Poincare Journal of Analysis & Applications
Vol. 9, No. 1 (2022), 91-103

BIVARIATE λ- BERNSTEIN TYPE OPERATORS VIA (p,q)-CALCULUS

BASHKIM TRUPAJ† AND GAZMEND NAFEZI

Date of Receiving	:	20. 07. 2021
Date of Revision	:	16. 06. 2022
Date of Acceptance	:	18. 06. 2022

Abstract. In this paper we have established an extension of the bivariate generalization of the (p,q)-Bernstein type operators involving parameter λ. We have gave some inequalities for the operators by means of partial and full modulus of continuity and obtain a Lipschitz type theorem.

1. Introduction

Let $h \in C(S)$ with $S = [0,1]$, $\lambda \in [-1,1]$ and $m_1 \in \mathbb{N}$. In 2018, Cai et al. [16] proposed a new generalization of Bernstein operators based on a fixed real parameter $\lambda \in [-1,1]$ as

$$\mathcal{B}^{\lambda}_{m_1}(h;y_2) = \sum_{k=0}^{m_1} \overline{\Omega}^{(\lambda)}_{m_1,k}(y_2) h\left(\frac{k}{m_1}\right), \quad x \in S \tag{1.1}$$

where the basis functions $\overline{\Omega}^{(\lambda)}_{m_1,k}(y_2)$ are defined as:

$$\begin{aligned}
\overline{\Omega}^{(\lambda)}_{m_1,0}(y_2) &= \Omega_{m_1,0}(y_2) - \frac{\lambda}{m_1+1}\Omega_{m_1+1,1}(y_2)\\
\overline{\Omega}^{(\lambda)}_{m_1,k}(y_2) &= \Omega_{m_1,k}(y_2) + \lambda\left(\frac{m_1-2j+1}{m_1^2-1}\Omega_{m_1+1,k}(y_2) - \frac{m_1-2j-1}{m_1^2-1}\Omega_{m_1+1,k+1}(y_2)\right),\\
&\quad 1 \le k \le m_1-1\\
\overline{\Omega}^{(\lambda)}_{m_1,m_1}(y_2) &= \Omega_{m_1,m_1}(y_2) - \frac{\lambda}{m_1+1}\Omega_{m_1+1,m_1}(y_2),
\end{aligned} \tag{1.2}$$

The authors have studied the established of some Korovkin type approximation properties and the degree of approximation by means of the modulus of continuity, Voronovskaja-type results and shape preserving properties for these operators. This

2010 *Mathematics Subject Classification.* 41A25, 26A15.

Key words and phrases. $\lambda-$Bernstein operators, modulus of continuity, Lipschitz class, (p,q)-Calculus.

Communicated by. Dan Bărbosu

† *Corresponding author*

work took attention of researchers from approximation theory for a short time. Since that time, lots of researchers have put forth many relevant studies on this issue and numerous articles can be given interrelated with their work [1, 2, 3, 4, 7, 8, 16, 18, 22, 15]. In the past two decades there has been a widespread application of (p, q)-calculus in the area of approximation theory. Mursaleen [20] was the first who introduced a (p, q)-analogue of the Bernstein operators. Later Mursaleen applied (p,q)-calculus to construct (p,q)-analogue of divided differences. Kang et al. [19] and some other researchers (see [25], [27], [9] and [6] etc.) studied the approximation properties of q-Bernstein polynomials. Malik et al. [21] presented a new analogue of Bernstein-Durrmeyer operators based on certain variants. Furthermore, (p, q)-analogue of Bernstein-Durrmeyer operators was defined and studied by Sharma [24]. Cai et al. [12] introduced a Bivariate tensor product (p, q) (p, q)-analogue of Kantorovich-type Bernstein-Stancu-Schurer operators and studied some convergence properties of these operators. Again, Cai et. al. [13] gave a new generalized (p, q)-Bernstein operator and studied the rate of approximation with shape preserving properties and monotonicity of the operator. Now, we give some basic definitions based on the (p,q) -calculus, which are used in this paper.
Let p, q be any(fixed) positive real number and $n \in \mathbb{N}$, the (p, q)-integer $[n]_{p,q}$ is defined as

$$[n]_{p,q} := \begin{cases} \frac{p^n-q^n}{p-q}, & \text{if } p \neq q \neq 1, \\ \frac{1-q^n}{1-q}, & \text{if } p = 1, \\ n, & \text{if } p = q = 1. \end{cases}$$

The (p, q)-factorial $[n]_{p,q}!$ is defined by

$$[n]_q! := \begin{cases} [n]_{p,q}[n-1]_{p,q}...1, & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases}$$

Also, for any n and k in $\mathbb{N} \cup \{0\}$ such that $0 \leqslant k \leqslant n$, the (p, q)-binomial coefficient is given by

$$\binom{n}{k}_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

Further, the (p, q)-analogue of Pochhammer symbol is defined by

$$(1 + x)^n_{p,q} := \prod_{i=0}^{n-1}(p^i + q^i x), \quad \text{for } n \geq 1.$$

Cai et al. [14] considered the generalized Bernstein type operators based on parameters $q-$ analogue and for fixed real parameter $\lambda \in [-1, 1]$ as

$$B^{\lambda}_{m_1,p_1,q_1}(f; y_1) = \sum_{k_1=0}^{m_1} \overline{\Omega}^{(\lambda,q_1)}_{m_1,k_1}(y_1) f\left(\frac{[k_1]_{p_1,q_1}}{p_1^{k_1-m_1}[m_1]_{p_1,q_1}}\right), y_1 \in [0, 1] \tag{1.3}$$

where the basis functions $\overline{\Omega}^{(\lambda,q_1)}_{m_1,k}(y_2)$ are defined as:

$$\overline{\Omega}^{(\lambda,p_1,q_1)}_{m_1,0}(y_1) = \Omega^{(p_1,q_1)}_{m_1,0}(y_1) - \frac{\lambda}{p_1^{1-m_1}[m_1]_{p_1,q_1+1}}\Omega^{(p_1,q_1)}_{m_1+1,1}(y_2)$$

$$\overline{\Omega}^{(\lambda,p_1,q_1)}_{m_1,j}(y_1) = \Omega^{(p_1,q_1)}_{m_1,j}(y_2) + \lambda\Bigg(\frac{p_1^{1-m_1}[m_1]_{p_1,q_1} - 2p_1^{1-j}[j]_{p_1,q_1} + 1}{p_1^{2-2m_1}[m_1]^2_{p_1,q_1} - 1}\Omega_{m_1+1,j}(y_1) -$$

