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AN APPLICATION OF JENSEN INEQUALITY IN STUDYING NÖRLUND SUMMABILITY OF FOURIER SERIES

PRINCE ASHVINBHAI BHUVA† AND BHIKHA LILA GHODADRA

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Abstract. In this paper we give sufficient conditions for the absolute Nörlund summability of Fourier series of functions of the class $\phi\Lambda\mathrm{BV}([-\pi, \pi])$. The crucial step of the proof follows from Jensen and Hölder inequalities.

1. Introduction

Absolute summability of Fourier series is one of the important problems in Fourier Analysis. Many authors (see, e.g., [4, 6, 7, 8, 10, 13, 15]) have studied absolute Nörlund summability of Fourier series. In 1942, McFadden [10] established the theorem for absolute Nörlund summability of Fourier series to class of functions of $\mathrm{Lip}(\alpha)$ and $\mathrm{Lip}(\alpha, p)$. Then this theorem was extended by many authors like S. N. Lal [6, 7], S. N. Lal and Siya Ram [8], M. Izumi and S. Izumi [4], S. M. Shah [13] and R. G. Vyas and K. N. Darji [15]. S. M. Shah extended this theorem for functions of bounded variation whereas S. N. Lal and Siya Ram extended this theorem for functions of r-bounded variation. Lastly this theorem was extended by R. G. Vyas and K. N. Darji for functions of Λ-bounded variation and r-Λ-bounded variation. Here, we study the absolute Nörlund summability of Fourier series of functions of $\phi\Lambda$-bounded variation.

2. Preliminaries

Let $\{p_n\}$ be a sequence of non-negative numbers and let $\{P_n\}$ denote the sequence of partial sums of $\sum_{n=0}^{\infty} p_n$. We assume that $P_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, $P_{-1} = p_{-1} = 0$, $P_{-2} = p_{-2} = 0$. We will call $\{t_n\}$ the Nörlund transform of an arbitrary sequence $\{S_n\}$, where

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k, \ n = 0, 1, 2, \ldots.$$

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† *Corresponding author*

The sequence $\{S_n\}$ is said to be summable (N, p_n) to s, if $\lim_{n\to\infty} t_n = s$. It is said to be absolute summable (N, p_n) or summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \tag{2.1}$$

McFadden [10] proved that summability $|N, p_n|$ implies summability (N, p_n).

The series $\sum a_n$ with the sequence of partial sums S_n is said to be summable (N, p_n) if sequence S_n is summable (N, p_n). It is said to be absolutely summable (N, p_n) or summable $|N, p_n|$, if sequence $\{S_n\}$ satisfies condition (2.1).

Remark 2.1. Note that, if $p_0 = 1$ and $p_n = 0$ for $n \in \mathbb{N}$, then P_n=1 for all $n \in \mathbb{N} \cup \{0\}$ and hence $t_n = S_n$. Therefore the series

$$\begin{aligned}\sum a_n \text{ is summable } (N, p_n) &\iff \{S_n\} \text{ is summable } (N, p_n)\\ &\iff \{t_n\} \text{ converges}\\ &\iff \{S_n\} \text{ converges}\\ &\iff \sum a_n \text{ converges.}\end{aligned}$$

So, the summability (N, p_n) of $\sum a_n$ reduces to ordinary convergence. Also, in this particular case as $t_n = S_n$, (2.1) becomes

$$\infty > \sum_{n=1}^{\infty} |S_n - S_{n-1}| = \sum_{n=1}^{\infty} |a_n|.$$

Therefore, the summability $|N, p_n|$ of $\sum a_n$ reduces to absolute convergence of $\sum a_n$.

In what follows, f is a 2π-periodic function, $\mathbb{T} = [-\pi, \pi)$, T is a finite collection of non-overlapping subintervals $T_n = [x_n, y_n]$ in $\overline{\mathbb{T}}$, $f(T_n) = f(y_n) - f(x_n)$, $\mathbb{L}$ is the class of all non-decreasing sequences $\Lambda = \{\lambda_n\}$ of positive numbers such that the series $\sum_n \frac{1}{\lambda_n}$ diverges, $\Lambda_N = \sum_{k=1}^{N} \frac{1}{\lambda_k}$, $\Theta_N = \sum_{k=1}^{N} \frac{1}{\lambda_k^2}$, $P(t) = P_{[t]}$ and $p(t) = p_{[t]}$ where $[t]$ denotes the integer part of t and A denotes a positive constant not necessarily the same at each occurrence.

The modulus of continuity of a function f defined on the interval $\overline{\mathbb{T}}$ is defined as

$$\omega(\delta) \equiv \omega(f, \delta) = \sup\{|f(x+h) - f(x)| : x \in \overline{\mathbb{T}},\ 0 \le h < \delta\},\ \delta > 0.$$

We know that $\omega(f; \delta)$ is monotonically increasing function and for any positive number λ, we have

$$\omega(f, \lambda\delta) \le (\lambda + 1)\omega(f, \delta).$$

The integral modulus of continuity of order $p \ge 1$ of $f \in L^1(\overline{\mathbb{T}})$ is defined as

$$\omega_p(f; \delta) = \sup\left\{\left(\int_{-\pi}^{\pi} |f(x+h) - f(x)|^p dx\right)^{\frac{1}{p}} : 0 \le h < \delta\right\},\ \delta > 0.$$

For a function $f \in L^1(\overline{\mathbb{T}})$, its Fourier series is defined as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

where a_n and b_n are given by the usual Euler-Fourier formulae.

If S_n is the nth partial sum of Fourier series of f then,

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_n(t)dt,$$

where $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt$ is the Dirichlet kernel.

In 1972, S. M. Shah [13] proved the following result concerning absolute Nörlund summability for functions of bounded variation. To state this result, first we recall the definition of a function of bounded variation.

Definition 2.2. A function $f : \overline{\mathbb{T}} \to \mathbb{R}$ is said to be of bounded variation over the interval $\overline{\mathbb{T}}$, in symbols: $f \in BV(\overline{\mathbb{T}})$, if

$$V(f;\overline{\mathbb{T}}) = \sup_T \left\{ \sum_n |f(T_n)| \right\} < \infty.$$

Theorem 2.3. *Let $\{p_n\}$ be a non-negative, non-increasing sequence such that $\lim_{n\to\infty} p_n = 0, \{p_n - p_{n+1}\}$ is non-increasing and*

$$c_1 n^\gamma \psi(n) \le P_n \le c_2 n^\gamma \psi(n), \tag{2.2}$$

where c_1 and c_2 are fixed positive constants, γ is a fixed constant, $0 \le \gamma < \frac{1}{2}$, and $\psi(x)$ is positive on $[0,\infty)$ and slowly oscillating in the sense of Karamata (see [5], [9]*), that is, $\frac{\psi(cx)}{\psi(x)} \to 1$ as $x \to \infty$ for all $c > 0$. If $f \in BV(\overline{\mathbb{T}})$,*

$$\sum_{n=1}^{\infty} \frac{1}{nP_n}\left(\omega\left(f;\frac{1}{n}\right)\right)^{\frac{1}{2}} < \infty, \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(f;\frac{1}{n})}{n} < \infty, \tag{2.4}$$

then the Fourier series of f is summable $|N, p_n|$.

R. G. Vyas and K. N. Darji [15] gave the following result for functions of Λ-bounded variation. To state this result, first we recall the definition of a function of Λ-bounded variation.

