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NORM INEQUALITIES FOR DUNKL-TYPE FRACTIONAL INTEGRAL AND FRACTIONAL MAXIMAL OPERATORS IN THE DUNKL-FOFANA SPACES

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Abstract. We establish some new properties of the Dunkl-Wiener amalgam spaces defined on the real line. These results allow us to obtain the boundedness of Dunkl-type fractional integral and fractional maximal operators in the Dunkl-Fofana spaces.

1. Introduction

Let us start with some notations that will be used throughout this paper. Let $k > -\frac{1}{2}$ be a fixed number and μ be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu(x) = \left(2^{k+1}\Gamma(k+1)\right)^{-1} |x|^{2k+1}\, dx.$$

For $1 \leq p \leq \infty$, we denote by $L^p(\mu)$ the Lebesgue space associated with the measure μ, while $L^{p,\infty}(\mu)$ is the weak $L^p(\mu)$-space. We denote by $L^0(\mu)$ the complex vector space of equivalence classes (modulo equality μ-almost everywhere) of complex-valued functions μ-measurable on $\mathbb{R}$. For $f \in L^p(\mu)$, $\|f\|_p$ stands for the classical norm of f. For any subset A of $\mathbb{R}$, χ_A denotes the characteristic function of A. For $x \in \mathbb{R}$ and for $r > 0$, we set

$$B(x, r) = \{y \in \mathbb{R} : \max\{0, |x| - r\} < |y| < |x| + r\}$$

if $x \neq 0$ and

$$B_r = B(0, r) = (-r, r).$$

The letter C will be used for non-negative constants not depending on the relevant variables, and this constant may change from one occurrence to another.

The study of the boundedness properties of certain integral type operators in the topological vector spaces where they act is an important problem in harmonic analysis. Many useful results concerning this topic, which were established in classical Fourier

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analysis have been generalized within the framework of Dunkl analysis (see, for example [3], [8], [9], [10], [12], and the references therein). Recently, P. Nagacy and J. Feuto [15] introduced the Dunkl-Wiener amalgam spaces and the Dunkl-Fofana spaces on the real line as described below. As an application, they established the boundedness of the Hardy-Littlewood maximal operator associated with the Dunkl operator in the Dunkl-Fofana spaces. Recall that on the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$.
For $1 \leq q, p \leq \infty$, the Dunkl-Wiener amalgam space $(L^q, L^p)(\mu)$ is defined as the subspace of $L^0(\mu)$ such that $\|f\|_{q,p} = {}_1\|f\|_{q,p} < \infty$, where for $r > 0$,

$$
{}_r\|f\|_{q,p} = \begin{cases} \left\| \left[\int_{\mathbb{R}} (\tau_y |f|^q)\chi_{B_r}(x) d\mu(x)\right]^{\frac{1}{q}} \right\|_p & \text{if} \quad q < \infty, \\ \left\| \left\| f\chi_{B(y,r)} \right\|_\infty \right\|_p & \text{if} \quad q = \infty, \end{cases} \tag{1.1}
$$

with the $L^p(\mu)$-norm taken with respect to the variable y. Here and in the sequel, $\tau_y (y \in \mathbb{R})$ stands for the Dunkl translation operator (see Section 2 for more details).
The space $(L^q, L^p)(\mu)$ is a complex Banach space endowed with the norm $\|\cdot\|_{q,p}$.
Let $1 \leq q \leq \alpha \leq p \leq \infty$. The space $(L^q, L^p)^\alpha(\mu)$ is defined as the subspace of $L^0(\mu)$ such that $\|f\|_{q,p,\alpha} < \infty$, where

$$
\|f\|_{q,p,\alpha} = \sup_{r>0} (\mu(B_r))^{\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p}} \, {}_r\|f\|_{q,p}.
$$

It is proved in [15] that the map $f \mapsto \|f\|_{q,p,\alpha}$ is a norm on $(L^q, L^p)^\alpha(\mu)$. In addition, for $\alpha \in \{p, q\}$, $(L^q, L^p)^\alpha(\mu) = L^\alpha(\mu)$.

In this paper we obtain several new properties of the spaces $(L^q, L^p)(\mu)$ by examining their relationship with other topological vector spaces. We also define an equivalent norm to $\|\cdot\|_{q,p}$. Furthermore, we show that $((L^q, L^p)^\alpha(\mu), \|\cdot\|_{q,p,\alpha})$ is a complex Banach space. These results allow us to establish norm inequalities in the spaces $(L^q, L^p)^\alpha(\mu)$ for the Dunkl-type fractional integral operator, defined by

$$
I_\beta f(x) = \int_{\mathbb{R}} \tau_x f(z)|z|^{\beta-(2k+2)} d\mu(z),
$$

and the fractional maximal operator

$$
M_\beta f(x) = \sup_{r>0} (\mu(B(0,r))^{\frac{\beta}{2k+2}-1} \int_{B(0,r)} \tau_x |f|(y) d\mu(y),
$$

where $0 < \beta < 2k + 2$.
In the limiting case $\beta = 0$, the fractional maximal operator reduces to the Hardy-Littlewood maximal operator associated with the Dunkl operator, denoted by M.
Our main results read as follows.

Theorem 1.1. *Let $1 < q \leq \alpha \leq p < \infty$ and $0 < \beta < \frac{2k+2}{\alpha}$. Put*

$$
\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \frac{1}{\bar{p}} = \frac{1}{p}(1 - \frac{\alpha\beta}{2k+2}) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q}(1 - \frac{\alpha\beta}{2k+2}).
$$

Then

$$
\|I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,p,\alpha}^{1-\frac{\alpha\beta}{2k+2}} \|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}, \qquad f \in (L^q, L^p)^\alpha(\mu)
$$

and

$$
\|I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,p,\alpha}, \qquad f \in (L^q, L^p)^\alpha(\mu).
$$

For $q = 1$, we prove the following

Theorem 1.2. *Let* $1 < \alpha < p < \infty$ *and* $0 < \beta < \frac{2k+2}{\alpha}$*. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \frac{1}{\bar{p}} = \frac{1}{p}(1 - \frac{\alpha\beta}{2k+2}) \textit{ and } \frac{1}{\bar{q}} = 1 - \frac{\alpha\beta}{2k+2}.$$

Then

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{1,p,\alpha}^{1-\frac{\alpha\beta}{2k+2}}\|f\|_{1,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}, \qquad f \in (L^1, L^p)^\alpha(\mu),$$

where

$$\|I_\beta f\|_{(L^{\bar{q},\infty},\, L^{\bar{p}})^{\alpha^*}(\mu)} = \sup_{r>0} \mu(B(0,r))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}}\|\|I_\beta f \tau_{(-.)}^{\frac{1}{\bar{q}}}\chi_{B_r}\|_{L^{\bar{q},\infty}(\mu)}\|_{\bar{p}}$$

and

$$\|I_\beta f\|_{(L^{\bar{q},\infty},\, L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{1,p,\alpha}, \qquad f \in (L^1, L^p)^\alpha(\mu).$$

As a consequence of Theorems 1.1, we obtain the corresponding boundedness results for M_β.

Corollary 1.3. Let $1 < q \leq \alpha \leq p < \infty$ and $0 < \beta < \frac{2k+2}{\alpha}$. Put

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \quad \frac{1}{\bar{p}} = \frac{1}{p}(1 - \frac{\alpha\beta}{2k+2}) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q}(1 - \frac{\alpha\beta}{2k+2}).$$

Then

$$\|M_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,p,\alpha}, \quad f \in (L^q, L^p)^\alpha(\mu).$$

As a consequence of Theorem 1.2, we obtain the following weak-type inequality for M_β.

Corollary 1.4. Let $1 < \alpha < p < \infty$ and $0 < \beta < \frac{2k+2}{\alpha}$. Put

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \quad \frac{1}{\bar{p}} = \frac{1}{p}(1 - \frac{\alpha\beta}{2k+2}) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q}(1 - \frac{\alpha\beta}{2k+2}).$$

Then

$$\|M_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{1,p,\alpha}, \quad f \in (L^1, L^p)^\alpha(\mu).$$

Remark 1.5. Note that

(1) Theorem 1.1 and Corollary 1.3 in the case $\alpha = p$ or $\alpha = q$ were proved in [8].
(2) Theorem 1.1 and Theorem 1.2 are generalizations, within the framework of Dunkl analysis, of Theorem 4.5 and Corollary 4.4 established in [6] in the one-dimensional case.

The remainder of this note is organized as follows. Section 2 contains the prerequisite for Dunkl analysis on the real line. In Section 3 we gather some basic results about the spaces $(L^q, L^p)(\mu)$. New properties of the spaces $(L^q, L^p)(\mu)$ and $(L^q, L^p)^\alpha(\mu)$ are established in Section 4. Section 5 is devoted to the proof of Theorems 1.1 and 1.2 as well as Corollaries 1.3 and 1.4.

