gamma] \\ &= \emptyset. \end{aligned}$$

Hence, $B \cap B^\star_\gamma = \emptyset$ and by hypothesis, $B^\star_\gamma = \emptyset$. This shows that $(A - A^\star_\gamma)^\star_\gamma = \emptyset$.

(2) $\Rightarrow$ (3) Note that $A = (A - A^\star_\gamma) \cup (A \cap A^\star_\gamma)$. Since γ is monotone, $A^\star_\gamma = (A - A^\star_\gamma)^\star_\gamma \cup (A \cap A^\star_\gamma)^\star_\gamma$ is satisfied. By hypothesis, we have $(A - A^\star_\gamma)^\star_\gamma = \emptyset$ and hence, $A^\star_\gamma = \emptyset \cup (A \cap A^\star_\gamma)^\star_\gamma = (A \cap A^\star_\gamma)^\star_\gamma$.

(3) $\Rightarrow$ (1) Suppose that $A \cap A^\star_\gamma = \emptyset$. By hypothesis, $(A \cap A^\star_\gamma)^\star_\gamma = A^\star_\gamma$. Therefore, $(A \cap A^\star_\gamma)^\star_\gamma = \emptyset^\star_\gamma = \emptyset$. $\square$

Note that the condition that γ is a monotone operator is only used in the proof of (2) $\Rightarrow$ (3).

Corollary 4.7. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is a subset of X, then the following properties hold:

(1) If $\tau \sim_\gamma \mathcal{I}$ and $A \cap A^\star_\gamma = \emptyset$, then $A^\star_\gamma = \emptyset$.
(2) If $\tau \sim_\gamma \mathcal{I}$, then $(A - A^\star_\gamma)^\star_\gamma = \emptyset$.
(3) If $\tau \sim_\gamma \mathcal{I}$, then $(A \cap A^\star_\gamma)^\star_\gamma = A^\star_\gamma$.

Theorem 4.8. *Let (X, τ) be a topological space, U be a γ-open set, W be an open set, γ be a regular operator and $U \cap W \neq \emptyset$, then $Int_\gamma(\gamma(W) \cap U) \neq \emptyset$.*

Proof. Suppose that $Int_\gamma(\gamma(W) \cap U) = \emptyset$. Then

$$\begin{aligned} X &= X - Int_\gamma(\gamma(W) \cap U) \\ &= Cl_\gamma[X - (\gamma(W) \cap U)] \\ &= Cl_\gamma[(X - \gamma(W)) \cup (X - U)]. \end{aligned}$$

Since γ is a regular operator, we have $Cl_\gamma[(X - \gamma(W)) \cup (X - U)] = Cl_\gamma[(X - \gamma(W))] \cup Cl_\gamma[(X - U)]$. Consequently, $X = Cl_\gamma[(X - \gamma(W))] \cup Cl_\gamma[(X - U)]$.

Let $x \in U \cap W \subset X$. If $x \in Cl_\gamma(X - \gamma(W))$, then W is an open set containing x and so, $\gamma(W) \cap (X - \gamma(W)) \neq \emptyset$, which is impossible.

Now, suppose that $x \in Cl_\gamma[(X - U)]$. Since $x \in U$ and U is γ-open set, we conclude there exists an open set V such that $x \in V \subset \gamma(V) \subset U$. Therefore, $\gamma(V) \cap (X - U) \neq \emptyset$, which is impossible. $\square$

Theorem 4.9. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and γ an regular operator associated with τ. If $\gamma(\emptyset)=\emptyset$, then the following properties are equivalent:*

(1) $X = X^\star_\gamma$.
(2) $\gamma(\tau)\cap\mathcal{I}=\emptyset$, *where* $\gamma(\tau)=\{\gamma(V) : V\in\tau\}$.
(3) $Int_\gamma(A)=\emptyset$, *for all* $A\in\mathcal{I}$.
(4) $U\subset U^\star_\gamma$, *for all* $U\in\tau_\gamma$, *where* τ_γ *is the collection of all* γ*-open sets.*

Proof. (1) $\Rightarrow$ (2) Suppose that there exists a subset A of X such that $A\in\gamma(\tau)\cap\mathcal{I}$. This shows that $A\in\mathcal{I}$, furthermore there exists some open set $V\in\tau$ such that $A=\gamma(V)$. Since $\gamma(V)\neq\emptyset$ and $\gamma(\emptyset)=\emptyset$, we have $V\neq\emptyset$. Let $x\in V\subset X=X^\star_\gamma$, thus $\gamma(V)\cap X\notin\mathcal{I}$. This is $\gamma(V)\notin\mathcal{I}$, which is impossible.