$$\frac{p_1^{1-m_1}[m_1]_{(p_1,q_1)} - 2q_1p_1^{-j}[j]_{p_1,q_1} - 1}{p_1^{2-2m_1}[m_1]^2_{p_1,q_1} - 1}\Omega^{(p_1,q_1)}_{m_1+1,j+1}(y_1)\Bigg),\ 1 \le j \le m_1 - 1$$

$$\overline{\Omega}^{(\lambda,p_1,q_1)}_{m_1,m_1}(y_1) = \Omega^{(p_1,q_1)}_{m_1,m_1}(y_1) - \frac{\lambda}{p_1^{1-m_1}[m_1]_{p_1,q_1} + 1}\Omega^{(p_1,q_1)}_{m_1+1,m_1}(y_1), \tag{1.4}$$

and $\Omega^{(p_1,q_1)}_{m_1,j}(y_1) = \binom{m_2}{k}_{q_1} y^j \prod_{s=0}^{m_1}(1 - q_1^s y_1)$. Note that for $p_1 = 1$, these operators reduce to (p,q)-analogue λ-Bernstein operators (1.1) and for $p_1 = q_1 = \lambda = 1, (1.3)$ reduce to Bernstein operators [11].
Therefore, linear operators, in particular the limit (p,q)-Bernstein operator, are of significant interest for applications.

Our aim in the present paper is to construct an extension of the bivariate λ, q-Bernstein type operators involving parameters and obtain degree of approximation by means of the Lipschitz type space for functions two variables.

2. Construction of the bivariate λ-Bernstein type $(p,q)-$analogue operators

The aim of this section is to construct a bivariate extension of the $\lambda-$ Bernstein (p,q)-operators. Let $p_1 \ge 0, p_2 \ge 0$ be two given integeers and let

$B^{\lambda_j}_{m_j,p_j,q_j} : C(S) \to C(S),\ j = 1,2,$

defined for any $m_1, m_2, p_1, p_2, q_1, q_2 \in \mathbb{N}, \lambda_1, \lambda_2 \in (0,1)$ and any $g \in C(S), h \in C(S)$, respectively by

$$B^{\lambda_1}_{m_1,p_1,q_1}(g;y_1) = \sum_{k_1=0}^{m_1} \overline{\Omega}^{(\lambda,q_1)}_{m_1,k_1}(y_1) g\left(\frac{[k_1]_{p_1,q_1}}{p_1^{k-m_1}[m_1]_{p_1,q_1}}\right), y_1 \in [0,1] \tag{2.5}$$

$$B^{\lambda_2}_{m_2,p_2,q_2}(h;y_2) = \sum_{k_2=0}^{m_2} \overline{\Omega}^{(\lambda_2,q_2)}_{m_2,k_2}(y_2) h\left(\frac{[k_2]_{p_2,q_2}}{p_2^{k_2-m_2}[m_2]_{p_2,q_2}}\right), y_2 \in [0,1] \tag{2.6}$$

where $\overline{\Omega}^{(\lambda_i,q_i)}_{m_i,k_i}(y_i), i = 1,2,$ defined as relation (1.4).
For $S^2 = [0,1] \times [0,1]$ let $C(S^2)$ be the space of all continuous functions on S^2.
The parametric extensions (for this term, see [10], [17] or [5]) of (2.5) and (2.6) are the operators $B^{\lambda_1}_{m_1,p_1,q_1}, B^{\lambda_2}_{m_2,p_2,q_2} : C(S^2) \to C(S^2)$ defined for any $m_1, m_2 \in \mathbb{N}$ and any $h \in C(S^2)$ as follows:

$$B^{\lambda_1,y_1}_{m_1,p_1,q_1}(h;y_1;y_2) = \sum_{k_1=0}^{m_1} \overline{\Omega}^{(\lambda,q_1)}_{m_1,k_1}(y_1) h\left(\frac{[k]_{p_1,q_1}}{p_1^{k-m_1}[m_1]_{p_1,q_1}}, y_2\right), \tag{2.7}$$

$$B^{\lambda_2,y_2}_{m_2,p_2,q_2}(h;y_1;y_2)=\sum_{k_2=0}^{m_2}\overline{\Omega}^{(\lambda_2,q_2)}_{m_2,k_2}(y_2)h\left(y_1,\frac{[k_2]_{p_2,q_2}}{p_2^{k_2-m_2}[m_2]_{p_2,q_2}}\right). \tag{2.8}$$

For $h\in C(S^2)$ and $\lambda_1,\lambda_2\in[-1,1]$, the bivariate extension of the operator (1.3) is defined by

$$\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2)=\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,k_1,k_2}(y_1,y_2)$$
$$h\left(\frac{[k]_{p_1,q_1}}{p_1^{k-m_1}[m_1]_{p_1,q_1}},\frac{[k_2]_{p_2,q_2}}{p_2^{k_2-m_2}[m_2]_{p_2,q_2}}\right), \tag{2.9}$$

where $\{q_{m_i}\}_{m_i\in N}$ and $\{p_{m_i}\}_{m_i\in N}$ is a sequence satisfying $0<q_{m_i}<p_{m_i}\leq 1$ and

$$q_{m_i}\to 1,\ q_{m_i}^{m_i}\to c_i;\ \text{and } p_{m_i}\to 1,\ p_{m_i}^{m_i}\to d_i;\ \text{for } i=1,2.$$

Further, $\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,j_1,j_2}(y_1,y_2)=\Omega^{(\lambda_1)}_{m_1,j_1}(y_1)\Omega^{(\lambda_2)}_{m_2,j_2}(y_2)$, where $\Omega^{(\lambda_1)}_{m_1,j_1}(y_1)$ and $\Omega^{(\lambda_2)}_{m_2,j_2}(y_2)$ are defined similarly as $\Omega^{(\lambda)}_{m_1,j}(y)$ in (1.4).

We denote

$$\begin{aligned}&\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((u-y_1)^s(v-y_2)^t;y_1,y_2)=\mathcal{B}^{\lambda_1}_{m_1,p_1,q_1}((u-y_1)^s;y_1)\\ &\times\mathcal{B}^{\lambda_2}_{m_2,p_2,q_2}((v-y_2)^t;y_2)\\ =\ &\mu_{m_1,\lambda_1,s}(p_{m_1},q_{m_1},y_1)\mu_{m_2,\lambda_2,t}(p_{m_2},q_{m_2},y_2)\quad s,t\in N\cup\{0\}.\end{aligned} \tag{2.10}$$

Lemma 2.1. [14] *For the operators $\mathcal{B}^{\lambda}_{m_1,p_1,q_1}$, we have*

(i) $\mathcal{B}^{\lambda}_{m_1,p_1,q_1}(1;y_1)=1$;