2. Prerequisite on Dunkl analysis on the real line

The Dunkl operator associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ is defined by

$$\Lambda_k f(x) = \frac{df}{dx}(x) + \frac{2k+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

For $\lambda \in \mathbb{C}$, the Dunkl kernel denoted by $\mathfrak{E}_k(\lambda)$ (see [4]), is the only solution of the initial value problem

$$\Lambda_k f(x) = \lambda f(x),\ f(0) = 1, \qquad x \in \mathbb{R}.$$

It is given by the formula

$$\mathfrak{E}_k(\lambda x) = j_k(i\lambda x) + \frac{\lambda x}{2(k+1)} j_{k+1}(i\lambda x), \qquad x \in \mathbb{R},$$

where

$$j_k(z) = 2^k\Gamma(k+1)\frac{J_k(z)}{z^k} = \Gamma(k+1)\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{n!2^{2n}\Gamma(n+k+1)}, \qquad z \in \mathbb{C}$$

is the normalized Bessel function of the first kind and of order k. Notice that $\Lambda_{-\frac{1}{2}} = \frac{d}{dx}$ and $\mathfrak{E}_{-\frac{1}{2}}(\lambda x) = e^{\lambda x}$. It is also proved (see [16]) that $|\mathfrak{E}_k(ix)| \leq 1$ for every $x \in \mathbb{R}$.

The Dunkl kernel $\mathfrak{E}_k$ gives rise to an integral transform on $\mathbb{R}$ denoted $\mathcal{F}_k$ and called Dunkl transform (see [2]). For $f \in L^1(\mu)$

$$\mathcal{F}_k f(\lambda) = \int_{\mathbb{R}} \mathfrak{E}_k(-i\lambda x) f(x) d\mu(x), \qquad \lambda \in \mathbb{R}.$$

We have the following properties of the Dunkl transform given in [2] (see also [5]).

Proposition 2.1. (1) Let $f \in L^1(\mu)$. If $\mathcal{F}_k(f)$ is in $L^1(\mu)$, then we have the following inversion formula :

$$f(x) = C\int_{\mathbb{R}} \mathfrak{E}_k(ixy)\mathcal{F}_k(f)(y) d\mu(y).$$

(2) The Dunkl transform has a unique extension to an isometric isomorphism on $L^2(\mu)$.

Definition 2.2. Let $y \in \mathbb{R}$ be given. The generalized translation operator $f \mapsto \tau_y f$ is defined on $L^2(\mu)$ by the equation

$$\mathcal{F}_k(\tau_y f)(x) = \mathfrak{E}_k(ixy)\mathcal{F}_k(f)(x), \qquad x \in \mathbb{R}.$$

Mourou proved in [14] that generalized translation operators have the following properties:

Proposition 2.3. (1) The operator τ_x, $x \in \mathbb{R}$, is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself,

(2) For $f \in \mathcal{E}(\mathbb{R})$ and $x, y \in \mathbb{R}$
   (a) $\tau_x f(y) = \tau_y f(x)$,
   (b) $\tau_0 f = f$,
   (c) $\tau_x \circ \tau_y = \tau_y \circ \tau_x$.

Proposition 2.4 (see Soltani [17])**.** For x, $\lambda \in \mathbb{R}$ and $f \in L^1(\mu)$,

$$\mathcal{F}_k(\tau_x f)(\lambda) = \mathfrak{E}_k(ix\lambda)\mathcal{F}_k(f)(\lambda).$$

For $x \in \mathbb{R}$, let $B(x,r) = \{y \in \mathbb{R} : \max\{0,|x|-r\} < |y| < |x|+r\}$ if $x \neq 0$ and $B_r := B(0,r) =]-r,r[$. We have $\mu(B_r) = d_k r^{2k+2}$ where $d_k = \left[2^{k+1}(k+1)\Gamma(k+1)\right]^{-1}$.

For $x \in \mathbb{R}$ and $r > 0$, the map $y \mapsto \tau_x \chi_{B_r}(y)$ is supported in $B(x,r)$ and

$$0 \le \tau_x \chi_{B_r}(y) \le \min\left\{1, \frac{2C_\kappa}{2\kappa+1}\left(\frac{r}{|x|}\right)^{2\kappa+1}\right\}, \qquad y \in B(x,r), \tag{2.1}$$

as proved in [9].

The following property of the translation will be useful.

Proposition 2.5. There exists positive real numbers C_1 and C_2 such that for all $(y,\ f) \in \mathbb{R} \times L^0(\mu)$ we have

$$\begin{aligned} C_1 \int_{\mathbb{R}} |f|^q(x)\chi_{I(y,1)}(x)d\mu(x) &\le \int_{\mathbb{R}} \tau_y |f|^q(x)\chi_{B(0,1)}(x)d\mu(x) \\ &\le C_2 \int_{\mathbb{R}} |f|^q(x)\chi_{I(y,1)}(x)d\mu(x). \end{aligned}$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_k$ of f and g by

$$f *_k g(x) = \int_{\mathbb{R}} \tau_x f(-y)g(y)d\mu(y).$$

The generalized convolution $*_k$ is associative and commutative (see [16]). We also have the below result.

Proposition 2.6 (see Soltani [17])**.** (1) For all $x \in \mathbb{R}$, the operator τ_x extends to $L^p(\mu)$, $p \ge 1$, and

$$\|\tau_x f\|_p \le 4\|f\|_p \tag{2.2}$$

for all $f \in L^p(\mu)$.

(2) Assume that $p,q,r \in [1,\infty]$ and satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then the generalized convolution defined on $\mathcal{C}_c \times \mathcal{C}_c$, extends to a continuous map from $L^p(\mu) \times L^q(\mu)$ to $L^r(\mu)$, and we have

$$\|f *_k g\|_r \le 4\|f\|_p \|g\|_q .$$

It is also proved in [7] that if $f \in L^1(\mu)$ and $g \in L^p(\mu)$, $1 \le p < \infty$, then

$$\tau_x(f *_k g) = \tau_x f *_k g = f *_k \tau_x g, x \in \mathbb{R}.$$

For any measurable function f on $\mathbb{R}$, we define its distribution and rearrangement functions

$$\begin{aligned} d_{f,k}(\lambda) &= \mu\left(\left\{\ x \in \mathbb{R} : |f(x)| > \lambda\ \right\}\right), \\ f_k^*(\lambda) &= \inf\left\{\ t > 0 : d_{f,k}(t) \le \lambda\ \right\}. \end{aligned}$$

For $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, define

$$\|f\|_{L^{p,q}(\mu)} = \begin{cases} \left[\int_0^\infty (s^{\frac{1}{p}} f_k^*(s))^q \frac{ds}{s}\right]^{\frac{1}{q}} & \text{if} \quad 1 \le p,\ q < \infty \\ \sup\limits_{s>0} s^{\frac{1}{p}} f_k^*(s) & \text{if} \quad 1 \le p \le \infty, q = \infty. \end{cases}$$

The generalized Lorentz space $L^{p,\,q}(\mu)$ is defined as the set of all measurable functions f such that $\|f\|_{L^{p,\,q}(\mu)} < \infty$. (See [10], [13]).
A particular case of this space is the weak-Lebesgue space $L^{p,\,\infty}(\mu)$.

Proposition 2.7 (see [10]). If $f \in L^{p_1,\,q_1}(\mu)$, $g \in L^{p_2,\,q_2}(\mu)$ and $\frac{1}{p_1} + \frac{1}{p_2} > 1$, then $f *_k g \in L^{p_3,\,q_3}(\mu)$ where $\frac{1}{p_3} + 1 = \frac{1}{p_1} + \frac{1}{p_2}$ and $q_3 \geq 1$ is any number such that $\frac{1}{q_3} \leq \frac{1}{q_1} + \frac{1}{q_2}$. Moreover,

$$\|f *_k g\|_{L^{p_3,\,q_3}(\mu)} \leq 3p_3 \|f\|_{L^{p_1,\,q_1}(\mu)} \|g\|_{L^{p_2,\,q_2}(\mu)}.$$

3. Preliminaries results on the Dunkl-Wiener amalgam spaces

The propositions of this section are proved in [15].
Hölder inequality remains valid in the Dunkl-Wiener amalgam spaces as stated below.

Proposition 3.1. Let $1 \leq q_1,\, p_1 \leq \infty$ and $1 \leq q_2,\, p_2 \leq \infty$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \leq 1 \text{ and } \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \leq 1.$$

If $f \in (L^{q_1},\, L^{p_1})(\mu)$ and $g \in (L^{q_2},\, L^{p_2})(\mu)$ then $fg \in (L^q,\, L^p)(\mu)$.
Moreover,

$$\|fg\|_{q,\,p} \leq \|f\|_{q_1,\,p_1} \|g\|_{q_2,\,p_2}.$$

The following result shows that $L^s(\mu) \subset (L^q,\, L^p)(\mu)$ when $1 \leq q \leq s \leq p \leq \infty$.