Definition 2.4. Let $\Lambda = \{\lambda_n\} \in \mathbb{L}$. A function $f : \overline{\mathbb{T}} \to \mathbb{R}$ is said to be of Λ-bounded variation over the interval $\overline{\mathbb{T}}$, in symbols: $f \in \Lambda BV(\overline{\mathbb{T}})$, if

$$V_\Lambda(f;\overline{\mathbb{T}}) = \sup_T \left\{ \sum_n \frac{|f(T_n)|}{\lambda_n} \right\} < \infty.$$

Theorem 2.5. *Let $\{p_n\}$ be a non-negative, non-increasing sequence such that $\lim_{n\to\infty} p_n = 0, \{p_n - p_{n+1}\}$ is non-increasing, and*

$$\sum_{k=n}^{\infty} \frac{P_k^2}{k^2} \le A\frac{P_n^2}{n}. \tag{2.5}$$

If $f \in \Lambda BV(\overline{\mathbb{T}})$,

$$\sum_{n=1}^{\infty} \frac{1}{P_n}\left(\frac{\omega\left(f;\frac{\pi}{n}\right)}{n\Lambda_{2n}}\right)^{\frac{1}{2}} < \infty, \tag{2.6}$$

and (2.4) *holds, then the Fourier series of f is summable $|N, p_n|$.*

Now we define $\phi\Lambda$-bounded variation [see, e.g., [11]], which is as follows.

Definition 2.6. Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing convex function and let $\Lambda = \{\lambda_n\} \in \mathbb{L}$. A function f defined on the interval $\overline{\mathbb{T}}$ is said to be of $\phi\Lambda$-bounded variation (that is, $f \in \phi\Lambda BV(\overline{\mathbb{T}})$) if

$$V_{\phi\Lambda}(f,\overline{\mathbb{T}}) = \sup_T \left\{ \sum_n \frac{\phi(|f(T_n)|)}{\lambda_n} \right\} < \infty.$$

Remark 2.7. In above definition, putting $\phi(x) = x$ and $\Lambda = \{1\}$ we get class of functions of bounded variation, putting $\phi(x) = x^r$ $(r \geq 1)$ and $\Lambda = \{1\}$ we get class of functions of r-bounded variation, putting $\phi(x) = x$ and we get class of functions of Λ-bounded variation, putting $\phi(x) = x^r$ $(r \geq 1)$ we get class of functions of r-Λ-bounded variation and putting $\Lambda = \{1\}$ we get class of functions of ϕ-bounded variation.

Remark 2.8. If $f \in \phi\Lambda BV(\overline{\mathbb{T}})$, then f is bounded. In fact, let $f \in \phi\Lambda BV(\overline{\mathbb{T}})$. Then for any $x \in \overline{\mathbb{T}}$, we have

$$\begin{aligned}
|f(x)| - |f(0)| \leq |f(x) - f(0)| &\implies \phi(|f(x)| - |f(0)|) \leq \phi(|f(x) - f(0)|) \\
&\implies \phi(|f(x)| - |f(0)|) \leq \lambda_1 \frac{\phi(|f(x) - f(0)|)}{\lambda_1} \\
&\implies \phi(|f(x)| - |f(0)|) \leq \lambda_1 V_{\phi\Lambda}(f,\bar{I}) = A, \text{ say,} \\
&\implies |f(x)| \leq \phi^{-1}(A) + |f(0)| \leq A.
\end{aligned}$$

Thus f is bounded on $\overline{\mathbb{T}}$.

Remark 2.9. Since $\phi(x)$ is strictly increasing and convex on $[0,\infty)$, $\phi^{-1}(x)$ is strictly increasing and

$$\phi^{-1}(bx) \leq b\phi^{-1}(x) \quad (b > 1).$$

Definition 2.10. The function ϕ is said to be a Δ_2-function if there is a constant $d \geq 2$ such that $\phi(2x) \leq d\phi(x)$ for all $x \geq 0$.

Notation. In this paper, we use the following notations:

$$\begin{aligned}
g_x(t) &= f(x+t) + f(x-t) - 2f(x), \\
\alpha(t) &= \sum_{n=0}^{\infty} p_n \cos nt, \\
\beta(t) &= \sum_{n=0}^{\infty} p_n \sin nt, \\
\alpha_n(x) &= \int_0^\pi g_x(t)\alpha(t)\cos nt dt, \\
\beta_n(x) &= \int_0^\pi g_x(t)\beta(t)\sin nt dt.
\end{aligned}$$

We note that if p_n decreases to 0 as n tends to ∞, then $\sum_{n=0}^{\infty} p_n \cos nt$ converges everywhere except for $x \equiv 0 \pmod{2\pi}$ and $\sum_{n=0}^{\infty} p_n \sin nt$ converges everywhere (see [1, pp. 87]). Using above notation of $g_x(t)$ we have

$$S_k(x) - f(x) = \frac{1}{\pi} \int_0^\pi g_x(t) D_k(t) dt. \tag{2.7}$$

3. Main Results

We prove the following results. In these results we assume that, $\{p_n\}$ is a non-negative, non-increasing sequence such that $\lim_{n\to\infty} p_n = 0, \{p_n - p_{n+1}\}$ is non-increasing, and

$$\sum_{k=n}^{\infty} \frac{P_k^4}{k^2} \le A\frac{P_n^4}{n}. \tag{3.1}$$

Theorem 3.1. *If $f \in \phi\Lambda BV(\overline{\mathbb{T}})$, where ϕ is a Δ_2-function with constant $d \ge 2$,*

$$\sum_{n=1}^{\infty} \frac{1}{P_n}\left(\omega\left(f;\frac{\pi}{n}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2n}}}\right)\Big/ n\right)^{\frac{1}{2}} < \infty, \tag{3.2}$$

$$\sum_{k=1}^{\infty} \frac{d^{\log_2 P_k^4}}{k^2} < \infty, \tag{3.3}$$

and (2.4) *holds, then the Fourier series of f is summable $|N, p_n|$.*

Remark 3.2. We note that there are sequences $\{p_n\}$ which satisfies the conditions of our theorem. For example, take $p_0 = 1$ and $p_n = \frac{1}{n(n+1)}$ for $n \in \mathbb{N}$. Then clearly $\{p_n\}$ is non-negative, non-increasing, and converges to zero. Since $p_n - p_{n+1} = \frac{2}{n(n+1)(n+2)}$, for $n \ge 1$, $p_0 - p_1 = \frac{1}{2}$ and $p_1 - p_2 = \frac{1}{3}$, $\{p_n - p_{n+1}\}$ is non-increasing. Also, we have $P_0 = 1$ and for $n \ge 1$,

$$P_n = p_0 + \sum_{k=1}^{n} p_k = 1 + \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 + \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 2 - \frac{1}{n+1}.$$

Therefore, as $\log_2 x \le x$ for $x > 0$, we have

$$\sum_{k=1}^{\infty} \frac{d^{\log_2 P_k^4}}{k^2} = \sum_{k=1}^{\infty} \frac{d^{\log_2 (2-\frac{1}{k+1})^4}}{k^2} \le \sum_{k=1}^{\infty} \frac{d^{(2-\frac{1}{k+1})^4}}{k^2} \le \sum_{k=1}^{\infty} \frac{d^{16}}{k^2} < \infty,$$

and for $n \ge 1$,

$$\sum_{k=n}^{\infty} \frac{P_k^4}{k^2} = \sum_{k=n}^{\infty} \frac{(2-\frac{1}{k+1})^4}{k^2} \le 16\sum_{k=n}^{\infty} \frac{1}{k^2} \le \frac{A}{n} \le \frac{A}{n}\left(2 - \frac{1}{n+1}\right)^4 = A\frac{P_n^4}{n}.$$

Therefore, $\{p_n\}$ satisfies (3.1) and (3.3).

Putting $\Lambda = \{1\}$ in Theorem 3.1, we get the following result as a corollary for the class of functions of ϕ-bounded variation.