(ii) $$\begin{aligned}\mathcal{B}^{\lambda}_{m_1,p_1,q_1}(t;y_1)&=y_1+\frac{2\lambda[m_1+1]_{p_1,q_1}y_1^2(1-y_1^{n-1})}{p_1^m(p_1^{2-2m_1}[m_1]^2_{q_1}-1)}(1-\frac{q_1}{p_1})+\frac{\lambda p_1^{m_1}(1-y_1^{m_1+1})}{q_1[m_1]_{p_1,q_1}(p_1^{1-m_1}[m_1]_{p_1,q_1}-1)}+\\ &\frac{\lambda[m_1+1]_{p_1,q_1}y_1(1-y_1^{m_1})}{[m_1]_{p_1,q_1}(p_1^{1-m_1}[m_1]_{p_1,q_1}-1)}(\frac{1}{p_1}-\frac{1}{q_1}-\frac{2}{p_1(p_1^{1-m_1}[m_1]_{p_1,q_1}+1})\\ &-\frac{\lambda\prod_{s=0}^{m_1}(p_1^s-q_1^sy_1)}{p_1^{\frac{m_1(m_1-1)}{2}}q_1[m_1]_{p_1,q_1}(p_1^{1-m_1}[m_1]_{p_1,q_1}-1)}.\end{aligned}$$

(iii) $$\begin{aligned}\mathcal{B}^{\lambda}_{m_1,p_1,q_1}(t^2;y_1)&=y_1^2+\frac{p_1^{m_1-1}y_1(1-y_1)}{[m_1]_{p_1,q_1}}+\frac{\lambda(q_1^2-p_1^2)[m_1+1]_{p_1,q_1}y_1^2(1-y_1^{n-1})}{p_1^2q_1[m_1]_{p_1,q_1}(p_1^{1-m_1}[m_1]_{p_1,q_1}+1)}\\ &-\frac{p_1^{2m_1}\lambda(1-y_1^{m_1+1})}{q_1^2[m_1]^2_{p_1,q_1}(p_1^{1-m_1}[m_1]_{p_1,q_1}-1)}+\frac{2\lambda(p_1^2q_1-p_1^3-p_1q_1^2-q_1^3)[m_1+1]_{p_1,q_1}y_1^2(1-y_1^{m_1-1})}{p_1^3q_1[m_1]_{p_1,q_1}(p_1^{2-2m_1}[m_2]^2_{p_1,q_1}-1)}\\ &+\frac{p_1^{m_1-2}(p_1^2+q_1^2)\lambda[m_1+1]_{p_1,q_1}y_1(1-y_1^{m_1})}{q_1^2[m_1]^2_{p_1,q_1}(p_1^{1-m_1}[m_2]_{p_1,q_1}+1)}+\frac{p_1\lambda\prod_{s=0}^{m_1}(p_1^s-q_1^sy_1)}{p_1^{\frac{(m_1-1)(m1-2)}{2}}q_1^2[m_1]^2_{p_1,q_1}(p_1^{1-m_1}[m_1]_{p_1,q_1}-1)}\\ &+\frac{2q_1(p_1^2-q_1^2\lambda[m_1+1]_{p_1,q_1}[m_1-1]_{p_1,q_1}y_1^3(1-y_1^{m_1-2})}{p_1^{m_1+2}[m_1]_{p_1,q_1}(p_1^{2-2m_1}[m_1]^2_{p_1,q_1}-1)}\\ &+\frac{2p_1^{m_1-1}(2q_1-p_1)\lambda[m_1+1]_{p_1,q_1}y_1}{q_1^2[m_1]^2_{p_1,q_1}(p_1^{2-2m_1}[m_1]^2_{p_1,q_1}-1)}(\frac{\prod_{s=0}^{m_1-1}(p_1^s-q_1^sy_1)}{p_1^{\frac{m_1(m_1-1)}{2}}}-(1-y_1^{m_1})).\end{aligned}$$

Let $e_{rs}(y_1,y_2)=t_1^rt_2^s,\ (r,s)\in\mathbb{N}_0\times\mathbb{N}_0$, with $r+s\leq 2$. In order to obtain the main results, we need the following lemmas:

Lemma 2.2. *For the operators* $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$, *we have*

(i) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(1;y_1,y_2)=1$;

(ii) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{10};y_1,y_2) = y_1 + \frac{[m_1+1]_{q_1}y_1(1-y_1^{m_1})\lambda_1}{[m_1]_{q_1}([m_1]_{q_1}-1)} - \frac{2[m_1+1]_{q_1}y_1\lambda_1}{[m]^2_{q_1}-1}[\frac{1-y_1^{m_1}}{[m_1]_{q_1}} + q_1y_1(1-y_1^{m_1-1})] + \frac{\lambda_1}{q_1[m_1]_{q_1}([m_1]_{q_1}+1)}[1-\prod_{i=0}^{m_1}(1-q_1^iy_1) - y_1^{m_1+1} - [m_1+1]_{q_1}y_1(1-y_1^{m_1})] + \frac{\lambda_1}{[m_1]^2_{q_1}-1}\{2[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1}) - \frac{2[m_1+1]_{q_1}y_1}{q_1[m_1]_{q_1}}(1-y_1^{m_1}) + \frac{2}{q_1[m_1]_{q_1}}[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}]\}$;

(iii) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{20};y_1,y_2) = y_1^2 + \frac{y_1(1-y_1)}{[m_1]_{q_1}} + \frac{y_1\lambda_1[m_1+1]_{q_1}}{[m_1]_{q_1}}[q_1y_1(1-y_1^{m_1-1}) + \frac{1-y_1^{m_1}}{[m_1]_{q_1}}] - \frac{2[m_1+1]_{q_1}\lambda_1}{[m_1]_{q_1}([m_1]^2_{q_1}-1)}[\frac{y_1(1-y_1^{m_1})}{[m_1]_{q_1}} + q_1(2+q_1)y_1^2(1-y_1^{m_1-1}) + q_1^3[m_1-1]_{q_1}y_1^3(1-y_1^{m_1-1})] - \frac{\lambda_1}{q_1[m_1]_{q_1}([m_1]_{q_1}+1)}\{[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1}) - \frac{[m_1+1]_{q_1}y_1(1-y_1^{m_1})}{q_1[m_1]_{q_1}} + \frac{1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}}{q_1[m_1]_{q_1}}\} + \frac{2\lambda_1}{[m_1]_{q_1}([m_1]^2_{q_1}-1)}\{q_1[m_1-1]_{q_1}[m_1+1]_{q_1}y_1^3(1-y_1^{m_1-2}) - \frac{(1-q_1)[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})}{q_1} + \frac{[m_1+1]_{q_1}y_1(1-y_1^{m_1})}{q_1^2[m_1]_{q_1}} - \frac{1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}}{q_1^2[m_1]_{q_1}}\} + \frac{2}{q_1[m_1]_{q_1}}[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}]$;