Proposition 3.2. Let $1 \leq q \leq s \leq p \leq \infty$. We have

$$\|f\|_{q,\,p} \leq 4^{\frac{1}{q}} \|f\|_s \, \mu(B_1)^{\frac{1}{p} - \frac{1}{s} + \frac{1}{q}}. \tag{3.1}$$

The family of spaces $(L^q,\, L^p)(\mu)$ is decreasing with respect to the q-power. More precisely, we have the following result.

Proposition 3.3. Let $1 \leq q_1 \leq q_2 \leq \infty$ and $1 \leq p \leq \infty$. For any locally μ-integrable function f, we have

$$\|f\|_{q_1,p} \leq \mu(B_1)^{\frac{1}{q_1} - \frac{1}{q_2}} \|f\|_{q_2,p}. \tag{3.2}$$

4. New properties of the spaces $(L^q,\, L^p)(\mu)$ and $(L^q,\, L^p)^\alpha(\mu)$

For $(x,\, r) \in \mathbb{R} \times (0,\, \infty)$ and for $\delta > 0$, we set

$$I(x,\, r) = (x - r,\, x + r) \quad \text{and} \quad \delta I = I(x,\, \delta r).$$

We will examine the relationship between the Dunkl-Wiener amalgam spaces and other topological vector spaces.

Proposition 4.1. Let $1 \leq p \leq \infty$. Then

$$\|f\|_{p,\,p} = \mu(B_1)^{\frac{1}{p}} \|f\|_p$$

and consequently, $(L^p,\, L^p)(\mu) = L^p(\mu)$.

Proof. The case where $p=\infty$ is proved in [15]. So we suppose that $p<\infty$. We have

$$\begin{aligned}\|f\|_{p,\,p} &= \left[\int_{\mathbb{R}}\int_{\mathbb{R}}|f|^p(x)\tau_{-y}\chi_{B(0,1)}(x)d\mu(x)d\mu(y)\right]^{\frac{1}{p}}\\ &= \left[\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\tau_x\chi_{B(0,1)}(y)d\mu(y)\right]|f|^p(x)d\mu(x)\right]^{\frac{1}{p}}\\ &= (\mu(B_1))^{\frac{1}{p}}\left[\int_{\mathbb{R}}|f|^p(x)d\mu(x)\right]^{\frac{1}{p}}.\end{aligned}$$

□

Proposition 3.3 and Proposition 4.1 yield the following remark.

Remark 4.2. For $1\le p\le q\le\infty$, we have $(L^q,\,L^p)(\mu)\subset L^p(\mu)$.

Let us denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space. We endow it with the topology induced by the family of semi-norms given by

$$\mathcal{N}_m(f)=\sup_{x\in\mathbb{R}}(1+|x|)^m\sum_{n\le m}\left|\frac{d^{(n)}}{dx^n}f(x)\right|,\qquad f\in\mathcal{S}(\mathbb{R}),$$

where m and n are non-negative integers.

Proposition 4.3. Let $1\le q,p\le\infty$. Then $\mathcal{S}(\mathbb{R})$ is continuously embedded in $(L^q,L^p)(\mu)$.

Proof. Let m be a non-negative integer and $f\in\mathcal{S}(\mathbb{R})$. We have

$$|f(x)|\le(1+|x|)^{-m}\mathcal{N}_m(f),\quad x\in\mathbb{R}. \tag{4.1}$$

• Suppose that $q<\infty$.
Let $y\in\mathbb{R}$. It follows from (4.1) that

$$|f(x)|\tau_{-y}^{\frac{1}{q}}\chi_{B_1}(x)\le(1+|x|)^{-m}\mathcal{N}_m(f)\tau_{-y}^{\frac{1}{q}}\chi_{B_1}(x),\qquad x\in\mathbb{R}.$$

Therefore,

$$\|f\tau_{-y}^{\frac{1}{q}}\chi_{B_1}\|_q\le\mathcal{N}_m(f)\|(1+|\cdot|)^{-m}\tau_{-y}^{\frac{1}{q}}\chi_{B_1}\|_q. \tag{4.2}$$

Since

$$\frac{1}{4}(1+|x|)\le 1+|y|\le 4(1+|x|),\qquad y\in\mathbb{R},\ x\in B(y,1), \tag{4.3}$$

we get from (4.2) that

$$\begin{aligned}\|f\tau_{-y}^{\frac{1}{q}}\chi_{B_1}\|_q &\le 4^m\mathcal{N}_m(f)(1+|y|)^{-m}\|\tau_{-y}^{\frac{1}{q}}\chi_{B_1}\|_q\\ &\le 4^m\mathcal{N}_m(f)(1+|y|)^{-m}[\mu(B_1)]^{\frac{1}{q}}\end{aligned}$$

according to [1, Theorem 6.3.4]. Hence, for all $m>\frac{2k+2}{p}$,

$$\|f\|_{q,p}=\|\|f\tau_{(-\cdot)}^{\frac{1}{q}}\chi_{B_1}\|_q\|_p\le 4^m\mathcal{N}_m(f)[\mu(B_1)]^{\frac{1}{q}}\|(1+|\cdot|)^{-m}\|_p<\infty.$$

Therefore, for all $m>\frac{2k+2}{p}$,

$$\|f\|_{q,p}\le C(q,m,k,p)\mathcal{N}_m(f).$$

• Suppose that $q = \infty$.
Let $y \in \mathbb{R}$. From (4.1) we have

$$|f(x)\chi_{B(y,1)}(x)| \leq (1+|x|)^{-m}\mathcal{N}_m(f)\chi_{B(y,1)}(x), \quad x \in \mathbb{R}.$$

It follows from (4.3) that

$$\begin{array}{rcl} \|f\chi_{B(y,1)}\|_\infty & \leq & \mathcal{N}_m(f)\|(1+|\cdot|)^{-m}\chi_{B(y,1)}\|_\infty \\ & \leq & 4^m\mathcal{N}_m(f)(1+|y|)^{-m}\|\chi_{B(y,1)}\|_\infty \\ & = & 4^m\mathcal{N}_m(f)(1+|y|)^{-m}. \end{array}$$

Hence, for all $m > \frac{2k+2}{p}$,

$$\|f\|_{\infty,p} = \|\|f\chi_{B(y,1)}\|_\infty\|_p \leq 4^m\mathcal{N}_m(f)\|(1+|\cdot|)^{-m}\|_p < \infty.$$

Therefore, for all $m > \frac{2k+2}{p}$,

$$\|f\|_{q,\,p} \leq C(q,\,m,\,k,\,p)\mathcal{N}_m(f)$$

and the result follows. □

The following result asserts that for $1 \leq q < s \leq p < \infty$ we have

$$L^{s,\,\infty}(\mu) \subset (L^q,\,L^p)(\mu).$$

Proposition 4.4. Let $1 \leq q < s \leq p < \infty$. Then there exists a constant $C = C(p,\,q,\,s)$ such that

$$\|f\|_{q,\,p} \leq C\|f\|_{L^{s,\,\infty}(\mu)}, \qquad f \in L^0(\mu).$$

Proof. Let $f \in L^0(\mu)$ and $1 \leq q < s \leq p < \infty$.
The result is obvious if f is not an element of $L^{s,\infty}(\mu)$. So we suppose that $f \in L^{s,\infty}(\mu)$. We have

$$\frac{ps}{qs-qp+ps} > 1 \quad \text{and} \quad \frac{qs-pq+ps}{ps} + \frac{q}{s} = \frac{q}{p} + 1$$

so that according to Proposition 2.7,

$$\begin{array}{rcl} \|f\|_{q,p}^q & = & \||f|^q *_k \chi_{B(0,1)}\|_{L^{\frac{p}{q}}(\mu)} \\ & \leq & \dfrac{3p}{q}\mu(B(0,1))^{\frac{qs-qp+ps}{ps}}(\|f\|_{L^{s,\infty}(\mu)})^q. \end{array}$$

□

Proposition 4.5. Assume that $1 \leq r_1,\,r_2,\;s_1,\,s_2 \leq \infty$. Let $\theta \in (0,\,1)$ satisfying

$$\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} \quad \text{and} \quad \frac{1}{s} = \frac{\theta}{s_1} + \frac{1-\theta}{s_2}. \tag{4.4}$$