Corollary 3.3. If $f \in \phi BV(\overline{\mathbb{T}})$, where ϕ is a Δ_2-function with constant $d \ge 2$,

$$\sum_{n=1}^{\infty} \frac{1}{P_n}\left(\left(\omega\left(f;\frac{\pi}{n}\right)\right)\phi^{-1}\left(\frac{1}{\sqrt{n}}\right)\Big/ n\right)^{\frac{1}{2}} < \infty, \tag{3.4}$$

and (2.4) and (3.3) hold, then the Fourier series of f is summable $|N, p_n|$.

Putting $\phi(x) = x$ in Theorem 3.1, we get the following result as a corollary for the class of functions of Λ-bounded variation.

Corollary 3.4. If $f \in \Lambda BV(\overline{\mathbb{T}})$,

$$\sum_{n=1}^{\infty} \frac{1}{P_n} \left(\frac{\omega\left(f; \frac{\pi}{n}\right)}{n\sqrt{\Theta_{2n}}} \right)^{\frac{1}{2}} < \infty, \tag{3.5}$$

and (2.4) holds, then the Fourier series of f is summable $|N, p_n|$.

Finally, putting $\phi(x) = x$ and $\Lambda = \{1\}$ in Theorem 3.1, we get the following result as a corollary for the class of functions of bounded variation.

Corollary 3.5. If $f \in BV(\overline{\mathbb{T}})$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}} P_n} \left(\omega\left(f; \frac{\pi}{n}\right) \right)^{\frac{1}{2}} < \infty, \tag{3.6}$$

and (2.4) holds, then the Fourier series of f is summable $|N, p_n|$.

Remark 3.6. We note that, in the case of a function of bounded variation, our Condition (3.1) is weaker than Condition (2.2) of Theorem 2.3 because if (2.2) holds then by [3, Lemma 5], we have

$$\sum_{k=n}^{\infty} \frac{P_k^4}{k^2} \leq \sum_{k=n}^{\infty} \frac{c_2^4 k^{4\gamma} \psi^4(k)}{k^2} = \sum_{k=n}^{\infty} c_2^4 k^{4\gamma-2} \psi^4(k) \leq A n^{4\gamma-1} \psi^4(n) \leq A \frac{P_n^4}{n}.$$

But our crucial Condition (3.6) of Corollary 3.5 is stronger than the crucial Condition (2.3) of Theorem 2.3 because

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left(\omega\left(f; \frac{1}{n}\right) \right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}} P_n} \left(\omega\left(f; \frac{\pi}{n}\right) \right)^{\frac{1}{2}}.$$

We also note that, in the case of a function of Λ-bounded variation, our crucial Condition (3.5) of Corollary 3.4 is stronger than the crucial Condition (2.6) of Theorem 2.5 because

$$\sqrt{\Theta_{2n}} = \left(\sum_{k=1}^{2n} \frac{1}{\lambda_k^2} \right)^{\frac{1}{2}} \leq \left(\sum_{k=1}^{2n} \frac{1}{\lambda_k} \right) = \Lambda_{2n},$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{P_n} \left(\frac{\omega(f; \frac{\pi}{n})}{n\Lambda_{2n}} \right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} \frac{1}{P_n} \left(\frac{\omega(f; \frac{\pi}{n})}{n\sqrt{\Theta_{2n}}} \right)^{\frac{1}{2}}.$$

Further, our Condition (3.1) is stronger than of Condition (2.5), in view of Lemma 4.6.

We would like to draw the attention of readers that O. P. Varshney [14, p. 594], S. M. Shah [12, p. 249], S. M. Shah [13, p. 314], S. N. Lal and S. Ram [8, p. 91–92], R. G. Vyas and K. N. Darji [15, p. 157] have used that $\alpha^2(t+h)|g_x(t+h) - g_x(t-h)|^2$ is π-periodic, which is not true in general because

$$\alpha^2(t) = \left(\sum_{n=1}^{\infty} p_n \cos nt \right)^2,$$

$$\alpha^2(t+\pi) = \left(\sum_{n=1}^{\infty} (-1)^n p_n \cos nt \right)^2,$$

and $g_x(t)$ is 2π-periodic as $f(t)$ is 2π-periodic. So, we feel that there is a lacuna in the proofs of above results where it is used that $\alpha^2(t+h)|g_x(t+h)-g_x(t-h)|^2$ is π-periodic. Thus, though our results may look weaker, are worth to study as far as the correct mathematics is concern.

4. Proof of the result

We need the following lemmas to prove our theorem.

Lemma 4.1 ([10, Lemma 5.11]). *If $\{p_n\}$ is non-negative and non-increasing, then for $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n, we have*

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| \le P(t^{-1}).$$

Lemma 4.2 ([10, Lemma 5.13]). *If (i) $\{p_n\}$ is non-negative and non-increasing and (ii) $\{p_n - p_{n+1}\}$ is non-increasing, then*

$$\frac{(n+1)^2(p_n-p_{n+1})}{P_n} \le A.$$

Lemma 4.3 ([10, see proof of Lemma 5.31]). *If $\{p_n\}$ is non-negative and non-increasing, then*

$$P_{2^\nu} \le AP_{2^{\nu-1}} \text{ for } \nu = 1, 2, \ldots.$$

Lemma 4.4 ([10, Lemma 5.20]). *If $\{p_n\}$ is non-negative and non-increasing, and if we write $\gamma(t) = \sum_{k=0}^{\infty} p_k e^{ikt}$, then, for t in $[h, \pi]$,*

$$|\gamma(t+2h) - \gamma(t)| \le Aht^{-1}P(h^{-1}).$$

The following lemma is almost proved in [2], but for the sake of completeness we give details.

Lemma 4.5. *Suppose ϕ is convex on $[0,\infty)$. If ϕ is a Δ_2 function with constant $d \ge 2$ then for $a > 0$, $\phi(ax) \le [a + d^{\log_2 a+1}]\phi(x)$, for all $x \ge 0$.*

Proof. Since ϕ is convex on $[0,\infty)$ and $\phi(0) = 0$, for any $0 < a < 1$ and $x \ge 0$ we have

$$\phi(ax) = \phi(ax + (1-a)0) \le a\phi(x) + (1-a)\phi(0) = a\phi(x).$$

Further, as $\phi(2x) \le d\phi(x)$, for all $x \ge 0$, by using induction we get

$$\phi(2^n x) \le d^n \phi(x),\ x \ge 0,\ n \in \mathbb{N}.$$

Next, if $a \ge 1$ is any real number, choosing $n \in \mathbb{N}$ such that $2^{n-1} \le a < 2^n$ we get $0 < \frac{a}{2^n} < 1$. Therefore for all $x \ge 0$ we have

$$\phi(ax) = \phi\left(\frac{a}{2^n}2^n x\right) \le \frac{a}{2^n}\phi(2^n x) \le 1 \cdot d^n \phi(x) \le d^{\log_2 a+1}\phi(x).$$

So, for any $a \ge 0$ and $x \ge 0$, we have

$$\phi(ax) \le \max\{a, d^{\log_2 a+1}\}\phi(x) \le [a + d^{\log_2 a+1}]\phi(x).$$

Lemma 4.6. *Under the Hypothesis* (3.1) *of Theorem 3.1 there exists a constant A such that, for $n \in \mathbb{N}$,*

$$\sum_{k=n}^{\infty} \frac{P_k}{k^2} \leq A \frac{P_n}{n}$$

and

$$\sum_{k=n}^{\infty} \frac{P_k^2}{k^2} \leq A \frac{P_n^2}{n}.$$

Proof. Since $\{P_n\}$ is non-decreasing, for $n \in \mathbb{N}$, we have

$$\sum_{k=n}^{\infty} \frac{P_k}{k^2} = \sum_{k=n}^{\infty} \frac{P_k^4}{P_k^3 k^2} \leq \frac{1}{P_n^3} \sum_{k=n}^{\infty} \frac{P_k^4}{k^2} \leq \frac{A}{P_n^3} \frac{P_n^4}{n} = A \frac{P_n}{n}.$$

Similarly, we can prove second inequality.