(iv) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{01};y_1,y) = y_2 + \frac{[m_2+1]_{q_2}y_2(1-y_2^{m_2})\lambda_2}{[m_2]_{q_2}([m_2]_{q_2}-1)} - \frac{2[m_2+1]_{q_2}y_2\lambda_2}{[m]^2_{q_2}-1}[\frac{1-y_2^{m_2}}{[m_2]_{q_2}} + q_2y_2(1-y_2^{m_2-1})] + \frac{\lambda_2}{q_2[m_2]_{q_2}([m_2]_{q_2}+1)}[1-\prod_{i=0}^{m_2}(1-q_2^iy_2) - y_2^{m_2+1} - [m_2+1]_{q_2}y_2(1-y_2^{m_2})] + \frac{\lambda_2}{[m_2]^2_{q_2}-1}\{2[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1}) - \frac{2[m_2+1]_{q_2}y_2}{q_2[m_2]_{q_2}}(1-y_2^{m_2}) + \frac{2}{q_2[m_2]_{q_2}}[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}]\}$;

(v) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{02};y_1,y_2) = y_2^2 + \frac{y_2(1-y_2)}{[m_2]_{q_2}} + \frac{y_2\lambda_2[m_2+1]_{q_2}}{[m_2]_{q_2}([m_2]_{q_2}-1)}[q_2y_2(1-y_2^{m_2-1}) + \frac{1-y_2^{m_2}}{[m_2]_{q_2}}] - \frac{2[m_2+1]_{q_2}\lambda_2}{[m_2]_{q_2}([m_2]^2_{q_2}-1)}[\frac{y_2(1-y_2^{m_2})}{[m_2]_{q_2}} + q_2(2+q_2)y_2^2(1-y_2^{m_2-1}) + q_2^3[m_2-1]_{q_2}y_2^3(1-y_2^{m_2-1})] - \frac{\lambda_2}{q_2[m_2]_{q_2}([m_2]_{q_2}+1)}\{[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1}) - \frac{[m_2+1]_{q_2}y_2(1-y_2^{m_2})}{q_2[m_2]_{q_2}} + \frac{1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}}{q_2[m_2]_{q_2}}\} + \frac{2\lambda_2}{[m_2]_{q_2}([m_2]^2_{q_2}-1)}\{q_2[m_2-1]_{q_2}[m_2+1]_{q_2}y_2^3(1-y_2^{m_2-2}) - \frac{(1-q_2)[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})}{q_2} + \frac{[m_2+1]_{q_2}y_2(1-y_2^{m_2})}{q_2^2[m_2]_{q_2}} - \frac{1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}}{q_2^2[m_2]_{q_2}}\}$;

Proof. The results follow from linearity of the operators $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$ and Lemma 2.1. □

Corollary 2.3. Applying Lemma 2.2, we have

(i) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(t-y_1;y_1,y_2)=\frac{2\lambda[m_2+1]_{p_1,q-1}y_1^2(1-y_1^{m_2-1})}{p_1^{m_2}(p_1^{2-2m_2}[m_2]^2_{p_1}-1)}(1-\frac{q_1}{p_1})$
$+\frac{\lambda p_1^{m_2}(1-y_1^{m_2+1})}{q_1[m_2]_{p_1,q_1}(p_1^{1-m_2}[m_2]_{p_1,q_1}-1)}+\frac{\lambda[m_2+1]_{p_1,q_1}y_1(1-y_1^{m_2})}{[m_2]_{p_1,q_1}(p_1^{1-m_2}[m_2]_{p_1,q_1}-1)}(\frac{1}{p_1}$
$-\frac{1}{q_1}-\frac{2}{p_1(p_1^{1-m_2}[m_2]_{p_1,q_1}+1)})-\frac{\lambda\prod_{s=0}^{m_2}(p_1^s-q_1^s y_1)}{p_1^{\frac{m_1(m_1-1)}{2}}q_1[m_1]_{p_1,q_1}(p^{1-m_1}[m_2]_{p_1,q_1}-1)};$

(ii) $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t_1-y_1)^2;y_1,y_2)=\frac{y_1(1-y_1)}{[m_1]_{q_1}}+\frac{[m_1+1]_{q_1}y_1\lambda_1}{[m_1]_{q_1}([m_1]_{q_1}-1)}[q_1y_1(1-y_1^{m_1-1})+\frac{1-y_1^{m_1}}{[m_1]_{q_1}}-2y_1(1-y_1^{m_1})]-\frac{2[m_1+1]_{q_1}\lambda_1}{[m_1]_{q_1}([m_1]^2_{q_1}-1)}[\frac{y_1(1-y_1^{m_1})}{[m_1]_{q_1}}+q_1(2+q_1)y_1^2(1-y_1^{m_1-1})+q_1^3[m_1-1]_{q_1}y_1^3(1-y_1^{m_1-1})+\frac{4y_1[m_1+1]_{q_1}y_1\lambda_1}{[m_1]^2_{q_1}-1}[\frac{1-y_1^{m_1}}{[m_1]_{q_1}}+q_1y_1(1-y_1^{m_1-1})]]-\frac{\lambda_2}{q_1[m_1]_{q_1}([m_1]_{q_1}+1)}\{[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-\frac{[m_1+1]_{q_1}y_1(1-y_1^{m_1})}{q_1[m_1]_{q_1}}+\frac{1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}}{q_1[m_1]_{q_1}}-2y_1[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}-[m_1+1]_{q_1}y_1(1-y_1^{m_1})]\}+\frac{2\lambda_2}{[m_1]_{q_1}([m_1]^2_{q_1}-1)}\{q_1[m_1-1]_{q_1}[m_1+1]_{q_1}y_1^3(1-y_1^{m_1-2})-\frac{(1-q_1)[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})}{q_1}+\frac{[m_1+1]_{q_1}y_1(1-y^{m_1})}{q_1^2[m_1]_{q_1}}-\frac{1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}}{q_1^2[m_1]_{q_1}}\}-2y_1\frac{\lambda_1}{[m_1]^2_{q_1}-1}\{2[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-\frac{2[m_1+1]_{q_1}y_1}{q_1[m_1]_{q_1}}(1-y_1^{m_1})+\frac{2}{q_1[m_1]_{q_1}}[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}]\};$

(iii) $\lim_{m_1\to\infty}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t_1-y_1)^2;y_1,y_2)\leq o\left(\frac{1}{[m_1]_{p_1,q_1}}\right)(1+y_1+y_1^2),\ as\ m_1\to\infty$

(iv) $\lim_{m_2\to\infty}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t_2-y_2)^2;y_1,y_2)\leq o\left(\frac{1}{[m_2]_{p_2,q_2}}\right)(1+y_2+y_2^2),\ as\ m_2\to\infty.$