Then, for all $f \in L^0(\mu)$ we have

$$\|f\|_{s,r} \leq \|f\|_{s_1,r_1}^\theta\|f\|_{s_2,r_2}^{1-\theta}.$$

Proof. Let $1 \le r_1,\, r_2,\ s_1,\, s_2 \le \infty$ and $\theta \in (0,\,1)$ satisfying (4.4). Let $f \in L^0(\mu)$.
• Suppose that $s_1 < \infty$ and $s_2 < \infty$.
Let $y \in \mathbb{R}$. By Hölder inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}} |f|^s (x)\tau_{-y}\chi_{B_1}(x)d\mu(x)\\
=\ &\int_{\mathbb{R}} |f|^{s(1-\theta)+s\theta} (x)(\tau_{-y}\chi_{B_1})^{\frac{s\theta}{s_1}+\frac{s(1-\theta)}{s_2}}(x)d\mu(x)\\
\le\ &\left(\int_{\mathbb{R}} \left(|f|^{s\theta}(x)(\tau_{-y}\chi_{B_1})^{\frac{s\theta}{s_1}}(x)\right)^{\frac{s_1}{s\theta}} d\mu(x)\right)^{\frac{s\theta}{s_1}} \times\\
&\left(\int_{\mathbb{R}} \left(|f|^{s(1-\theta)}(x)(\tau_{-y}\chi_{B_1})^{\frac{s(1-\theta)}{s_2}}(x)\right)^{\frac{s_2}{s(1-\theta)}} d\mu(x)\right)^{\frac{s(1-\theta)}{s_2}}.
\end{aligned}$$

By applying again Hölder inequality, it follows that

$$\begin{aligned}
&\left\|\left(\int_{\mathbb{R}} |f|^s (x)\tau_{-(\cdot)}\chi_{B_1}(x)d\mu(x)\right)^{\frac{1}{s}}\right\|_r\\
\le\ &\left\|\left(\int_{\mathbb{R}} |f|^{s_1} (x)(\tau_{-(\cdot)}\chi_{B_1})(x)d\mu(x)\right)^{\frac{\theta}{s_1}}\right\|_{r_1}^{\theta} \times\\
&\left\|\left(\int_{\mathbb{R}} |f|^{s_2} (x)(\tau_{-(\cdot)}\chi_{B_1})(x)d\mu(x)\right)^{\frac{1-\theta}{s_2}}\right\|_{r_2}^{1-\theta}.
\end{aligned}$$

Hence,

$$\|f\|_{s,\,r} \le \|f\|_{s_1,\,r_1}^{\theta}\|f\|_{s_2,\,r_2}^{1-\theta}.$$

• Suppose that $s_1 = s_2 = s = \infty$.
If $r = \infty$ then $r_1 = r_2 = \infty$. In this case, the result is obvious. So we suppose that $r < \infty$.
It follows from Hölder inequality that

$$\begin{aligned}
\|f\|_{\infty,r}^r &= \int_{\mathbb{R}} \|f\chi_{B(y,1)}\|_{\infty}^{r\theta}\|f\chi_{B(y,1)}\|_{\infty}^{r(1-\theta)}d\mu(y)\\
&\le \|\|f\chi_{B(\cdot,1)}\|_{\infty}\|_{r_1}^{r\theta}\|\|f\chi_{B(\cdot,1)}\|_{\infty}\|_{r_2}^{r(1-\theta)}.
\end{aligned}$$

Hence,

$$\|f\|_{\infty,r} \le \|f\|_{\infty,r_1}^{\theta}\|f\|_{\infty,r_2}^{1-\theta}.$$

• Suppose that $s_1 = \infty$ and $s_2 < \infty$.
From Hölder inequality, we have

$$\begin{aligned}
\|f\|_{s,r} &= \left\| \left(\int_{\mathbb{R}} |f|^{s(1-\theta)+s\theta} (x) \chi_{B(\cdot,1)}(x) \tau_{-(\cdot)} \chi_{B_1}(x) d\mu(x) \right)^{\frac{1}{s}} \right\|_r \\
&\leq \left\| \|f\chi_{B(\cdot,1)}\|_\infty^\theta \left(\int_{\mathbb{R}} |f|^{s_2} (x) \tau_{-(\cdot)} \chi_{B_1}(x) d\mu(x) \right)^{\frac{1-\theta}{s_2}} \right\|_r \\
&\leq \| \|f\chi_{B(\cdot,1)}\|_\infty \|_{r_1}^\theta \left\| \left(\int_{\mathbb{R}} |f|^{s_2} (x) \tau_{-(\cdot)} \chi_{B_1}(x) d\mu(x) \right)^{\frac{1}{s_2}} \right\|_{r_2}^{1-\theta}.
\end{aligned}$$

Hence,

$$\|f\|_{s,r} \leq \|f\|_{\infty,r_1}^\theta \|f\|_{s_2,r_2}^{1-\theta}.$$

• Suppose that $s_1 < \infty$ and $s_2 = \infty$.
This case is similar to the previous one. □

Now, we focus on the existence of an equivalent norm to $\|\cdot\|_{q,\,p}$. For this purpose, we set

$$Q_l = [l,\, l+1), \qquad l \in \mathbb{Z}.$$

It is clear that $\{Q_l \,:\, l \in \mathbb{Z}\}$ is a disjoint family which covers $\mathbb{R}$.
For any $\delta > 0$ and for any integer l, we denote by δQ_l the half-open interval $[a_l,\, b_l)$, where $a_l = l + \frac{1-\delta}{2}$ and $b_l = l + \frac{1+\delta}{2}$.

Lemma 4.6. *Let l be an integer. The following assertions hold.*

(1) For all $y \in \mathbb{R}$, the set $L_y = \left\{ \; i \in \mathbb{Z} : Q_i \cap I(y,1) \neq \emptyset \; \right\}$ has at most three elements.
(2) For all $y \in Q_l$, $Q_l \subset I(y,1) \subset 3Q_l$.
(3) μ is doubling with respect to Q_l.

Proof. We give the proof of the last assertion because the others are trivial.
Let l be an integer. Put $u = l + \frac{1}{2}$. It is proved in [15] that for any $(x,\, r) \in \mathbb{R} \times (0,\, \infty)$, μ is doubling for $I(x,\, r)$. Then there exists a constant $C > 0$ such that

$$\mu(2I(u,\, \frac{1}{2})) \leq C\mu(I(u,\, \frac{1}{2})).$$

As

$$\mu(2Q_l) = \mu(2I(u,\, \frac{1}{2})) \quad \text{and} \quad \mu(I(u,\, \frac{1}{2})) = \mu(Q_l),$$

we deduce that

$$\mu(2Q_l) \leq C\mu(Q_l).$$

This ends the proof. □

For $1 \leq p,\, q \leq \infty$ and for any μ-measurable function f on $\mathbb{R}$, set

$$\widetilde{\|f\|}_{q,p} = \begin{cases} \left(\sum\limits_{l\in\mathbb{Z}} \mu(Q_l) \, \|f\chi_{Q_l}\|_q^p \right)^{\frac{1}{p}} & \text{if} \quad p < \infty, \\ \sup\limits_{l\in\mathbb{Z}} \|f\chi_{Q_l}\|_q & \text{if} \quad p = \infty. \end{cases}$$

Proposition 4.7. Let $1 \leq p,\, q \leq \infty$. Then, there are positive constants C_1 and C_2 such that

$$C_1 \widetilde{\|f\|}_{q,\,p} \leq \|f\|_{q,\,p} \leq C_2 \widetilde{\|f\|}_{q,\,p}, \quad f \in L^0(\mu).$$

Proof. Let f be any μ-measurable function on $\mathbb{R}$.
1^{rst} case. We suppose that $q < \infty$.
According to Proposition 2.5, there exists a constant $C > 0$ not depending on f such that

$$\int_{\mathbb{R}} \tau_y |f|^q(x) \chi_{B(0,1)}(x) d\mu(x) \leq C \int_{\mathbb{R}} |f|^q(x) \chi_{I(y,1)}(x) d\mu(x).$$

• Suppose $p < \infty$. Then

$$\begin{aligned} \|f\|_{q,\,p} &\leq C(q) \left[\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f|^q(x) \chi_{I(y,1)}(x) d\mu(x) \right]^{\frac{p}{q}} d\mu(y) \right]^{\frac{1}{p}} \\ &\leq C(q) \left[\sum_{l \in \mathbb{Z}} \int_{Q_l} \left[\sum_{i \in L_y} \int_{Q_i} |f|^q(x) \chi_{I(x,1)}(y) d\mu(x) \right]^{\frac{p}{q}} d\mu(y) \right]^{\frac{1}{p}}. \end{aligned}$$