Proof of Theorem 3.1. As proceeding in the [10, Lemma 5.16], we have

$$\begin{aligned}
\pi|t_n(x) - t_{n-1}(x)| \leq & \frac{|\alpha_n(x)|}{P_{n-1}} + \frac{|\beta_n(x)|}{P_{n-1}} + \frac{1}{P_{n-1}} \left| \int_0^{\frac{1}{n}} g_x(t) \sum_{k=n}^{\infty} p_k \cos[(n-k)t] dt \right| \\
& + \frac{p_n}{P_n P_{n-1}} \left| \int_0^{\frac{1}{n}} g_x(t) \sum_{k=0}^{n-1} P_k \cos[(n-k)t] dt \right| \\
& + \frac{1}{P_{n-1}} \left| \int_{\frac{1}{n}}^{\pi} \frac{g_x(t)}{2 \sin \frac{t}{2}} \sum_{k=n}^{\infty} (p_k - p_{k+1}) \sin \left[\left(k - n + \frac{1}{2} \right) t \right] dt \right| \\
& + \frac{p_n}{P_n P_{n-1}} \left| \int_{\frac{1}{n}}^{\pi} \frac{g_x(t)}{2 \sin \frac{t}{2}} \sum_{k=0}^{n-1} p_k \sin \left[\left(n - k + \frac{1}{2} \right) t \right] dt \right| \\
& + \frac{1}{2} \frac{p_n}{P_{n-1}} \left(1 - \frac{P_{n-1}}{P_n} \right) \left| \int_{\frac{1}{n}}^{\pi} g_x(t) dt \right| \\
= & \sum_{k=1}^{7} A_k(n), \text{ say.}
\end{aligned} \tag{4.1}$$

In order to prove the theorem, by (4.1) we have to show that

$$\sum_{n=2}^{\infty} A_k(n) < \infty, \text{ for } k = 1 \text{ to } 7.$$

First we show that $\sum_{n=2}^{\infty} A_1(n) < \infty$. Since $|Rez| \leq |z|$ for $z \in \mathbb{C}$ and $|e^{int}| = 1$ for $t \in \mathbb{R}$, using Lemma 1, for $0 \leq t \leq \pi$, we have

$$\alpha^2(t) = |\alpha(t)|^2 = \left| \sum_{k=0}^{\infty} p_k \cos kt \right|^2 \leq \left| \sum_{k=0}^{\infty} p_k e^{-ikt} \right|^2 = \left| \sum_{k=0}^{\infty} p_k e^{i(n-k)t} \right|^2 \leq P^2(t^{-1}). \tag{4.2}$$

Since $\alpha^2(t)$ is even, from this inequality, we get

$$\int_0^{2\pi} \alpha^2(t)dt = \int_{-\pi}^{\pi} \alpha^2(t)dt = 2\int_0^{\pi} \alpha^2(t)dt \le 2\int_0^{\pi} P^2(t^{-1})dt.$$

Put $\frac{1}{t} = y,\ 0 < t \le \pi$. Then $\frac{dt}{dy} = -\frac{1}{y^2};\ t \to 0_+ \iff y \to \infty$; and $t \to \pi \iff \frac{1}{\pi}$. So, above inequality becomes

$$\begin{aligned}\int_0^{2\pi} \alpha^2(t)dt \le \int_{\infty}^{\frac{1}{\pi}} P^2(y)\Big(-\frac{1}{y^2}\Big)dy &= \int_{\frac{1}{\pi}}^{\infty} P^2(t)\Big(\frac{1}{t^2}\Big)dt \\ &= \int_{\frac{1}{\pi}}^{1} P^2(t)\Big(\frac{1}{t^2}\Big)dt + \int_1^{\infty} P^2(t)\Big(\frac{1}{t^2}\Big)dt \\ &\le \int_{\frac{1}{\pi}}^{1} P_0^2\pi^2 dt + \sum_{k=1}^{\infty}\int_k^{k+1} P^2(t)\Big(\frac{1}{t^2}\Big)dt \\ &\le A + \sum_{k=1}^{\infty}\int_k^{k+1} \frac{P_k^2}{k^2}dt \\ &= A + \sum_{k=1}^{\infty}\frac{P_k^2}{k^2} \le A + A\frac{P_1^2}{1} = A,\end{aligned}$$

by Lemma 4.6. Therefore $\alpha(t) \in L^2$. Since $f \in \phi BV(\overline{\mathbb{T}})$, it is bounded. So, it follows that $g_x(t)\alpha(t) \in L^2$.

Let $0 < h \le \pi$ be fixed and let a_n $(n = 0, 1, 2, \dots)$, b_n $(n = 1, 2, \dots)$ be the Fourier coefficients of $g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t-h)$. Since $g_x(t)$ and $\alpha(t)$ are even,

$$\begin{aligned}g_x(-t+h)\alpha(-t+h) - g_x(-t-h)\alpha(-t-h) &= g_x(t-h)\alpha(t-h) - g_x(t+h)\alpha(t+h) \\ &= -(g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t-h)).\end{aligned}$$

Therefore $g_x(t+h)\alpha(t+h)-g_x(t-h)\alpha(t-h)$ is odd, and hence $a_n = 0$ for $n = 0, 1, 2, \dots$. Also, for $n = 1, 2, \dots,$

$$\begin{aligned}b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} [g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t-h)]\sin nt dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(t+h)\alpha(t+h)\sin nt dt - \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(t-h)\alpha(t-h)\sin nt dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(t)\alpha(t)\sin n(t-h)dt - \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(t)\alpha(t)\sin n(t+h)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(t)\alpha(t)[\sin nt\cos nh - \cos nt\sin nh - \sin nt\cos nh - \cos nt\sin nh]dt \\ &= -\frac{2}{\pi}\int_{-\pi}^{\pi} g_x(t)\alpha(t)\cos nt\sin nh dt \\ &= -\frac{4}{\pi}\sin nh\int_0^{\pi} g_x(t)\alpha(t)\cos nt dt \\ &= -\frac{4}{\pi}\sin nh\alpha_n(x).\end{aligned}$$

Hence the Fourier series of $g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t-h)$ is

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}\alpha_n(x)\sin nh\sin nt.$$

So, in view of Bessel's inequality, we get

$$\sum_{n=1}^{\infty}\Big(-\frac{4}{\pi}\alpha_n(x)\sin nh\Big)^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t-h)|^2dt.$$

Since the integrand of right hand side is even we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}|\alpha_n(x)\sin nh|^2\\
&\leq A\int_0^{\pi}|g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t-h)|^2dt\\
&= A\int_0^{\pi}|g_x(t+h)\alpha(t+h) - g_x(t-h)\alpha(t+h) + g_x(t-h)\alpha(t+h)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - g_x(t-h)\alpha(t-h)|^2dt.
\end{aligned}$$

Since $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b \in \mathbb{R}$, from above inequality we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}|\alpha_n(x)\sin nh|^2\\
&\leq A\int_0^{\pi}\alpha^2(t+h)|g_x(t+h) - g_x(t-h)|^2dt + A\int_0^{\pi}g_x^2(t-h)|\alpha(t+h) - \alpha(t-h)|^2dt\\
&= A\int_0^{\pi}\alpha^2(t+h)|g_x(t+h) - g_x(t-h)|^2dt + A\int_{-h}^{\pi-h}g_x^2(t)|\alpha(t+2h) - \alpha(t)|^2dt\\
&\leq A\int_0^{\pi}\alpha^2(t+h)|g_x(t+h) - g_x(t-h)|^2dt + A\int_{-h}^{\pi}g_x^2(t)|\alpha(t+2h) - \alpha(t)|^2dt\\
&= A\int_0^{\pi}\alpha^2(t+h)|g_x(t+h) - g_x(t-h)|^2dt + A\int_{-h}^{h}g_x^2(t)|\alpha(t+2h) - \alpha(t)|^2dt\\
&\qquad\qquad\qquad\qquad\qquad + A\int_{h}^{\pi}g_x^2(t)|\alpha(t+2h) - \alpha(t)|^2dt.
\end{aligned}$$