Lemma 2.4. *Let $h\in C(S^2)$. Then $||\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((h);y_1,y_2)||\leq||h||$, where $||.||$ is the sup-norm in $C(S^2)$.*

Proof. Using Lemma 2.2, we have

$$|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2)|\leq||h||\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{00};y_1,y_2)=||h||.$$

□

Lemma 2.5. *The generalized bivariate (p,q)-analogue of $\lambda-$Bernstein operator (1.3) interpolates the function h in the four corners of the unit square, i.e.,*

$$\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(f;0,0)=f(0,0);\ \ \mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(f;0,0)=f(0,1)$$

$$\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(f;1,0)=f(1,0);\ \ \mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(f;1,1)=f(1,1).$$

Proof. The result is derived directly from paper [10]- Lemma 2.5. □

3. Main Results

In order to get the rate of convergence and degree of approximation for the operators $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$, we consider $p_1=p_{m_1},p_2=p_{m_2}$ and $q_1=q_{m_1},q_2=q_{m_2}$ such that

$0 < q_{m_1} < p_{m_1} \leq 1$ and $0 \leq q_{m_2} < p_{m_2} \leq 1$ satisfying

$$\lim_{m_1\to\infty} q_{m_1}^{m_1} \to a, \lim_{m_2\to\infty} q_{m_2}^{m_2} \to b, \lim_{n\to\infty} p_{m_1}^{m_1} \to c, \lim_{m_2\to\infty} p_{m_2}^{m_2} \to d \tag{3.11}$$

and

$$\lim_{m_1\to\infty} p_{m_1} \to 1, \lim_{m_2\to\infty} p_{m_2} \to 1, \lim_{m_2\to\infty} q_{m_2} \to 1, \lim_{m_2\to\infty} q_{m_2} \to 1, \tag{3.12}$$

where $0 \leq a, b < c, d < 1$. Here, we recall the following result due to Volkov [16]:

Theorem 3.1. [26] *Let I and J be compact intervals of the real line. Let L_{m_2,m_1} : $\mathcal{B}(I \times J) \to \mathcal{B}(I \times J), (m_2, m_1) \in N \times N$ be linear positive operators. If*

$$\lim_{m_1,m_2\to\infty} L_{m_2,m_1}(e_{ij}) = e_{x,y,}(i, j) \in \{(0, 0), (1, 0), (0, 1)\}$$

and

$$\lim_{m_1,m_2\to\infty} L_{m_2,m_1}(e_{20} + e_{02}) = e_{20} + e_{02},$$

uniformly on $I \times J$, then the sequence $(L_{m_2,m_1} f)$ converges to f uniformly on $I \times J$ for any $f \in \mathcal{B}(I \times J)$

Theorem 3.2. *Let $e_{ij}(y_1, y_2) = y_1^i y_2^j (0 \leq i+j \leq 2, i, j \in N)$ be the test functions defined on $J_1 \times J_2$ and $(p_{m_1}), (q_{m_1}), (p_{m_2}), (q_{m_2})$ be the sequences defined by* (3.11) *and* (3.12). *If*

$$\lim_{m_1,m_2\to\infty} \mathcal{B}_{m_1,m_2,p_{m_1},q_{m_1}}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}(e_{ij}; y_1, y_2) = e_{ij}(y_1, y_2), (i, j) \in \{(0, 0), (1, 0), (0, 1)\}$$

and

$$\lim_{m_1,m_2\to\infty} \mathcal{B}_{m_1,m_2,p_{m_1},q_{m_1}}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}(e_{20} + e_{20}; y_1, y_2) = e_{20}(y_1, y_2) + e_{02}(y_1, y_2),$$

uniformly on $J_1 \times J_2$, then

$$\lim_{m_1,m_2\to\infty} \mathcal{B}_{m_1,m_2,p_{m_1},q_{m_1}}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}(f; y_1, y_2) = f(y_1, y_2),$$

uniformly for any $f \in C(J_1 \times J_2)$.

Proof. Using Lemma 2.2, it is obvious for $i = j = 0$

$$\lim_{m_1,m_2\to\infty} \mathcal{B}_{m_1,m_2,p_{m_1},q_{m_1}}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}(e_{00}; y_1, y_2) = e_{00}(y_1, y_2).$$

For $i = 1$ and $j = 0$, we have

$$\lim_{m_1,m_2\to\infty} \mathcal{B}_{m_1,m_2,p_{m_1},q_{m_1}}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}(e_{10}; y_1, y_2) = y_1.$$

For $i = 0$ and $j = 1$, we have

$$\lim_{m_1,m_2\to\infty} \mathcal{B}_{m_1,m_2,p_{m_1},q_{m_1}}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}(e_{01}; y_1, y_2) = y_1,$$

and

$$\lim_{m_1,m_2\to\infty} \mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{20}+e_{02};y_1,y_2)$$
$$= \lim_{m_1,m_2\to\infty}\left\{\frac{p_1^{m_2-1}b_{m_2}}{[m_2]_{p_1,q_1}}y_1+\frac{q_1[m_2-1]_{p_1,q_1}}{[m_2]_{p_1,q_1}}y_2^2 + \frac{p_2^{m_1-1}b_{m_1}}{[m_1]_{p_2,q_2}}y_2+\frac{q_2[m_1-1]_{p_2,q_2}}{[m_1-2]_{p_2,q_2}}y_2^2\right\}$$

$$\lim_{m_2,m_1\to\infty}(\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(e_{20}+e_{02})(y_1,y_2)=e_{20}(y_1,y_2)+e_{02}(y_1,y_2).$$

From Theorem 3.1, we completes the proof of Theorem 3.2. □

In order to discuss the next results, let us recall the definitions of modulus of continuity and partial modulus of continuity

Definition 3.3. For $h\in C(S^2)$ and $\delta>0$, the full modulus of continuity in the bivariate case is defined as

$$\omega_c(h;\delta)=\max_{\sqrt{\sum\limits_{i=1}^{2}(t_i-y_i)^2\le\delta}}\{|h(t_1,t_2)-h(y_1,y_2)|:(t_1,t_2),(y_1,y_2)\in S^2\}. \tag{3.13}$$

For each fixed $i=1,2$, the partial modulus of continuity of h with respect to y_i is defined

$$\omega^{(1)}(h;\delta)=\sup\{|h(x_1,y)-h(x_2,y)|:y\in[0,1]\quad |x_1-x_2|\le\delta\}$$

and

$$\omega^{(2)}(h;\delta)=\sup\{|h(x,y_1)-h(x,y_2)|:x\in[0,1]\quad |y_1-y_2|\le\delta\}, \tag{3.14}$$

respectively.