By Lemma 4.6, we get

$$\begin{aligned} \|f\|_{q,p} &\leq C(q,\,p) \left[\sum_{i \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \int_{Q_l \cap 3Q_i} \|f\chi_{Q_i}\|_q^p \, d\mu(y) \right]^{\frac{1}{p}} \\ &\leq C(q,p) \left[\sum_{i \in \mathbb{Z}} \mu(Q_i) \, \|f\chi_{Q_i}\|_q^p \right]^{\frac{1}{p}} = C(q,p) \widetilde{\|f\|}_{q,\,p}. \end{aligned}$$

Applying again Lemma 4.6 and Proposition 2.5 we have

$$\begin{aligned} \widetilde{\|f\|}_{q,p} &= \left[\sum_{i \in \mathbb{Z}} \int_{Q_i} \|f\chi_{Q_i}\|_q^p \, d\mu(y) \right]^{\frac{1}{p}} \\ &\leq \left[\sum_{i \in \mathbb{Z}} \int_{Q_i} \left\| f\chi_{I(y,1)} \right\|_q^p d\mu(y) \right]^{\frac{1}{p}} = \left[\int_{\mathbb{R}} \left\| f\chi_{I(y,1)} \right\|_q^p d\mu(y) \right]^{\frac{1}{p}} \\ &\leq C(q) \left[\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \tau_y |f|^q(x) \chi_{B(0,1)}(x) d\mu(x) \right]^{\frac{p}{q}} d\mu(y) \right]^{\frac{1}{p}}. \end{aligned}$$

Hence,

$$\widetilde{\|f\|}_{q,p} \leq C(q) \, \|f\|_{q,p} \, .$$

• Suppose $p=\infty$. On the one hand, according to Proposition 2.5, there exists a constant $C>0$ not depending on f such that

$$\begin{aligned}\int_{B(0,1)}\tau_y|f|^q(x)d\mu(x) &\leq C\int_{I(y,1)}|f|^q(x)d\mu(x)\\ &\leq C\sum_{l\in\mathbb{Z}:Q_l\cap I(y,1)\neq\emptyset}\int_{Q_l}|f|^q(x)\chi_{I(y,1)}(x)d\mu(x)\\ &\leq C\sup_{l\in\mathbb{Z}}\|f\chi_{Q_l}\|_q^q\sum_{l\in\mathbb{Z}:Q_l\cap I(y,1)\neq\emptyset}1.\end{aligned}$$

Hence, $\quad \|f\|_{q,\infty}\leq C(q)\widetilde{\|f\|}_{q,\infty}$.
On the other hand, by applying again Lemma 4.6, we have

$$\|f\chi_{Q_l}\|_q\leq\|f\chi_{I(y,1)}\|_q,\qquad y\in Q_l.$$

Moreover, there exists a constant $C>0$ not depending on f such that

$$\int_{I(y,1)}|f|^q(x)d\mu(x)\leq C\int_{B(0,1)}\tau_y|f|^q(x)d\mu(x).$$

Hence, $\quad \widetilde{\|f\|}_{q,\infty}\leq C(q)\left\|f\right\|_{q,\infty}$.
2^{nd} case. We suppose that $q=\infty$.
• Suppose $p<\infty$. Thanks to Lemma 6.2 in [15] we have

$$\|f\|_{\infty,p}\quad\leq\quad \left\|\left\|f\chi_{I(\cdot,3)}\right\|_\infty\right\|_p+\left\|\left\|f\chi_{I(-\cdot,3)}\right\|_\infty\right\|_p=J_1+J_2.$$

Notice that

$$|f(x)\chi_{I(y,3)}(x)|\leq\sum_{l\in\mathbb{Z}}\left|f(x)\chi_{I(y,3)}(x)\chi_{Q_l}(x)\right|,\qquad x\in\mathbb{R}.$$

Since for every integer l,

$$I(x,3)\subset 7Q_l,\qquad x\in Q_l,$$

we get

$$\|f\chi_{I(y,3)}\|_\infty\leq\sum_{l\in\mathbb{Z}:Q_l\cap I(y,3)\neq\emptyset}\left\|f\chi_{Q_l}\right\|_\infty\chi_{7Q_l}(y).$$

Thus,

$$J_1\leq\left[\int_{\mathbb{R}}\left(\sum_{l\in\mathbb{Z}:Q_l\cap I(y,3)\neq\emptyset}\left\|f\chi_{Q_l}\right\|_\infty\chi_{7Q_l}(y)\right)^p d\mu(y)\right]^{\frac{1}{p}}.$$

As for all $y \in \mathbb{R}$ the set $\left\{ \ l \in \mathbb{Z} : Q_l \cap I(y,3) \neq \emptyset \ \right\}$ has at most seven elements, then by applying Hölder inequality we have

$$\begin{aligned} J_1 &\leq 7^{1-\frac{1}{p}} \left[\int_{\mathbb{R}} \sum_{l \in \mathbb{Z}} \|f\chi_{Q_l}\|_\infty^p \chi_{7Q_l}(y) d\mu(y) \right]^{\frac{1}{p}} \\ &\leq 7^{1-\frac{1}{p}} \left[\sum_{i \in \mathbb{Z}} \int_{Q_i} \sum_{l \in \mathbb{Z}} \|f\chi_{Q_l}\|_\infty^p \chi_{7Q_l}(y) d\mu(y) \right]^{\frac{1}{p}} \\ &\leq 7^{1-\frac{1}{p}} \left[\sum_{l \in \mathbb{Z}} \mu(7Q_l) \|f\chi_{Q_l}\|_\infty^p \right]^{\frac{1}{p}} . \end{aligned}$$

Since μ is doubling, we get $J_1 \leq C(p)\widetilde{\|f\|}_{\infty,p}$.
Also, $J_2 \leq C(p)\widetilde{\|f\|}_{\infty,p}$ because μ is a symmetric invariant measure. Thus the result follows.
For the reverse inequality, we apply Lemma 4.6 to get

$$\sum_{l \in \mathbb{Z}} \int_{Q_l} \|f\chi_{Q_l}\|_\infty^p d\mu(y) \leq \sum_{l \in \mathbb{Z}} \int_{Q_l} \left\|f\chi_{I(y,1)}\right\|_\infty^p d\mu(y).$$

Then the previous inequality becomes

$$\begin{aligned} & \sum_{l \in \mathbb{Z}} \int_{Q_l} \|f\chi_{Q_l}\|_\infty^p d\mu(y) \\ \leq & \int_{B(0,1)} \left\|f\chi_{I(y,1)}\right\|_\infty^p d\mu(y) + \int_{B(0,1)^c} \left\|f\chi_{I(y,1)}\right\|_\infty^p d\mu(y). \end{aligned}$$

Since $I(y,1) \subset B(y,1)$ for $|y| \geq 1$, we deduce that

$$\begin{aligned} \int_{B(0,1)^c} \left\|f\chi_{I(y,1)}\right\|_\infty^p d\mu(y) &\leq \int_{B(0,1)^c} \left\|f\chi_{B(y,1)}\right\|_\infty^p d\mu(y) \\ &\leq \int_{\mathbb{R}} \left\|f\chi_{B(y,1)}\right\|_\infty^p d\mu(y) = \|f\|_{\infty,p}^p . \end{aligned}$$

As $I(y,1) \setminus \{0\} \subset B(y,1)$ when $|y| < 1$, then

$$\begin{aligned} \int_{B(0,1)} \left\|f\chi_{I(y,1)}\right\|_\infty^p d\mu(y) &= \int_{B(0,1)} \left\|f\chi_{I(y,1)\setminus\{0\}}\right\|_\infty^p d\mu(y) \\ &\leq \int_{\mathbb{R}} \left\|f\chi_{B(y,1)}\right\|_\infty^p d\mu(y) = \|f\|_{\infty,p}^p . \end{aligned}$$

Hence, $\quad \widetilde{\|f\|}_{\infty,p} \leq 2^{\frac{1}{p}} \|f\|_{\infty,p}$.
• Suppose $p = \infty$.
By Proposition 4.1, $\|f\|_{\infty,\infty} = \|f\|_\infty$. To get the desired result, we will show that $\widetilde{\|f\|}_{\infty,\infty} = \|f\|_\infty$.
Since

$$\|f\chi_{Q_l}\|_\infty \leq \|f\|_\infty , \qquad l \in \mathbb{Z},$$

it follows that $\widetilde{\|f\|}_{\infty,\infty} \leq \|f\|_{\infty}$.
For the converse, we assume that $\|f\|_{\infty} > 0$, otherwise the result is obvious.
Let r be a real number satisfying $0 < r < \|f\|_{\infty}$. Then

$$0 < \mu\left(\{x \in \mathbb{R} : |f(x)| > r\}\right) = \mu\left(\bigcup_{l \in \mathbb{Z}} \{x \in Q_l \ : \ |f(x)| > r\}\right).$$

Thus, there exists an integer l satisfying $0 < \mu\left(\{x \in Q_l \ : \ |f(x)| > r\}\right)$. Hence,

$$r < \|f\chi_{Q_l}\|_{\infty} \leq \widetilde{\|f\|}_{\infty,\infty}.$$

Therefore, $\|f\|_{\infty} \leq \widetilde{\|f\|}_{\infty,\infty}$ and the claim follows. □

It is easy to see that the map $f \mapsto \widetilde{\|f\|}_{q,p}$ defines a norm on $(L^p, L^q)(\mu)$.