(2) $\Rightarrow$ (3) Suppose that $Int_\gamma(A)\neq\emptyset$ for some $A\in\mathcal{I}$. Taking $x\in Int_\gamma(A)$, then there exists an open set V such that $x\in V\subset\gamma(V)\subset A$. Since $A\in\mathcal{I}$ we obtain $\gamma(V)\in\mathcal{I}$, but $\gamma(V)\in\gamma(\tau)$. In consequence, $\gamma(V)\in\gamma(\tau)\cap\mathcal{I}$ and $\gamma(V)\neq\emptyset$. Therefore, $\gamma(\tau)\cap\mathcal{I}\neq\emptyset$.

(3) $\Rightarrow$ (4) Let U be a γ-open and W be an open set, containing x. If $Int_\gamma(\gamma(W)\cap U)=\emptyset$, then by Theorem 4.8, γ is a regular operator (contradiction). This show that $\gamma(W)\cap U\notin\mathcal{I}$.

(4) $\Rightarrow$ (1) Note that X is γ-open set and from hypothesis, we have $X\subset X^\star_\gamma$. On the other hand, always true that $X^\star_\gamma\subset X$. Therefore, $X=X^\star_\gamma$. □

In Theorem 4.9, the condition $\gamma(\emptyset)=\emptyset$ is only used in the proof of (1) $\Rightarrow$ (2) and the condition γ is a regular operator is only used in the proof of (3) $\Rightarrow$ (4).

Theorem 4.10. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space, γ an operator associated with τ and A be a subset of X. If $\mathcal{I}$ is γ-compatible with τ, then*

$$A^\star_\gamma\subset(A^\star_\gamma)^\star_\gamma.$$

Proof. It follows from Theorems 3.7, 3.11 and 4.4. □

Theorem 4.11. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space, γ an operator associated with τ and A,G be an subsets of X. If τ is γ-compatible with $\mathcal{I}$, then*

$$\begin{aligned} Cl(G\cap A)^\star_\gamma &= (G\cap A)^\star_\gamma\subset(G\cap(A)^\star_\gamma)^\star_\gamma\\ &\subset Cl_\gamma(G\cap(A)^\star_\gamma). \end{aligned}$$

Proof. From Theorem 3.7 we have $Cl(G\cap A)^\star_\gamma=(G\cap A)^\star_\gamma$. Since $\tau\sim_\gamma\mathcal{I}$ by Corollary 4.7, $(G\cap A)^\star_\gamma=((G\cap A)\cap(G\cap A)^\star_\gamma)^\star_\gamma$. Note that $G\cap A\subset G$ and $G\cap A\subset A$, thus $(G\cap A)^\star_\gamma\subset(G\cap(A)^\star_\gamma)^\star_\gamma$. Finally, by Theorem 3.7 we conclude

$$(G\cap(A)^\star_\gamma)^\star_\gamma\subset Cl_\gamma(G\cap(A)^\star_\gamma).$$

□

5. Generalized closure

In this section we introduce a notion of closure in terms of the generalized local function, study some basic properties and the relation that this closure has with those known in the literature.

Definition 5.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ be an operator associated with τ and A be a subset of X. We define $Cl^{\star}_{\gamma}(A)$ as the union of A with its generalized local function, this is,

$$Cl^{\star}_{\gamma}(A) = A \cup A^{\star}_{\gamma}.$$

The following theorem shows some properties that satisfies $Cl^{\star}_{\gamma}(A)$.

Theorem 5.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ be an operator associated with τ and A, B two a subsets of X, then the following properties hold:*

(1) If $A \subset B$, then $Cl^{\star}_{\gamma}(A) \subset Cl^{\star}_{\gamma}(B)$.
(2) $Cl^{\star}_{\gamma}(\emptyset) = \emptyset$.
(3) $A \subset Cl^{\star}_{\gamma}(A) \subset Cl_{\gamma}(A)$.
(4) If γ is a monotone operator, then

$$Cl^{\star}_{\gamma}(A \cup B) = Cl^{\star}_{\gamma}(A) \cup Cl^{\star}_{\gamma}(B).$$

(5) $Cl^{\star}_{\gamma}(A) \subset Cl^{\star}_{\gamma}(Cl^{\star}_{\gamma}(A))$.
(6) If $A \in \mathcal{I}$, then $Cl^{\star}_{\gamma}(A) = A$.

Proof. (1) Suppose that $x \in Cl^{\star}_{\gamma}(A)$, then $x \in A \cup A^{\star}_{\gamma}$. Taking $x \in A$ and since $A \subset B$ we have $x \in B \subset B \cup B^{\star}_{\gamma} = Cl^{\star}_{\gamma}(B)$. On the other hand, if $x \in A^{\star}_{\gamma}$ using Theorem 3.7, we obtain $A^{\star}_{\gamma} \subset B^{\star}_{\gamma}$. Hence, $x \in B^{\star}_{\gamma} \subset B \cup B^{\star}_{\gamma} = Cl^{\star}_{\gamma}(B)$.