The following version of the Shisha–Mond theorem [23] for estimating the rate of convergence is known.

Theorem 3.4. [23] *Let $L:C([a,b]\times[c,d])\longrightarrow C([a,b]\times[c,d])$ be a linear positive operator. For any $f\in C([a,b]\times[c,d])$, and $(y_1,y_2)\in[a,b]\times[c,d]$ and any $\delta_1\in[0,b-a],\delta_2\in[0,d-c]$, the following inequality*

$$|(LF)(x,y)-f(x,y)|\le|Le_{0,0}(x,y)-1||f(x,y)|+$$
$$+[Le_{0,0}(x,y)+\delta_1^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot-x))^2(x,y)}+\delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(*-y))^2(x,y)}+$$
$$+\delta_1^{-1}\delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot-x))^2(x,y)Le_{0,0}(x,y)(L(*-y))^2(x,y)}\Big]]\omega_c(f;\delta_1,\delta_2),$$

holds.

Theorem 3.5. *Let $h\in C(S^2)$ and $(y_1,y_2)\in S^2$. Then, for $(m_1,m_2)\in\mathbb{N}^2$ and for any $\delta_1,\delta_2>0$, we have*

$$|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}h(y_1,y_2)-h(y_1,y_2)|\le 4\omega_{total}(h;\delta_1,\delta_2) \tag{3.15}$$

where $\delta_1=\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t-y_1)^2,y_1,y_2)}$ *and* $\delta_1=\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((s-y_2)^2,y_1,y_2)}$.

Proof. From Theorem 3.4, we have

$$|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}} h(y_1,y_2) - h(y_1,y_2)| \leq \omega_c(f;\delta_1,\delta_2)$$
$$\Bigg[1+\delta_1^{-1}\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t-y_1)^2,y_1,y_2)}+\delta_2^{-1}\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((s-y_2)^2,y_1,y_2)}$$
$$+\delta_1^{-1}\delta_2^{-1}\sqrt{\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t-y_1)^2,y_1,y_2)}\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((s-y_2)^2,y_1,y_2)}}\Bigg].$$

On choosing

$$\delta_1=\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((t-y_1)^2,y_1,y_2)} \text{ and } \delta_1=\sqrt{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}((s-y_2)^2,y_1,y_2)},$$

we get the requires result. □

Theorem 3.6. *Let $h \in C(S^2)$, then the following inequality holds:*

$$|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)| \leq 2\left\{\omega^{(1)}(h;\sqrt{\mu_{m_1,\lambda_1,2}(q_{m_1},y_1)})\right.$$
$$\left.+\omega^{(2)}(h;\sqrt{\mu_{m_2,\lambda_2,2}(q_{m_2},y_2)})\right\}.$$

Proof. Using the defnition of partial modulus of continuity $\omega^{(i)}(f;\delta), i=1,2$, we may write

$$|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)|$$
$$\leq \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,j_1,j_2}(y_1,y_2)|h(t_1,t_2)-h(t_1,y_2)|$$
$$+\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,j_1,j_2}(y_1,y_2)|h(t_1,y_2)-h(y_1,y_2)|$$
$$\leq \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,j_1,j_2}(y_1,y_2)(\omega^{(2)}(h;|t_2-y_2|);y_1,y_2)$$
$$+\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,j_1,j_2}(y_1,y_2)(\omega^{(1)}(h;|t_1-y_1|);y_1,y_2)\}.$$

Applying Cauchy Schwarz inequality with $\delta_1 = \big(\mu_{m_1,\lambda_1,2}(q_{m_1},y_1)\big)^{\frac{1}{2}}$ and $\delta_2 = \big(\mu_{m_2,\lambda_2,2}(q_{m_2},y_2)\big)^{\frac{1}{2}}$, we obtain the desired result. □

Now, we want to give the quantitative result in terms of the Lipschitz class functions. For any functions $h: S^2 \to \mathbb{R}$ and $0<\varkappa\leq 1$, the function h is said to be in Lipschitz class $Lip_M\varkappa(S^2)$ if ∃ a $M>0$ such that

$$Lip_M\varkappa := \left\{h : |h(s_1,s_2)-h(y_1,y_2)| \leq M\frac{\|u-v\|^{\varkappa}}{(\|u\|+y_1+y_2)^{\frac{\gamma}{2}}}\right\},$$

$\forall\ s=(s_1,s_2), t=(y_1,y_2)\in S^2$, where $\|s-t\| = \left\{(s_1-y_1)^2+(s_2-y_2)^2\right\}^{1/2}$ is the Euclidean norm and M is a positive real constant.

The following theorem yields us a estimate of error for functions in $Lip_M\varkappa$, by the operators $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$.

Theorem 3.7. *Let $h \in Lip_M\varkappa$. Then for sufficiently large m and n and for all $y_1, y_2 \in S^2$, there holds the inequality*

$$\left\|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h) - h\right\|_{C(I^2)} \leq \frac{M}{(y_1+y_2)^{\frac{\varkappa}{2}}}\left\{[m_1]^{-1}_{q_{m_1}} + [m_2]^{-1}_{q_{m_2}}\right\}^{\varkappa/2}$$

where $M > 0$ is a constant.

Proof. Let $0 < \varkappa < 1$. Then, taking into account that $g \in Lip_M\varkappa$, and using the monotonicity and linearity of operators $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$, we have

$$\begin{aligned}\left|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2) - h(y_1,y_2)\right| &\leq \mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(|h(s_1,s_2)-h(y_1,y_2)|;y_1,y_2\right)\\ &\leq M\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(\frac{\|u-v\|^{\varkappa}}{(\|u\|+y_1+y_2)^{\frac{\varkappa}{2}}};y_1,y_2\right)\\ &\leq \frac{M}{(y_1+y_2)^{\frac{\varkappa}{2}}}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(\|u-v\|^{\varkappa};y_1,y_2\right)\end{aligned}$$

where $s = (s_1, s_2), t = (y_1, y_2) \in S^2$. Now, applying Holder's inequality with $u_1 = \frac{2}{\kappa_1}$, $u_2 = \frac{2}{2-\kappa_2}$ we have

$$\left|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2) - h(y_1,y_2)\right| \leq \frac{M}{(y_1+y_2)^{\frac{\varkappa}{2}}}\left\{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(\|s-t\|^2;y_1,y_2\right)\right\}^{\frac{\varkappa}{2}}$$

$$\leq \frac{M}{(y_1+y_2)^{\frac{\varkappa}{2}}}\left\{\mu_{m_1,\lambda_1,2}(q_{m_2},y_1) + \mu_{m_2,\lambda_2,2}(q_{m_2},y_2)\right\}^{\varkappa/2},$$

which leads to the required result on applying Corollary 2.3. Similarly the authentication for $\varkappa = 1$. $\square$

Let $C'(S^2)$ denote the space of continuous functions $h(y_1, y_2)$ on S^2 whose first order partial derivatives g'_{y_1} and g'_{y_2} are also continuous on S^2.