Lemma 4.8. *There is a constant $C > 0$ such that for all integers l we have*

$$\mu(I(0,1)) \leq C\mu(Q_l).$$

Proof. It is proved in [15] that

$$\mu(I(0,1)) \leq 2\mu(I(x,1)), \quad x \in \mathbb{R}.$$

Thus, for $x \in Q_l$, we have by Lemma 4.6 that

$$\mu(I(0,1)) \leq 2\mu(I(x,1)) \leq 2\mu(3Q_l).$$

The desired result follows from the fact that μ is doubling with respect to Q_l. □

Proposition 4.9. Let $1 \leq q, p_1, p_2 \leq \infty$ and $p_2 \geq p_1$. Then there exists a constant $C > 0$ such that

$$\|f\|_{q,p_2} \leq C \ \|f\|_{q,p_1}, \qquad f \in L^0(\mu).$$

Proof. Let $f \in L^0(\mu)$. If $p_1 = p_2$ then the result is obvious.
• We suppose that $p_1 < p_2 < \infty$. We have

$$\begin{aligned} \widetilde{\|f\|}_{q,p_2} &= \left[\sum_{i\in\mathbb{Z}} \left[\mu(Q_i)^{\frac{1}{p_2}} \|f\chi_{Q_i}\|_q\right]^{p_2}\right]^{\frac{1}{p_2}} \\ &\leq \left[\sum_{i\in\mathbb{Z}} \mu(Q_i)^{\frac{p_1}{p_2}} \|f\chi_{Q_i}\|_q^{p_1}\right]^{\frac{1}{p_1}} \\ &= \left[\sum_{i\in\mathbb{Z}} \mu(Q_i)^{\frac{p_1}{p_2}-1} \mu(Q_i) \|f\chi_{Q_i}\|_q^{p_1}\right]^{\frac{1}{p_1}}. \end{aligned}$$

An application of Lemma 4.8 and Proposition 4.7 leads to the desired result.
• Suppose $p_1 < p_2 = \infty$. For all $l \in \mathbb{Z}$,

$$\|f\chi_{Q_l}\|_q^{p_1} \leq \sum_{l\in\mathbb{Z}} 1 \, \|f\chi_{Q_l}\|_q^{p_1} \leq C\mu(I(0,1))^{-1} \sum_{l\in\mathbb{Z}} \mu(Q_l) \, \|f\chi_{Q_l}\|_q^{p_1}$$

according to Lemma 4.8. The result follows by Proposition 4.7. □

Proposition 4.7 and Proposition 4.9 lead to the following remark.

Remark 4.10. For $1 \leq q,\, p_1,\, p_2,\, \leq \infty$ and $p_2 \geq p_1$ we have

$$(L^q,\, L^{p_1})(\mu) \subset (L^q,\, L^{p_2})(\mu).$$

If in addition $p_2 = q$, then by Proposition 4.1,

$$(L^q,\, L^{p_1})(\mu) \subset L^q(\mu).$$

For $1 \leq q < \infty$, we denote by $L^q_c(\mu)$ the set of elements of $L^q(\mu)$ with compact support. For any element f of $L^0(\mu)$, we will denote by $supp(f)$ its support.
The following result shows that $L^q_c(\mu) \subset (L^q, L^p)(\mu)$ when $1 \leq q, p < \infty$.

Lemma 4.11. *Let $1 \leq q,\, p < \infty$. If $f \in L^q_c(\mu)$, then there is a constant $C > 0$ such that*

$$\widetilde{\|f\|}_{q,p} \leq C \|f\|_q \,.$$

Proof. Let $f \in L^q_c(\mu)$. Let us set $K = supp(f)$. We have

$$\begin{aligned}
\widetilde{\|f\|}_{q,p} &= \left[\sum_{l\in\mathbb{Z}:Q_l\cap K\neq\emptyset} \mu(Q_l)\, \|f\chi_{Q_l\cap K}\|_q^p\right]^{\frac{1}{p}} \\
&\leq \|f\chi_K\|_q \left[\sum_{l\in\mathbb{Z}:Q_l\cap K\neq\emptyset} \mu(Q_l)\right]^{\frac{1}{p}} = C\|f\|_q,
\end{aligned}$$

since the set $\{l \in \mathbb{Z} : Q_l \cap K \neq \emptyset\}$ has a finite number of elements. $\square$

Proposition 4.12. Let $1 \leq q,\, p < \infty$. Then $L^q_c(\mu)$ is a dense subspace of $(L^q, L^p)(\mu)$.

Proof. Let $1 \leq q,\, p < \infty$ and $f \in (L^q,\, L^p)(\mu)$. Let n be a positive integer. We consider the function f_n defined by

$$f_n(x) = f(x)\max(1 - |\frac{x}{n}|, 0), \quad x \in \mathbb{R}.$$

a) We have $suppf_n \subset B(0, n)$.
We will show that f_n belongs to $L^q(\mu)$ by examining two cases.
• Suppose that $p \leq q$.
We have

$$\begin{aligned}
\widetilde{\|f_n\|}_{q,\,q} &\leq \left[\sum_{l\in\mathbb{Z}} \left[(\mu(Q_l))^{\frac{1}{q}} \|f\chi_{Q_l}\|_q\right]^q\right]^{\frac{1}{q}} \leq \left[\sum_{l\in\mathbb{Z}} (\mu(Q_l))^{\frac{p}{q}} \|f\chi_{Q_l}\|_q^p\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{l\in\mathbb{Z}} (\mu(Q_l))^{\frac{p}{q}-1} \mu(Q_l)\, \|f\chi_{Q_l}\|_q^p\right]^{\frac{1}{p}}.
\end{aligned}$$

It follows that if $p = q$ then $\widetilde{\|f_n\|}_{q,\,q} \leq \widetilde{\|f\|}_{q,\,p}$. If $p < q$ then an application of Lemma 4.8 ensures the existence of a constant $C > 0$ such that $\widetilde{\|f_n\|}_{q,\,q} \leq C\widetilde{\|f\|}_{q,\,p}$.
We deduce from Proposition 4.1 that $f \in L^q(\mu)$.

• Suppose that $q < p$.
We have

$$\begin{aligned}\widetilde{\|f_n\|}_{q,\,q}^{q} &\leq \sum_{l\in\mathbb{Z}\,:\,Q_l\cap B(0,n)\neq\emptyset} \mu(Q_l)\int_{B(0,n)} |f(x)\chi_{Q_l}(x)|^q\, d\mu(x)\\ &\leq \sum_{l\in\mathbb{Z}\,:\,Q_l\cap B(0,n)\neq\emptyset} (\mu(Q_l))^{\frac{1}{s}+\frac{1}{s'}}\int_{B(0,n)} |f(x)\chi_{Q_l}(x)|^q\, d\mu(x),\end{aligned}$$

where $s = \frac{p}{q}$ and $s' = \frac{p}{p-q}$. Then, it follows from Hölder inequality that

$$\widetilde{\|f_n\|}_{q,\,q}^{q} \leq \left[\sum_{l\in\mathbb{Z}:Q_l\cap B(0,n)\neq\emptyset} \mu(Q_l)\right]^{\frac{1}{s'}} \left[\sum_{l\in\mathbb{Z}} \mu(Q_l)\,\|f\chi_{Q_l}\|_q^p\right]^{\frac{q}{p}}.$$

Since the set $\{l \in \mathbb{Z} : Q_l \cap B(0,n) \neq \emptyset\}$ has $2n$ elements, we have that

$$D_n = \left[\sum_{l\in\mathbb{Z}:Q_l\cap B(0,n)\neq\emptyset} \mu(Q_l)\right]^{\frac{1}{s'q}}$$

is a positive real number satisfying $\widetilde{\|f_n\|}_{q,q} \leq D_n\widetilde{\|f\|}_{q,\,p}$. It follows again from Proposition 4.1 that $f_n \in L^q(\mu)$.
b) We have,

$$\begin{aligned}\|f_n - f\|_{q,p} &= \left\|\left\|(f_n - f)\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q\right\|_p\\ &= \left\|\left\|f_n\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)} - f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q\right\|_p,\end{aligned}$$

where $\left\|\left\|(f_n - f)\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q\right\|_p$ denotes the $L^p(\mu)$-norm of the function

$$y \mapsto \left\|\left\|(f_n - f)\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q\right\|_p.$$