(2) Note that $Cl^{\star}_{\gamma}(\emptyset) = \emptyset \cup (\emptyset)^{\star}_{\gamma}$, then by Theorem 3.7, we have $(\emptyset)^{\star}_{\gamma} = \emptyset$ and $Cl^{\star}_{\gamma}(\emptyset) = \emptyset \cup \emptyset = \emptyset$.

(3) Since $A \subset A \cup A^{\star}_{\gamma}$, we have $A \subset Cl^{\star}_{\gamma}(A)$. On the other hand, using Theorem 3.7, $Cl^{\star}_{\gamma}(A) = A \cup A^{\star}_{\gamma} \subset A \cup Cl_{\gamma}(A) = Cl_{\gamma}(A)$.

(4) From monotone operator γ and Theorem 3.9, we obtain

$$\begin{aligned}
Cl^{\star}_{\gamma}(A \cup B) &= (A \cup B) \cup (A \cup B)^{\star}_{\gamma} \\
&= (A \cup B) \cup (A)^{\star}_{\gamma} \cup (B)^{\star}_{\gamma} \\
&= (A \cup (A)^{\star}_{\gamma}) \cup (B \cup (B)^{\star}_{\gamma}) \\
&= Cl^{\star}_{\gamma}(A) \cup Cl^{\star}_{\gamma}(B).
\end{aligned}$$

(5) It follows from (1).

(6) Since $A \in \mathcal{I}$ and using Theorem 3.7, we obtain $A^{\star}_{\gamma} = \emptyset$. Then, $Cl^{\star}_{\gamma}(A) = A \cup A^{\star}_{\gamma} = A \cup \emptyset = A$. □

Remark 5.3. Note that by varying the ideal $\mathcal{I}$ and the operator γ, also using the results obtained in the Remark 3.2, notions of closures already defined in the literature are recovered.

(1) If γ is the identity operator and $\mathcal{I} = \{\emptyset\}$, then $Cl^{\star}_{\gamma}(A) = A \cup Cl(A) = Cl(A)$.
(2) If γ is an arbitrary operator and $\mathcal{I} = \mathcal{P}(X)$, then $Cl^{\star}_{\gamma}(A) = A \cup \emptyset = A$.
(3) If γ is the identity operator and $\mathcal{I} = \mathcal{N}$, then $Cl^{\star}_{\gamma}(A) = A \cup Cl(Int(Cl(A)))$ [7].
(4) If γ is the closure operator and $\mathcal{I} = \{\emptyset\}$, then $Cl^{\star}_{\gamma}(A) = A \cup Cl_{\theta}(A) = Cl_{\theta}(A)$ [17].
(5) If γ is the interior closure operator and $\mathcal{I} = \{\emptyset\}$, then $Cl^{\star}_{\gamma}(A) = A \cup Cl_{\delta}(A) = Cl_{\delta}(A)$ [17].
(6) If γ is an arbitrary operator and $\mathcal{I} = \{\emptyset\}$, then $Cl^{\star}_{\gamma}(A) = A \cup Cl_{\gamma}(A) = Cl_{\gamma}(A)$ [6].

Definition 5.4. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ be an operator associated with τ. A subset A of X is said to be $\tau_\gamma^\star$-closed, if $A = Cl_\gamma^\star(A)$.

Theorem 5.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and γ be a monotone operator associated with τ. If A, B are subsets $\tau_\gamma^\star$-closed of X, then $A \cup B$ is a subset $\tau_\gamma^\star$-closed of X.*

Proof. By hypothesis, $A = Cl_\gamma^\star(A)$ and $B = Cl_\gamma^\star(B)$, then $A \cup B = Cl_\gamma^\star(A) \cup Cl_\gamma^\star(B)$. On the other hand, by Theorem 5.2 we obtain

$$Cl_\gamma^\star(A \cup B) = Cl_\gamma^\star(A) \cup Cl_\gamma^\star(B)$$

and $A \cup B = Cl_\gamma^\star(A \cup B)$. Therefore, $A \cup B$ is a set $\tau_\gamma^\star$. □

The theorem presented below characterizes the $\tau_\gamma^\star$-closed sets.