Our next result yields us the rate of approximation for continuously differentiable functions on S^2 by the operators $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$

Theorem 3.8. *Let $h \in C'(S^2)$. Then for sufficiently large m_1 and m_2, we have*

$$\left\|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h) - h\right\|_{C(S^2)} \leq A\left(\|\frac{\partial h}{\partial y_1}\|_{C(S^2)}[m_1]^{-1/2}_{q_{m_1}} + \|\frac{\partial h}{\partial y_2}\|_{C(S^2)}[m_2]^{-1/2}_{q_{m_2}}\right),$$

where A is some positive constant.

Proof. For $y_1, y_2 \in S^2$ be arbitrary, we may write

$$h(t_1,t_2) - h(y_1,y_2) = \int_{y_1}^{t_1}\frac{\partial h}{\partial \eta}(\eta,t_2)d\eta + \int_{y_2}^{t_2}\frac{\partial h}{\partial \phi}(y_1,\phi)d\phi, \quad \text{for } (t_1,t_2) \in S^2.$$

Hence, applying the operator $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$ on both sides of the above equation, we obtain

$$\begin{aligned}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2) - h(y_1,h_2) &= \mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(\int_{y_1}^{t_1}\tfrac{\partial h}{\partial \eta}(\eta,t_2)d\eta;y_1,y_2\right)\\ &+\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(\int_{y_2}^{t_2}\tfrac{\partial h}{\partial \phi}(y_1,\phi)d\phi;y_1,y_2\right).\end{aligned}$$

By using sup-norm on S^2

$$\begin{aligned}\left|\int_{y_1}^{t_1} h'_\eta(\eta,t_2)d\eta\right| &\leq \|\frac{\partial h}{\partial y_1}\|_{C(S^2)}|t_1-y_1|,\\ \left|\int_{y_2}^{t_2} \frac{\partial h}{\partial \phi}(y_1,\phi)d\phi\right| &\leq \|\frac{\partial h}{\partial y_2}\|_{C(S^2)}|t_2-y_2|.\end{aligned}$$

we get

$$\begin{aligned}\left|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2)-h(y_1,h_2)\right| &\leq \|\tfrac{\partial h}{\partial y_1}\|_{C(S^2)}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(|t_1-y_1|;y_1,y_2\right)\\ &+\|\tfrac{\partial h}{\partial y_2}\|_{C(S^2)}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(|t_2-y_2|;y_1,y_2\right).\end{aligned}$$

Hence, applying the Cauchy-Schwarz inequality and Corollary 2.3, we obtain

$$\begin{aligned}\left|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2) \right. &- \left. h(y_1,y_2)\right|\\ &\leq \|\frac{\partial h}{\partial y_1}\|_{C(S^2)}\left\{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_1-y_1)^2;y_1,y_2\right)\right\}^{1/2}\\ &+\|\frac{\partial h}{\partial y_2}\|_{C(S^2)}\left\{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_2-y_2)^2;y_1,y_2\right)\right\}^{1/2},\end{aligned}$$

from which the desired result is immediate. □

The following result yields the degree of approximation of h by $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$ in terms of the partial modul of continuity of the partial derivatives of h.

Theorem 3.9. *Let $h\in C'(S^2)$then for sufficiently large m_1 and m_2, we have*

$$\left\|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h)-h\right\|_{C(S^2)}\leq B\sum_{i=1}^{2}[m_i]_{q_{m_i}}^{-1/2}\left\{1+2\omega^{(i)}\left(\frac{\partial h}{\partial y_i};[m_i]_{q_{m_i}}^{-1/2}\right)\right\}$$

where $\omega^{(i)}\left(\frac{\partial h}{\partial y_i};.\right)$ are the partial modul of continuity of $\frac{\partial h}{\partial y_i}$ for $i=1,2$ and B is some positive constant.

Proof. If we use the mean value theorem in the following form, we have

$$\begin{aligned}h(t_1,t_2)-h(y_1,y_2) &= (t_1-y_1)\frac{\partial h}{\partial y_1}(\eta,y_2)+(t_2-y_2)\frac{\partial h}{\partial y_2}(y_1,\varkappa)\\ &= (t_1-y_1)\frac{\partial h}{\partial y_1}(y_1,y_2)+(t_1-y_1)\left(\frac{\partial h}{\partial y_1}(\eta,y_2)-\frac{\partial h}{\partial y_1}(y_1,y_2)\right)\\ &\quad+(t_2-y_2)\frac{\partial h}{\partial y_2}(y_1,y_2)+(t_2-y_2)\left(\frac{\partial h}{\partial y_2}(y_1,\varkappa)-\frac{\partial h}{\partial y_2}(y_1,y_2)\right),\end{aligned}$$

where $y_1<\eta<t_1$ and $y_2<\varkappa<t_2$. Applying the operator $\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$ to both sides, we deduce that

$$\begin{aligned}&\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2)-h(y_1,y_2)=\frac{\partial h}{\partial y_1}(y_1,y_2)\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\\ &+\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_1-y_1)\left(\frac{\partial h}{\partial y_1}(\eta,y_2)-\frac{\partial h}{\partial y_1}(y_1,y_2)\right);y_1,y_2\right)\end{aligned}$$

$$+\frac{\partial h}{\partial y_2}(y_1,y_2)\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}$$
$$+\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_2-y_2)\left(\frac{\partial h}{\partial y_2}(y_1,\varkappa)-\frac{\partial h}{\partial y_2}(y_1,y_2)\right);y_1,y_2\right).$$

Since $\frac{\partial h}{\partial y_1}$ and $\frac{\partial h}{\partial y_2}$ are continuous in S^2 there exist positive constants A_1 and A_2 such that $\left|\frac{\partial h}{\partial y_1}\right|\le A_1$ and $\left|\frac{\partial h}{\partial y_2}\right|\le A_2$, for all $(y_1,y_2)\in S^2$. Hence, applying the Cauchy-Schwarz inequality, we obtain

$$\left|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(h;y_1,y_2)-h(y_1,y_2)\right|\le\sum_{i=1}^{2}\left|\frac{\partial h}{\partial y_i}\right|\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}(|t_i-y_i|;y_1,y_2)$$
$$+\sum_{i=1}^{2}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left(|t_i-y_i|\left(1+\frac{|t_i-y_i|}{\delta_i}\right);y_1,y_2\right)\omega^{(i)}\left(\frac{\partial h}{\partial y_i};\delta_i\right)$$
$$\le\sum_{i=1}^{2}\left\{M_i\left\{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_i-y_i)^2;y_1,y_2\right)\right\}^{1/2}\right.$$
$$+\omega^{(i)}\left(\frac{\partial h}{\partial y_i};\delta_i\right)\left[\left\{\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_i-y_i)^2;y_1,y_2\right)\right\}^{1/2}\right.$$
$$\left.\left.+\frac{1}{\delta_i}\mathcal{B}^{\lambda_1,\lambda_2,p_{m_2},q_{m_2}}_{m_1,m_2,p_{m_1},q_{m_1}}\left((t_i-y_i)^2;y_1,y_2\right)\right]\right\}$$