Let $y \in \mathbb{R}$. Then $(f_n\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)})_{n\geq 1}$ is a μ-measurable sequence of functions which converges to $f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}$ μ-almost everywhere. In addition, for μ-almost every $y \in \mathbb{R}$,

$$\left|f_n\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right| \leq \left|f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right| \in L^q(\mu)$$

because $f \in (L^q, L^p)(\mu)$. Hence, applying the dominated convergence theorem we obtain

$$\lim_{n\to\infty}\left\|f_n\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)} - f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q = 0.$$

Moreover,

$$\left\|f_n\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)} - f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q \leq 2\left\|f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)}\right\|_q$$

and the function $y \mapsto \left\| f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)} \right\|_q$ belongs to $L^p(\mu)$ since $f \in (L^q, L^p)(\mu)$. Hence, applying again the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \left\| \left\| f_n\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)} - f\tau_{-y}^{\frac{1}{q}}\chi_{B(0,1)} \right\|_q \right\|_p = 0.$$

From parts a) and b), we conclude that $L_c^q(\mu)$ is a dense subspace of $(L^q,\, L^p)(\mu)$. $\square$

Let $1 \leq q, p \leq \infty$. For all $r > 0$, we define

$$(L^q, L^p)_r(\mu) = \left\{ \; f \in L^0(\mu) : \quad {}_r\|f\|_{q,p} < \infty \; \right\}$$

and for any μ-measurable function f on $\mathbb{R}$, we set

$$\widetilde{{}_r\|f\|_{q,p}} = \begin{cases} \left(\sum\limits_{l\in\mathbb{Z}} \mu(Q_l^r) \left\| f\chi_{Q_l^r} \right\|_q^p \right)^{\frac{1}{p}} & \text{if} \quad p < \infty, \\ \sup\limits_{l\in\mathbb{Z}} \left\| f\chi_{Q_l^r} \right\|_q & \text{if} \quad p = \infty, \end{cases}$$

where $Q_l^r = [rl,\, rl + r)$.

It is easy to see that an adaptation of the proofs of the propositions stated in [15, Section 3] and those given in this section (by replacing $\mu(B_1)$ and Q_l by $\mu(B_r)$ and Q_l^r respectively) yields the following result.

Proposition 4.13. Let $1 \leq q, p \leq \infty$ and $r > 0$. Then the following assertions hold.

(1) $(L^q, L^p)_r(\mu)$ is a complex subspace of the space $L^0(\mu)$.
(2) The maps $f \mapsto {}_r\|f\|_{q,p}$ and $f \mapsto \widetilde{{}_r\|f\|_{q,p}}$ define norms on $(L^q, L^p)_r(\mu)$.
(3) The norms ${}_r\|f\|_{q,p}$ and $\widetilde{{}_r\|f\|_{q,p}}$ are equivalent.
(4) $((L^q, L^p)_r(\mu),\, {}_r\|f\|_{q,p})$ is a complex Banach space.
(5) If $1 \leq p_1 \leq p_2 \leq \infty$ then there exists a constant $C > 0$ not depending on r such that

$${}_r\left\|f\right\|_{q,p_2} \leq C\mu(B(0,r))^{\frac{1}{p_2}-\frac{1}{p_1}}\; {}_r\left\|f\right\|_{q,p_1}, \qquad f \in L^0(\mu).$$

Now, we state some new properties of the spaces $(L^q, L^p)^\alpha(\mu)$.

Using the same arguments as in the proof of Proposition 3.1 with B_r instead of B_1, it is easy to see that Hölder inequality remains valid in the Dunkl-Fofana spaces. More precisely, we have the following

Proposition 4.14. Let (q_1, p_1, α_1) and (q_2, p_2, α_2) two elements of $[1, +\infty]^3$ such that $q_1 \leq \alpha_1 \leq p_1$ and $q_2 \leq \alpha_2 \leq p_2$.
If

$$\tfrac{1}{q_1} + \tfrac{1}{q_2} = \tfrac{1}{q} \leq 1, \quad \tfrac{1}{p_1} + \tfrac{1}{p_2} = \tfrac{1}{p} \leq 1 \text{ and } \tfrac{1}{\alpha_1} + \tfrac{1}{\alpha_2} = \tfrac{1}{\alpha},$$

then for all f and g of $L^0(\mu)$, we have

$$\|fg\|_{q,p,\alpha} \leq \|f\|_{q_1,p_1,\alpha_1}\|g\|_{q_2,p_2,\alpha_2}.$$

From Proposition 4.4 we deduce the following result.

Proposition 4.15. Let $1 \leq q < s \leq p < \infty$. Then

$$\|f\|_{q,p,\alpha} \leq C\|f\|_{L^{\alpha,\infty}(\mu)}, \qquad f \in L^0(\mu),$$

where $C > 0$ is a constant not depending on f.

The above result shows that $L^{\alpha,\infty}(\mu)$ is continuously embedded in $(L^q, L^p)^\alpha(\mu)$. The following proposition asserts that endowed with the norm $\|\cdot\|_{q,p,\alpha}$, $(L^q, L^p)^\alpha(\mu)$ is a complex Banach space.

Proposition 4.16. Let $1 \leq \alpha,\, p,\, q \leq \infty$ with $q \leq \alpha \leq p$. Then $((L^q, L^p)^\alpha(\mu),\, \|\cdot\|_{q,p,\alpha})$ is a complex Banach space.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence of elements of $(L^q, L^p)^\alpha(\mu)$.
Fix $\epsilon > 0$. Then there exists an integer N such that for all integers m and n we have

$$m,\, n > N \implies \|f_m - f_n\|_{q,p,\alpha} < \epsilon.$$

This implies that for r fixed and for $m,\, n > N$,

$$\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f_m - f_n\|_{q,p} < \epsilon.$$

Hence, $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence of $(L^q, L^p)_r(\mu)$. Since $(L^q, L^p)_r(\mu)$ is a complex Banach space, $(f_j)_{j\in\mathbb{N}}$ converges to $f \in (L^q, L^p)_r(\mu)$. In addition, for all $m, n > N$ we have

$$|\|f_m\|_{q,p,\alpha} - \|f_n\|_{q,p,\alpha}| \leq \|f_m - f_n\|_{q,p,\alpha} < \epsilon.$$

Therefore, $(\|f_j\|_{q,p,\alpha})_{j\in\mathbb{N}}$ is a Cauchy sequence on $\mathbb{R}$. Let us note $M \geq 0$ its limit. It follows that for $m > N$,

$$\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f\|_{q,p} \quad \leq \quad \|f_m\|_{q,p,\alpha} + \mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f - f_m\|_{q,p}.$$

By letting m go to ∞, we get

$$\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f\|_{q,p} \leq M.$$

Hence, $\|f\|_{q,p,\alpha} \leq M$ and $f \in (L^q, L^p)^\alpha(\mu)$.
Let m be an integer such that $m > N$. For all $r > 0$ and $l \in \mathbb{N}$, we have

$$\begin{aligned} & \mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f_m - f\|_{q,p} \\ \leq\; & \|f_m - f_{m+l}\|_{q,p,\alpha} + \mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f_{m+l} - f\|_{q,p} \\ \leq\; & \epsilon + \mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f_{m+l} - f\|_{q,p}. \end{aligned}$$

Since

$$\lim_{l\to\infty} \mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\,{}_r\|f_{m+l} - f\|_{q,p} = 0,$$

we have

$$\|f_m - f\|_{q,p,\alpha} \leq \epsilon.$$

This ends the proof. □

The family of spaces $(L^q, L^p)^\alpha(\mu)$ is increasing with respect to the p power as stated below.

Proposition 4.17. Let $1 \leq q \leq \alpha \leq p_1 \leq p_2 \leq \infty$. There exists a constant $C > 0$ such that

$$\|f\|_{q,p_2,\alpha} \leq C\|f\|_{q,p_1,\alpha}, \qquad f \in L^0(\mu),$$

and consequently, $(L^q, L^{p_1})^\alpha(\mu) \subset (L^q, L^{p_2})^\alpha(\mu)$.