Again using $(a+b)^2 \leq 2(a^2+b^2)$ in second term of right hand side, we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}|\alpha_n(x)\sin nh|^2 \\
&\le A\int_0^{\pi}\alpha^2(t+h)|g_x(t+h)-g_x(t-h)|^2dt + A\int_{-h}^{h}g_x^2(t)\alpha^2(t+2h)dt \\
&\qquad + A\int_{-h}^{h}g_x^2(t)\alpha^2(t)dt + A\int_h^{\pi}g_x^2(t)|\alpha(t+2h)-\alpha(t)|^2dt \\
&= \sum_{k=1}^{4}B_k(h), \text{ say.}
\end{aligned}
\tag{4.3}
$$

Next, we shall estimate $B_1\left(\frac{\pi}{2N}\right)$. Then

$$
\begin{aligned}
B_1\left(\frac{\pi}{2N}\right) &= A\int_0^{\pi}\alpha^2\left(t+\frac{\pi}{2N}\right)\left|g_x\left(t+\frac{\pi}{2N}\right)-g_x\left(t-\frac{\pi}{2N}\right)\right|^2dt \\
&\le A\int_{-\pi}^{\pi}\alpha^2\left(t+\frac{\pi}{2N}\right)\left|g_x\left(t+\frac{\pi}{2N}\right)-g_x\left(t-\frac{\pi}{2N}\right)\right|^2dt.
\end{aligned}
$$

Put $t-\frac{\pi}{2N}=s$. Then $t\to-\pi \iff s\to-\pi-\frac{\pi}{2N}$ and $t\to\pi \iff s\to\pi-\frac{\pi}{2N}$. In view of this substitution and the 2π-periodicity of the integrand, we get

$$
\begin{aligned}
B_1\left(\frac{\pi}{2N}\right) &\le A\int_{-\pi-\frac{\pi}{2N}}^{\pi-\frac{\pi}{2N}}\alpha^2\left(s+\frac{\pi}{N}\right)\left|g_x\left(s+\frac{\pi}{N}\right)-g_x(s)\right|^2ds \\
&= A\int_{-\pi}^{\pi}\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|^2dt \\
&= A\int_{-\pi}^{\pi}\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right| \\
&\qquad\times\left|f\left(x+t+\frac{\pi}{N}\right)+f\left(x-t-\frac{\pi}{N}\right)-2f(x)-f(x+t)-f(x-t)+2f(x)\right|dt \\
&\le A\omega\left(f;\frac{\pi}{N}\right)\int_{-\pi}^{\pi}\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|dt.
\end{aligned}
$$

Now, if $\omega\left(f;\frac{\pi}{N}\right)=0$ then $B_1\left(\frac{\pi}{2N}\right)=0=A\omega\left(f;\frac{\pi}{N}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2N}}}\right)$. So, assume that $\omega\left(f;\frac{\pi}{N}\right)>0$. Then by using properties of ϕ, Jensen's inequality, and Lemma 4.5, we have

$$
\begin{aligned}
\phi\left(\frac{B_1\left(\frac{\pi}{2N}\right)}{A\omega\left(f;\frac{\pi}{N}\right)}\right) &\le \phi\left(\int_{-\pi}^{\pi}\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|dt\right) \\
&= \phi\left(\frac{2\pi}{2\pi}\int_{-\pi}^{\pi}\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|dt\right) \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi\left(2\pi\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq A\int_{-\pi}^{\pi}\phi\left(\alpha^2\left(t+\frac{\pi}{N}\right)\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|\right)dt\\
&\leq A\int_{-\pi}^{\pi}\left[\alpha^2\left(t+\frac{\pi}{N}\right)+d^{\log_2\alpha^2\left(t+\frac{\pi}{N}\right)+1}\right]\phi\left(\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|\right)dt\\
&\leq A\int_{-\pi}^{\pi}\left[\alpha^2\left(t+\frac{\pi}{N}\right)+d^{\log_2\alpha^2\left(t+\frac{\pi}{N}\right)}\right]\phi\left(\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|\right)dt.
\end{aligned}
$$

Using Hölder's inequality, we get

$$
\begin{aligned}
\phi\left(\frac{B_1\left(\frac{\pi}{2N}\right)}{A\omega\left(f;\frac{\pi}{N}\right)}\right)&\leq A\left(\int_{-\pi}^{\pi}\left[\alpha^2\left(t+\frac{\pi}{N}\right)+d^{\log_2\alpha^2\left(t+\frac{\pi}{N}\right)}\right]^2dt\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int_{-\pi}^{\pi}\left[\phi\left(\left|g_x\left(t+\frac{\pi}{N}\right)-g_x(t)\right|\right)\right]^2dt\right)^{\frac{1}{2}}\\
&=AI_1^{\frac{1}{2}}I_2^{\frac{1}{2}},\ \text{say}.
\end{aligned}\tag{4.4}
$$

Put $t+\frac{\pi}{N}=s$ in I_1. Then $\frac{dt}{ds}=1$, $t\to-\pi \iff s\to-\pi+\frac{\pi}{N}$ and $t\to\pi \iff s\to\pi+\frac{\pi}{N}$. Using this substitution and the fact that $\alpha(t)$ is 2π-periodic and even, we get

$$
\begin{aligned}
I_1=\int_{-\pi+\frac{\pi}{N}}^{\pi+\frac{\pi}{N}}\left[\alpha^2(s)+d^{\log_2\alpha^2(s)}\right]^2ds&=\int_{-\pi}^{\pi}\left[\alpha^2(s)+d^{\log_2\alpha^2(s)}\right]^2ds\\
&=2\int_0^{\pi}\left[\alpha^2(t)+d^{\log_2\alpha^2(t)}\right]^2dt.
\end{aligned}
$$

Using (4.2) and the fact that, $(a+b)^2\leq 2(a^2+b^2)$ for $a,b\in\mathbb{R}$, from above equality we get

$$
I_1\leq 2\int_0^{\pi}\left[P^2(t^{-1})+d^{\log_2 P^2\left(t^{-1}\right)}\right]^2dt\leq 4\int_0^{\pi}\left[P^4(t^{-1})+d^{2\log_2 P^2\left(t^{-1}\right)}\right]dt.
$$

Put $\frac{1}{t}=s$. Then $\frac{dt}{ds}=-\frac{1}{s^2}$, $t\to\pi \iff s\to\frac{1}{\pi}$, and $t\to 0_+ \iff s\to\infty$. Therefore, in view of (3.1) and (3.3), we get

$$
\begin{aligned}
I_1&\leq A\int_{\infty}^{\frac{1}{\pi}}[P^4(s)+d^{2\log_2 P^2(s)}]\left(-\frac{1}{s^2}\right)ds\\
&=A\int_{\frac{1}{\pi}}^{\infty}\left[\frac{P^4(t)}{t^2}+\frac{d^{\log_2 P^4(t)}}{t^2}\right]dt\\
&=A\left\{\int_{\frac{1}{\pi}}^{1}\left[\frac{P^4(t)}{t^2}+\frac{d^{\log_2 P^4(t)}}{t^2}\right]dt+\int_1^{\infty}\left[\frac{P^4(t)}{t^2}+\frac{d^{\log_2 P^4(t)}}{t^2}\right]dt\right\}\\
&\leq A\left[(P_0^4\cdot\pi^2+d^{\log_2 P_0^4}\cdot\pi^2)\int_{\frac{1}{\pi}}^{1}1dt+\sum_{k=1}^{\infty}\frac{P_k^4}{k^2}+\sum_{k=1}^{\infty}\frac{d^{\log_2 P_k^4}}{k^2}\right]\\
&\leq A\left[1+\frac{P_1^4}{1}+\sum_{k=1}^{\infty}\frac{d^{\log_2 P_k^4}}{k^2}\right]\leq A.
\end{aligned}\tag{4.5}
$$