Theorem 5.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, γ be an operator associated with τ and A be an subset of X. A is a $\tau_\gamma^\star$-closed if and only if $A_\gamma^\star \subset A$.*

Proof. Suppose that A is a $\tau_\gamma^\star$-closed set. Thus, $A = Cl_\gamma^\star(A) = A \cup A_\gamma^\star$ and hence $A_\gamma^\star \subset A$. Conversely, let $A_\gamma^\star \subset A$. Since $Cl_\gamma^\star(A) = A \cup A_\gamma^\star$ and $A \cup A_\gamma^\star \subset A$, we have $Cl_\gamma^\star(A) \subset A$.

On the other hand, by Definition 5.1, we obtain $A \subset Cl_\gamma^\star(A)$ and hence $Cl_\gamma^\star(A) = A$. Therefore, A is a $\tau_\gamma^\star$-closed set. □

The following two examples shows that in a ideal topological space $(X, \tau, \mathcal{I})$, for any subset A of X the concepts $Cl_\gamma^\star(A)$ and $Cl(A)$ are independent.

Example 5.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$ with $\gamma(U) = X - Bd(U)$. We define $A = \{b\}$.

Note that $A \in \mathcal{I}$ and by Theorem 3.7, we have $A_\gamma^\star = \emptyset$. Thus, $Cl_\gamma^\star(A) = A \cup A_\gamma^\star = \{b\} \cup \emptyset = \{b\}$. On the other hand, $Cl(\{A\}) = \{b, d\}$ and $d \in Cl(A)$, but $d \notin Cl_\gamma^\star(A)$. Therefore, $Cl(A)$ is not a subset of $Cl_\gamma^\star(A)$.

Example 5.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ with γ defined as:

$$\gamma(U) = \begin{cases} Int(Cl(U)) & \text{for} \quad c \notin U \\ Cl(U) & \text{for} \quad c \in U \end{cases}$$

Let $A = \{b, d\}$. Then, $A_\gamma^\star = \{b, c, d\}$ and $Cl_\gamma^\star(A) = A \cup A_\gamma^\star = \{b, d\} \cup \{b, c, d\} = \{b, c, d\}$. On the other hand, $Cl(A) = \{b, d\}$ and hence $c \in Cl_\gamma^\star(A)$, but $c \notin Cl(A)$. Therefore, $Cl_\gamma^\star(A)$ is not a subset of $Cl(A)$.

The following example shows that $Cl_\gamma^\star(.)$ is not an idempotent operator.

Example 5.9. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$ with γ defined as:

$$\gamma(U) = \begin{cases} Cl(U) & \text{for} \quad a \in U \\ X - Bd(U) & \text{for} \quad a \notin U \end{cases}$$

Let $A = \{b, d\}$. Note, $A_\gamma^\star = \{b, c, d\}$ and also

$$Cl_\gamma^\star(A) = A \cup A_\gamma^\star = \{b, d\} \cup \{b, c, d\} = \{b, c, d\}.$$

On the other hand, $(Cl^\star_\gamma(A))^\star_\gamma = \{a,b,c,d\}$ and hence

$$\begin{aligned} Cl^\star_\gamma(Cl^\star_\gamma(A)) &= Cl^\star_\gamma(A) \cup (Cl^\star_\gamma(A))^\star_\gamma \\ &= \{b,c,d\} \cup \{a,b,c,d\} \\ &= \{a,b,c,d\}. \end{aligned}$$

This show that $Cl^\star_\gamma(Cl^\star_\gamma(A))$ is not a subset of $Cl^\star_\gamma(A)$.

Remark 5.10. Note that by Example 5.9, $Cl^\star_\gamma(.)$ is not a Kuratowski closure operator and hence not generate a topology on X where the closed sets are just the $\tau^\star_\gamma$-closed sets. However, from Theorem 5.2 $Cl^\star_\gamma(.)$ is a monotone operator associated to the topology τ. Therefore, by the Theorem 2.4 the collection: $\tau_{Cl^\star_\gamma} = \{A \subset X : \forall x \in A,\ \exists U \in \tau(x) \mid Cl^\star_\gamma(U) \subset A\}$, is a topology on X.

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(José Tormet) Departamento de Ciencias, Unidad de Estudios Básicos, Universidad de Oriente, Puerto La cruz, Venezuela.
E-mail address: `ajosetormet@gmail.com`

(Rafael Lozada-Yavina) Facultad de Ciencias Básicas, Universidad Católica del Maule, Talca, Chile &, Facultad de Ciencias e Ingeniería, Universidad Tecnológica del Perú, Lima, Perú.
E-mail address: `trobuyo@gmail.com`

(Margot Salas-Brown) Matemáticas Aplicadas y Ciencias de la Computación, Escuela de Iingeniería, Ciencia y Tecnología, Universidad del Rosario, Bogotá, Colombia.
E-mail address: `margot.salas@urosario.edu.co`