Choosing $\delta_i=\left([m_i]_{q_{m_i}}\right)^{-1/2}$, $i=1,2$ we get the required result. □

References

[1] T. Acar, A. Aral and S. A. Mohiuddine, Approximation by bivariate (p,q)-Bernstein-Kantorovich operators, Iran. J. Sci. Technol. Trans. Sci. 42 (2018), 655-662.

[2] T. Acar and A. Kajla, Degree of approximation for bivariate generalized Bernstein type operators, Results Math. 73(2018), Article No. 79, 20 pp.

[3] A. M. Acu, Stancu-Schurer-Kantorovich operators based on q-integers, Appl. Math. Comput. 259 (2015), 896-907.

[4] A. M. Acu, N, Manav and F. D. Sofonea, Approximation properties of λ-Kantorovich operators, J. Inequal. Appl. 2018, Paper No. 202, 12 pp.

[5] A. M. Acu and C. Muraru, Approximation properties of bivariate extension of q-Bernstein-Schurer-Kantorovich operators, Results Math. 67 (2015), no. 3-4, 265–279.

[6] E. Aliaga and B. Baxhaku, On the Approximation Properties of q- Analogue Bivariate $\lambda-$ Bernstein Type Operators, J. Funct. Spaces 2020, Art. ID 4589310, 11 pp.

[7] B. Baxhaku and P. N. Agrawal, Degree of approximation for bivariate extension of Chlodowsky-type q-Bernstein–Stancu–Kantorovich operators, Appl. Math. Comput. 306(2017), 56–-72.

[8] B. Baxhaku, R. Shukla and P. N. Agrawal, A tensor product of Kantorovich-Stancu type operators with shifted knots and their kth order generalization, Filomat 35(2021), no. 12, 4239-4255.

[9] B. Baxhaku and A. Kajla, Blending type approximation by bivariate generalized Bernstein type operators, Quaes. Math. 43(2020), no. 10, 1449-1465.

[10] D. Barbosu, Some generalized bivariate Bernstein operators, Math. Notes (Miskolc) 1 (2000), no. 1, 3–10.

[11] S. N. Bernstein, Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités, Communications de la Société Mathématique de Kharkov 2. Series XIII(1912), no. 1, 1–-2.
[12] Q. B. Cai, X. W. Xu and G. Zhou, Bivariate tensor product (p,q)-analogue of Kantorovich-type Bernstein-Stancu-Schurer operators, J. Inequal. Appl. 2017, Paper No. 284, 14 pp.
[13] Q. B. Cai and W. T. Cheng, Convergence of λ-Bernstein operators based on (p,q)-integers, J. Inequal. Appl. 2020, Paper No. 35, 17 pp.
[14] Q. B. Cai, G. Zhou and J. Li, Statistical approximation properties of λ-Bernstein operators based on q-integers, Open Mathematics, 17(2019), no. 1, 487-498.
[15] R. Chauhan and P. N. Agrawal, Degree of approximation by generalized Boolean sum of λ-Bernstein operators, in Mathematical analysis. I. Approximation theory, 95–109, Springer Proc. Math. Stat., 306, Springer, Singapore, 2018.
[16] X. Chen, J. Tan, Z. Liu and J. Xie, Approximation of functions by a new family of generalized Bernstein operators, J. Math. Anal. Appl. 450 (2017), 244-261.
[17] F. J. Delvos and W. Schempp, Boolean methods in interpolation and approximation, Pitman Research Notes in Mathematics Series, 230, Longman Scientific & Technical, Harlow, 1989.
[18] O. Doğru and V. Gupta, Korovkin-type approximation propeties of bivariate q-Meyer-König and Zeller operators, Calcolo, 43 (2006), 51-63.
[19] S. M.Kang, A. Rafiq, A. M. Acu, F. Ali, and Y.C. Kwun, Some approximation properties of (p,q)-Bernstein operators, J. Inequal. Appl. 2016, Paper No. 169, 10 pp.
[20] M. Mursaleen, K.J. Ansari and A. Khan, Corrigendum to: "Some approximation results by (p,q)-analogue of Bernstein-Stancu operators" [Appl. Math. Comput. 264(2015) 392–402], Appl. Math. Comput. 269 (2015), 744–746.
[21] N. Malik N, S. Araci and M.S. Beniwal, Approximation of Durrmeyer type operators depending on certain parameters, Abstr. Appl. Anal. 2017, Art. ID 5316150, 9 pp.
[22] R, Ruchi, B. Baxhaku, and P. N. Agrawal, GBS operators of bivariate Bernstein-Durrmeyer-Type on a triangle, Math. Methods Appl. Sci. 41 (2018), no. 7, 2673-2683.
[23] F. Stancu, Aproximarea functiilor de doua si mai multe variabile cu ajutorul operatorilor liniarisi pozitivi, PhD Thesis, Univ. Babes-Bolyai, Cluj-Napoca, 1984 (Romanian).
[24] H. Sharma, On Durrmeyer-type generalization of (p,q)-Bernstein operators, Arab. J. Math. 5(2016), no. 4, 239-248.
[25] T. Vedi-Dilek and E. Gemikonakli, Bivariate Chlodowsky-Stancu Variant of (p,q)-Bernstein-Schurer Operators, J. Funct. Spaces 2019, Art. ID 7587401, 11 pp.
[26] V. I. Volkov, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables (in Russian), Dokl. Akad. Nauk SSSR (NS) 115 (1957) 17-19.
[27] L. Zeng , Q. B. Cai and X. W. Xu, A-statistical convergence properties of Kantorovich type λ-Bernstein operators via (p,q)-Calculus, Mathematics, 8(2020), no. 2, Paper No. 291, 14 pp.

(Bashkim Trupaj) Department of Mathematics, University of Hasan Prishtina, Prishtina, Kosovo
E-mail address: `bashkim.trupaj@uni-pr.edu`

(Gazmend Nafezi) Department of Physics, University of Hasan Prishtina,Prishtina, Kosovo
E-mail address: `gazmend.nafezi@uni-pr.edu`