Since $g_x(t)$ is 2π-periodic, putting $t = s + \frac{(k-1)\pi}{N}$ in I_2 for $k = 1, 2, \ldots, 2N$, we get

$$\begin{aligned} I_2 &= \int_{-\pi}^{\pi} \left[\phi\left(\left|g_x\left(s+\frac{k\pi}{N}\right) - g_x\left(s+\frac{(k-1)\pi}{N}\right)\right|\right)\right]^2 ds \\ &= \int_{-\pi}^{\pi} \lambda_k^2 \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] \\ &\qquad \times \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] dt. \end{aligned} \tag{4.6}$$

Since g_x is of $\phi\Lambda$-bounded variation, it is bounded. Also as $\{\lambda_k\}$ is increasing we have

$$\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k} \le \frac{\phi(A)}{\lambda_1}.$$

Using this inequality in (4.6) we get

$$\begin{aligned} I_2 &\le \int_{-\pi}^{\pi} \lambda_k^2 \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] \frac{\phi(A)}{\lambda_1} dt \\ &\le A\lambda_k^2 \int_{-\pi}^{\pi} \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] dt. \end{aligned}$$

Since $\lambda_k^2 > 0$, we get

$$\frac{1}{\lambda_k^2} I_2 \le A \int_{-\pi}^{\pi} \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] dt.$$

Summing above inequality from $k = 1, 2, \ldots, 2N$, we get

$$\begin{aligned} I_2 \sum_{k=1}^{2N} \frac{1}{\lambda_k^2} &\le A \sum_{k=1}^{2N} \int_{-\pi}^{\pi} \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] dt \\ &= A \int_{-\pi}^{\pi} \sum_{k=1}^{2N} \left[\frac{\phi\left(\left|g_x\left(t+\frac{k\pi}{N}\right) - g_x\left(t+\frac{(k-1)\pi}{N}\right)\right|\right)}{\lambda_k}\right] dt \le A, \end{aligned} \tag{4.7}$$

because g_x is of $\phi\Lambda$-bounded variation. Using (4.5) and (4.7) in (4.4), we get

$$B_1\left(\frac{\pi}{2N}\right) \le A\omega\left(f; \frac{\pi}{N}\right) \phi^{-1}\left(\frac{A}{\sqrt{\Theta_{2N}}}\right) \le A\omega\left(f; \frac{\pi}{N}\right) \phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2N}}}\right). \tag{4.8}$$

Now, as ω is increasing, by (4.2), we have

$$\begin{aligned} B_2(h) &\leq A\int_{-h}^{h} g_x^2(t)P^2\Big(\frac{1}{t+2h}\Big)dt \\ &\leq A\int_{-h}^{h} \omega^2(t)P^2\Big(\frac{1}{t+2h}\Big)dt \\ &\leq A\omega^2(h)\int_{h}^{3h} P^2\big(t^{-1}\big)dt. \end{aligned}$$

Therefore

$$B_2(h) \leq A\omega^2(h)P^2(h^{-1})\int_{h}^{3h} dt \leq Ah\omega^2(h)P^2(h^{-1}). \tag{4.9}$$

Again, as ω is increasing and α is even, by (4.2), we have

$$\begin{aligned} B_3(h) \leq A\int_{-h}^{h} \omega^2(t)\alpha^2(t)dt \leq A\omega^2(h)\int_{-h}^{h}\alpha^2(t)dt = A\omega^2(h)\int_0^h \alpha^2(t)dt \\ \leq A\omega^2(h)\int_0^h P^2\big(t^{-1}\big)dt. \end{aligned}$$

Therefore, by Lemma 4.6, we have

$$\begin{aligned} B_3(h) &\leq A\omega^2(h)\int_{\frac{1}{h}}^{\infty}\frac{P^2(t)}{t^2}dt \\ &= A\omega^2(h)\Big[\int_{\frac{1}{h}}^{\left[\frac{1}{h}\right]+1}\frac{P^2(t)}{t^2}dt + \int_{\left[\frac{1}{h}\right]+1}^{\infty}\frac{P^2(t)}{t^2}dt\Big] \\ &\leq A\omega^2(h)\left[P^2\Big(\Big[\frac{1}{h}\Big]+1\Big)\int_{\frac{1}{h}}^{\left[\frac{1}{h}\right]+1}\frac{1}{t^2}dt + \sum_{k=\left[\frac{1}{h}\right]+1}^{\infty}\frac{P_k^2}{k^2}\right] \\ &\leq A\omega^2(h)\Big[P^2\Big(\Big[\frac{1}{h}\Big]+1\Big)\Big[-\Big(\Big[\frac{1}{h}\Big]+1\Big)^{-1}+h\Big] + \frac{P^2\big(\big[\frac{1}{h}\big]+1\big)}{\big[\frac{1}{h}\big]+1}\Big] \\ &\leq A\omega^2(h)\Big[hP^2\Big(\Big[\frac{1}{h}\Big]+1\Big) + \frac{P^2\big(\big[\frac{1}{h}\big]+1\big)}{\big[\frac{1}{h}\big]+1}\Big] \\ &\leq A\omega^2(h)hP^2\Big(\Big[\frac{1}{h}\Big]+1\Big) \\ &= A\omega^2(h)hP^2(h^{-1})\frac{P^2\big(\big[\frac{1}{h}\big]+1\big)}{P^2(h^{-1})} \leq A\omega^2(h)hP^2(h^{-1}), \end{aligned} \tag{4.10}$$

because

$$\frac{P\big(\big[\frac{1}{h}\big]+1\big)}{P(h^{-1})} \leq \frac{P\big(\big[\frac{1}{h}\big]\big)+p(\big[\frac{1}{h}\big]+1)}{P\big(\big[\frac{1}{h}\big]\big)} \leq 1+\frac{p\big(\big[\frac{1}{h}\big]\big)}{P\big(\big[\frac{1}{h}\big]\big)} \leq 2.$$

Since $|\operatorname{Re} z| \le |z|$ for $z \in \mathbb{C}$, we have

$$\begin{aligned}
|\alpha(t+2h)-\alpha(t)| &= \left|\sum_{n=0}^{\infty} p_n \cos n(t+2h) - \sum_{n=0}^{\infty} p_n \cos nt\right| \\
&= \left|\operatorname{Re}\left(\sum_{n=0}^{\infty} p_n e^{in(t+2h)} - \sum_{n=0}^{\infty} p_n e^{int}\right)\right| \\
&\le \left|\sum_{n=0}^{\infty} p_n e^{in(t+2h)} - \sum_{n=0}^{\infty} p_n e^{int}\right| \\
&= |\gamma(t+2h)-\gamma(t)|,
\end{aligned}$$

where $\gamma(t)$ is as defined in Lemma 4.4. Using above inequality, Lemma 4.4, and $|g_x(t)| \le \omega(t)$ in $B_4(h)$, we have

$$\begin{aligned}
B_4(h) \le A\int_h^{\pi} g_x^2(t)|\gamma(t+2h)-\gamma(t)|^2 dt &\le A\int_h^{\pi} g_x^2(t) h^2 t^{-2} P^2(h^{-1}) dt \\
&\le A h^2 P^2(h^{-1}) \int_h^{\pi} \omega^2(t) t^{-2} dt \\
&= A h^2 P^2(h^{-1}) \int_{\frac{1}{\pi}}^{\frac{1}{h}} \omega^2(t^{-1}) dt. \qquad (4.11)
\end{aligned}$$

Taking $h = \frac{\pi}{2N}$ in (4.3), by (4.8)–(4.11) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\alpha_n(x)\sin\left(\frac{n\pi}{2N}\right)\right|^2 &\le A\Bigg[\omega\left(f;\frac{\pi}{N}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2N}}}\right) + N^{-1}\omega^2\left(\frac{\pi}{2N}\right)P^2\left(\frac{2N}{\pi}\right) \\
&\qquad + N^{-2}P^2\left(\frac{2N}{\pi}\right)\int_{\frac{1}{\pi}}^{\frac{2N}{\pi}} \omega^2(t^{-1})dt\Bigg] \\
&\le A\left[\omega\left(f;\frac{\pi}{N}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2N}}}\right) + N^{-2}P^2\left(\frac{2N}{\pi}\right)\int_{\frac{1}{\pi}}^{\frac{2N}{\pi}} \omega^2(t^{-1})dt\right], \qquad (4.12)
\end{aligned}$$

because

$$\begin{aligned}
N^{-2}P^2\left(\frac{2N}{\pi}\right)\int_{\frac{1}{\pi}}^{\frac{2N}{\pi}} \omega^2(t^{-1})dt &\ge N^{-2}P^2\left(\frac{2N}{\pi}\right)\omega^2\left(\frac{\pi}{2N}\right)\left[\frac{2N}{\pi}-\frac{1}{\pi}\right] \\
&\ge N^{-2}P^2\left(\frac{2N}{\pi}\right)\omega^2\left(\frac{\pi}{2N}\right)\left[\frac{2N}{\pi}-\frac{N}{\pi}\right] \\
&= (N\pi)^{-1}P^2\left(\frac{2N}{\pi}\right)\omega^2\left(\frac{\pi}{2N}\right).
\end{aligned}$$

Note that for $\mu \in \mathbb{N}$, we have

$$\begin{aligned}
2^{\mu-1}+1 \le n \le 2^{\mu} &\implies \frac{(2^{\mu-1}+1)\pi}{2^{\mu+1}} \le \frac{n\pi}{2^{\mu+1}} \le \frac{2^{\mu}\pi}{2^{\mu+1}} \\
&\implies \frac{\pi}{4} + \frac{\pi}{2^{\mu+1}} \le \frac{n\pi}{2^{\mu+1}} \le \frac{\pi}{2} \\
&\implies \frac{\pi}{4} < \frac{n\pi}{2^{\mu+1}} \le \frac{\pi}{2} \\
&\implies \frac{1}{\sqrt{2}} < \sin\left(\frac{n\pi}{2^{\mu+1}}\right) \le 1 \\
&\implies 1 < 2\sin^2\left(\frac{n\pi}{2^{\mu+1}}\right).
\end{aligned}$$

Therefore, using (4.12) for $N = 2^{\mu}$ and the fact that, $|a+b|^p \le |a|^p + |b|^p$, for $0 \le p < 1$ and $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
&\left(\sum_{n=2^{\mu-1}+1}^{2^{\mu}} |\alpha_n(x)|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{2}\left(\sum_{n=2^{\mu-1}+1}^{2^{\mu}} \left|\alpha_n(x)\sin\left(\frac{n\pi}{2^{\mu+1}}\right)\right|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{2}\left(\sum_{n=1}^{\infty} \left|\alpha_n(x)\sin\left(\frac{n\pi}{2^{\mu+1}}\right)\right|^2\right)^{\frac{1}{2}} \\
&\le A\left[\omega\left(f;\frac{\pi}{2^{\mu}}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right) + 2^{-2\mu}P^2\left(\frac{2^{\mu+1}}{\pi}\right)\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}} \omega^2(t^{-1})dt\right]^{\frac{1}{2}} \\
&\le A\left[\omega\left(f;\frac{\pi}{2^{\mu}}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right]^{\frac{1}{2}} + A\left[2^{-\mu}P\Big(\frac{2^{\mu+1}}{\pi}\Big)\Big(\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}} \omega^2(t^{-1})dt\Big)^{\frac{1}{2}}\right].
\end{aligned} \tag{4.13}$$

Since $\{p_n\}$ is non-negative and non-increasing, by Lemma 4.3, we have

$$\sum_{n=2^{\mu-1}+1}^{2^{\mu}} \frac{1}{P_{n-1}^2} \le \frac{1}{P_{2^{\mu-1}}^2}[2^{\mu} - 2^{\mu-1}] = \frac{2^{\mu-1}}{P_{2^{\mu-1}}^2} \le \frac{2^{\mu}}{2}\cdot\frac{A}{P_{2^{\mu}}^2} \le A\frac{2^{\mu}}{P_{2^{\mu}}^2} \tag{4.14}$$

and

$$\frac{P\left(\frac{2^{\mu+1}}{\pi}\right)}{P_{2^{\mu}}} \le \frac{P_{2^{\mu+1}}}{P_{2^{\mu}}} \le A. \tag{4.15}$$

Therefore, using Holder's inequality and (4.13)–(4.15), we have

$$\begin{aligned}
&\sum_{n=2^{\mu-1}+1}^{2^\mu}\frac{|\alpha_n(x)|}{P_{n-1}}\\
&\le\left(\sum_{n=2^{\mu-1}+1}^{2^\mu}|\alpha_n(x)|^2\right)^{\frac12}\left(\sum_{n=2^{\mu-1}+1}^{2^\mu}\frac{1}{P_{n-1}^2}\right)^{\frac12}\\
&\le A\frac{2^{\frac{\mu}{2}}}{P_{2^\mu}}\left[\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}+2^{-\mu}P\Big(\frac{2^{\mu+1}}{\pi}\Big)\Big(\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}}\omega^2(t^{-1})dt\Big)^{\frac12}\right]\\
&\le A\left[\frac{2^{\frac{\mu}{2}}}{P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}+2^{-\frac{\mu}{2}}\frac{P\big(\frac{2^{\mu+1}}{\pi}\big)}{P_{2^\mu}}\Big(\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}}\omega^2(t^{-1})dt\Big)^{\frac12}\right]\\
&\le A\left[\frac{2^{\frac{\mu}{2}}}{P_{2^u}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}+2^{-\frac{\mu}{2}}\Big(\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}}\omega^2(t^{-1})dt\Big)^{\frac12}\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{n=2}^\infty A_1(n)&=\sum_{\mu=1}^\infty\sum_{n=2^{\mu-1}+1}^{2^\mu}\frac{|\alpha_n(x)|}{P_{n-1}}\\
&\le A\sum_{\mu=1}^\infty\frac{2^{\frac{\mu}{2}}}{P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}+A\sum_{\mu=1}^\infty2^{-\frac{\mu}{2}}\Big(\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}}\omega^2(t^{-1})dt\Big)^{\frac12}.
\end{aligned}\tag{4.16}$$

Observe that

$$\sum_{\mu=1}^\infty\frac{2^{\frac{\mu}{2}}}{P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}=2\sum_{\mu=1}^\infty\frac{2^{\mu-1}}{2^{\frac{\mu}{2}}P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}.\tag{4.17}$$

Note that for $2^{\mu-1}\le m<2^\mu$, we have $\omega(f;\frac{\pi}{2^\mu})\le\omega(f;\frac{\pi}{m})$, $\Theta_{2m}=\sum_{k=1}^{2m}\frac{1}{\lambda_k^2}\le\sum_{k=1}^{2(2^\mu)}\frac{1}{\lambda_k^2}=\Theta_{2^{\mu+1}}$, and $P_m\le P_{2^\mu}$. Therefore

$$\begin{aligned}
&\sum_{m=2^{\mu-1}}^{2^\mu-1}\frac{1}{P_m}\left(\omega\left(f;\frac{\pi}{m}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2m}}}\right)\Big/m\right)^{\frac12}\\
&\ge\sum_{m=2^{\mu-1}}^{2^\mu-1}\frac{1}{P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\Big/2^\mu\right)^{\frac12}\\
&=\frac{2^{\mu-1}}{P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\Big/2^\mu\right)^{\frac12}\\
&=\frac{2^{\mu-1}}{2^{\frac{\mu}{2}}P_{2^\mu}}\left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac12}.
\end{aligned}\tag{4.18}$$

Using (3.2) and (4.18) in (4.17) we get

$$\begin{aligned}
\sum_{\mu=1}^{\infty} \frac{2^{\frac{\mu}{2}}}{P_{2^\mu}} \left(\omega\left(f;\frac{\pi}{2^\mu}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2^{\mu+1}}}}\right)\right)^{\frac{1}{2}}
&\le 2\sum_{\mu=1}^{\infty}\sum_{m=2^{\mu-1}}^{2^\mu-1} \frac{1}{P_m}\left(\omega\left(f;\frac{\pi}{m}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2m}}}\right)\Big/ m\right)^{\frac{1}{2}} \\
&= 2\sum_{n=1}^{\infty}\frac{1}{P_n}\left(\omega\left(f;\frac{\pi}{n}\right)\phi^{-1}\left(\frac{1}{\sqrt{\Theta_{2n}}}\right)\Big/ n\right)^{\frac{1}{2}} \\
&\le A. \qquad (4.19)
\end{aligned}$$

Using the fact that $\omega(f;\delta)$ is increasing and $\omega(f;\lambda\delta) \le (\lambda+1)\omega(f;\delta)$ for $\lambda > 0$, we have

$$\begin{aligned}
\int_{\frac{1}{\pi}}^{\frac{2^{\mu+1}}{\pi}} \omega^2(t^{-1})dt
&= \int_{\frac{1}{2}}^{2^\mu} \omega^2\left(\frac{\pi}{2t}\right)\frac{2}{\pi}dt \\
&\le A\int_{\frac{1}{2}}^{2^\mu} \omega^2(t^{-1})dt \\
&= A\int_{\frac{1}{2}}^{2} \omega^2(t^{-1})dt + A\int_{2}^{2^\mu}\omega^2(t^{-1})dt \\
&\le A\omega^2(2)\int_{\frac{1}{2}}^{2} dt + A\sum_{k=2}^{2^\mu-1}\int_k^{k+1}\omega^2(t^{-1})dt \\
&\le A + A\sum_{k=2}^{2^\mu-1}\omega^2(k^{-1}) \\
&\le A + A\sum_{k=2}^{2^\mu}\omega^2(k^{-1}). \qquad (4.20)
\end{aligned}$$

Using (4.19) and (4.20) in (4.16) and the inequality $|a+b|^p \le |a|^p + |b|^p$ for $0 \le p < 1$ and $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
\sum_{n=2}^{\infty} A_1(n) &\le A + A\sum_{\mu=1}^{\infty} 2^{-\frac{\mu}{2}} + A\sum_{\mu=1}^{\infty}2^{-\frac{\mu}{2}}\left(\sum_{k=2}^{2^\mu}\omega^2(k^{-1})\right)^{\frac{1}{2}} \\
&\le A + A + A\sum_{\mu=1}^{\infty}2^{-\frac{\mu}{2}}\sum_{m=1}^{\mu}\left(\sum_{k=2^{m-1}+1}^{2^m}\omega^2(k^{-1})\right)^{\frac{1}{2}}. \qquad (4.21)
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{\mu=1}^{\infty} 2^{-\frac{\mu}{2}} \sum_{m=1}^{\mu} \Bigg(\sum_{k=2^{m-1}+1}^{2^m} \omega^2(k^{-1}) \Bigg)^{\frac{1}{2}} \\
&\quad \leq \sum_{\mu=1}^{\infty} 2^{-\frac{\mu}{2}} \sum_{m=1}^{\mu} \omega\left(\frac{1}{2^{m-1}}\right) \Bigg(\sum_{k=2^{m-1}+1}^{2^m} 1 \Bigg)^{\frac{1}{2}} \\
&\quad = \sum_{\mu=1}^{\infty} 2^{-\frac{\mu}{2}} \sum_{m=1}^{\mu} 2^{\frac{m-1}{2}} \omega\left(\frac{1}{2^{m-1}}\right) \\
&\quad = \sum_{\mu=1}^{\infty} 2^{-\frac{\mu}{2}} \sum_{m=1}^{\mu} \frac{2^{\frac{m}{2}}}{2^{\frac{1}{2}}} \omega\left(\frac{2}{2^{m}}\right) \\
&\quad \leq A \sum_{\mu=1}^{\infty} 2^{-\frac{\mu}{2}} \sum_{m=1}^{\mu} 2^{\frac{m}{2}} \omega\left(\frac{1}{2^{m}}\right) \\
&\quad = A \sum_{m=1}^{\infty} 2^{\frac{m}{2}} \omega\left(\frac{1}{2^{m}}\right) \sum_{\mu=m}^{\infty} 2^{-\frac{\mu}{2}} \\
&\quad \leq A \sum_{m=1}^{\infty} 2^{\frac{m}{2}} \omega\left(\frac{1}{2^{m}}\right) 2^{-\frac{m}{2}} \\
&\quad = A \sum_{m=1}^{\infty} \omega\left(\frac{1}{2^{m}}\right).
\end{aligned}$$

Using above inequality in (4.21), we get

$$\sum_{n=2}^{\infty} A_1(n) \leq A + A \sum_{m=1}^{\infty} \omega\left(\frac{1}{2^m}\right). \tag{4.22}$$

Since $\omega(f;\delta)$ is increasing, we have

$$\sum_{m=1}^{\infty} \omega\left(\frac{1}{2^m}\right) = 2 \sum_{m=1}^{\infty} \sum_{k=2^{m-1}}^{2^m-1} \frac{\omega\left(\frac{1}{2^m}\right)}{2^m} \leq 2 \sum_{m=1}^{\infty} \sum_{k=2^{m-1}}^{2^m-1} \frac{\omega\left(\frac{1}{k}\right)}{k} = 2 \sum_{n=1}^{\infty} \frac{\omega\left(\frac{1}{n}\right)}{n}.$$

Using (2.4) and above inequality in (4.22), we get

$$\sum_{n=2}^{\infty} A_1(n) \leq A + A \sum_{m=1}^{\infty} \frac{\omega\left(\frac{1}{m}\right)}{m} \leq A.$$

Similarly we can prove that $\sum_{n=2}^{\infty} A_2(n) < \infty$, by using the fact that the Fourier series of $g_x(t+h)\beta(t+h) - g_x(t-h)\beta(t-h)$ is

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \beta_n(x) \sin nh \cos nt.$$

Also for $k = 3, 4, ..., 7$, it follows from the proof of [8, Theorem] that

$$\sum_{n=2}^{\infty} A_k(n) < \infty,$$

and hence the result follows. □

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(Prince Ashvinbhai Bhuva and Bhikha Lila Ghodadra) Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara - 390 002 (Gujarat), India.

E-mail address, Prince Ashvinbhai Bhuva: `the.princebhuva@gmail.com`

E-mail address, Bhikha Lila Ghodadra: `bhikhu_ghodadra@yahoo.com`