Proof. Let $f \in L^0(\mu)$. Since for $p_1 = p_2$ the result is obvious, we give the proof for $p_1 < p_2$.
Let $r > 0$.
• Suppose that $p_2 < \infty$.
According to Proposition 4.13, there exists a constant $C > 0$, not depending on r and f, such that

$$_r\|f\|_{q,p_2} \leq C\mu(B(0,r))^{\frac{1}{p_2}-\frac{1}{p_1}}\ _r\|f\|_{q,p_1}.$$

This implies that

$$\mu(B(0,r))^{-\frac{1}{p_2}}\ _r\|f\|_{q,p_2} \leq C\mu(B(0,r))^{-\frac{1}{p_1}}\ _r\|f\|_{q,p_1}.$$

Multiplying the both sides of the above inequality by $(\mu_k(I(0,r)))^{\frac{1}{\alpha}-\frac{1}{q}}$, we get

$$\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p_2}}\ _r\|f\|_{q,p_2} \leq C\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p_1}}\ _r\|f\|_{q,p_1}.$$

Hence $\quad \|f\|_{q,p_2,\alpha} \leq C\|f\|_{q,p_1,\alpha}$.
• Suppose that $p_2 = \infty$.
By Proposition 4.13, there exists a constant $C > 0$, not depending on r and f, such that

$$_r\|f\|_{q,\infty} \leq C(\mu(B(0,r)))^{-\frac{1}{p_1}}\ _r\|f\|_{q,p_1}.$$

This implies that

$$\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}}\ _r\|f\|_{q,\infty} \leq C\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p_1}}\ _r\|f\|_{q,p_1}.$$

Hence $\quad \|f\|_{q,\infty,\alpha} \leq C\|f\|_{q,p_1,\alpha}$.
We conclude that $(L^q, L^{p_1})^\alpha(\mu) \subset (L^q, L^{p_2})^\alpha(\mu)$. □

5. Proof of the main results

For $1 < q < \infty$, we will denote by q' its conjugate exponent: $\frac{1}{q} + \frac{1}{q'} = 1$. We will need the following results.

Proposition 5.1. ([15]) Let $1 < q \leq \alpha \leq p \leq \infty$. There exists a constant $C > 0$ such that

$$\|Mf\|_{q,p,\alpha} \leq C\|f\|_{q,p,\alpha}, \quad f \in L^1_{loc}(\mu).$$

Proposition 5.2. ([15]) Let $1 \leq \alpha \leq p \leq \infty$. There exists $C > 0$ such that

$$\|Mf\|_{(L^{1,\infty},L^p)^\alpha(\mu)} \leq C\|f\|_{1,p,\alpha}, \quad f \in L^1_{loc}(\mu).$$

Proof of Theorem 1.1. Let $1 < q \leq \alpha \leq p < \infty$, $0 < \beta < \frac{2k+2}{\alpha}$ and $f \in (L^q, L^p)^\alpha(\mu)$. We assume that $f \neq 0$ μ-almost everywhere, otherwise the result is obvious.
Let $r > 0$ and $x \in \mathbb{R}$. Then

$$\begin{aligned} I_\beta f(x) &= \int_{B(0,r)} \tau_x f(z)|z|^{\beta-2k-2} d\mu(z) + \\ &\quad \int_{B(0,r)^c} \tau_x f(z)|z|^{\beta-2k-2} d\mu(z) = A(x) + B(x). \end{aligned}$$

Following the ideas of part (1) of the proof of Theorem 4 in [8], we have

$$\begin{aligned}
|A(x)| &= \left|\sum_{i=0}^{\infty}\int_{B(0,2^{-i}r)\backslash B(0,2^{-i-1}r)}\tau_x f(z)|z|^{\beta-2k-2}d\mu(z)\right| \\
&= \left|\sum_{i=0}^{\infty}\int_{\mathbb{R}} f(z)\tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{-i}r)\backslash B(0,2^{-i-1}r)})(z)d\mu(z)\right| \\
&\leq \sum_{i=0}^{\infty}\int_{\mathbb{R}} |f(z)|\tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{-i}r)\backslash B(0,2^{-i-1}r)})(z)d\mu(z) \\
&\quad \text{(according to [1, Theorem 6.3.4]).}
\end{aligned}$$

It follows that

$$\begin{aligned}
|A(x)| &\leq \sum_{i=0}^{\infty}\int_{B(0,2^{-i}r)\backslash B(0,2^{-i-1}r)}\tau_x|f(z)||z|^{\beta-2k-2}d\mu(z) \\
&\leq \sum_{i=0}^{\infty}(2^{-i-1}r)^{\beta-2k-2}d_k(2^{-i}r)^{2k+2}Mf(x) \\
&\leq d_k 2^{2k+2-\beta}r^{\beta}Mf(x)\sum_{i=0}^{\infty}(2^{-1})^{\beta}.
\end{aligned}$$

Hence,

$$|A(x)| \leq Cr^{\beta}Mf(x). \tag{5.1}$$

On the other hand, we have

$$\begin{aligned}
&|B(x)| \\
&= \left|\sum_{i=0}^{\infty}\int_{B(0,2^{i+1}r)\backslash B(0,2^{i}r)}\tau_x f(z)|z|^{\beta-2k-2}d\mu(z)\right| \\
&= \left|\sum_{i=0}^{\infty}\int_{\mathbb{R}} f(z)\tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{i+1}r)\backslash B(0,2^{i}r)})(z)d\mu(z)\right|.
\end{aligned}$$

By Hölder inequality we have

$$\begin{aligned}
&\left|\sum_{i=0}^{\infty}\int_{\mathbb{R}} f(z)\tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{i+1}r)\backslash B(0,2^{i}r)})(z)d\mu(z)\right| \\
\leq\ & \sum_{i=0}^{\infty}\int_{\mathbb{R}} |f(z)|\tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{i+1}r)\backslash B(0,2^{i}r)})^{\frac{1}{q}+\frac{1}{q'}}(z)d\mu(z) \\
\leq\ & \sum_{i=0}^{\infty}\left(\int_{\mathbb{R}} |f(z)|^q\tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{i+1}r)\backslash B(0,2^{i}r)})(z)d\mu(z)\right)^{\frac{1}{q}} \\
\times\ & \left(\int_{\mathbb{R}} \tau_{-x}(|\cdot|^{\beta-2k-2}\chi_{B(0,2^{i+1}r)\backslash B(0,2^{i}r)})(z)d\mu(z)\right)^{\frac{1}{q'}}.
\end{aligned}$$

So, thanks to [1, Theorem 6.3.4], we write that

$$\begin{aligned} |B(x)| &\leq \sum_{i=0}^{\infty}\left(\int_{2^i r\leq|z|<2^{i+1}r}\tau_x|f(z)|^q|z|^{\beta-2k-2}d\mu(z)\right)^{\frac{1}{q}}\times \\ &\quad \left(\int_{2^i r\leq|z|<2^{i+1}r}|z|^{\beta-2k-2}d\mu(z)\right)^{\frac{1}{q'}} \\ &\leq \sum_{i=0}^{\infty}(2^i r)^{\frac{\beta-2k-2}{q}}\left(\int_{2^i r\leq|z|<2^{i+1}r}\tau_x|f(z)|^q d\mu(z)\right)^{\frac{1}{q}}\times \\ &\quad \left((2^i r)^{\beta-2k-2}\mu(B(0,2^{i+1}r))\right)^{\frac{1}{q'}}. \end{aligned}$$

As

$$\left(\int_{2^i r\leq|z|<2^{i+1}r}\tau_x|f(z)|^q d\mu(z)\right)^{\frac{1}{q}} \leq \left\|\left(\int_{B(0,2^{i+1}r)}\tau_x|f(z)|^q d\mu(z)\right)^{\frac{1}{q}}\right\|_{\infty},$$

it follows that

$$\begin{aligned} |B(x)| &\leq \sum_{i=0}^{\infty}(2^i r)^{\beta-2k-2}\mu(B(0,2^{i+1}r))^{\frac{1}{q'}}\mu(B(0,2^{i+1}r))^{-\frac{1}{\alpha}+\frac{1}{q}} \\ &\times \mu(B(0,2^{i+1}r))^{\frac{1}{\alpha}-\frac{1}{q}}\left\|\left(\int_{B(0,2^{i+1}r)}\tau_x|f(z)|^q d\mu(z)\right)^{\frac{1}{q}}\right\|_{\infty} \\ &\leq \|f\|_{q,\infty,\alpha}\sum_{i=0}^{\infty}(2^i r)^{\beta-2k-2}\mu(B(0,2^{i+1}r))^{1-\frac{1}{\alpha}}. \end{aligned}$$

Since

$$\mu(B(0,2^{i+1}r))^{1-\frac{1}{\alpha}} = d_k^{1-\frac{1}{\alpha}}(2^{i+1}r)^{(2k+2)(1-\frac{1}{\alpha})},$$

we have

$$\begin{aligned} &\sum_{i=0}^{\infty}(2^i r)^{\beta-2k-2}\mu(B(0,2^{i+1}r))^{1-\frac{1}{\alpha}} \\ &= d_k^{1-\frac{1}{\alpha}}2^{(2k+2)(1-\frac{1}{\alpha})}r^{\beta-\frac{2k+2}{\alpha}}\sum_{i=0}^{\infty}(2^i)^{\beta-\frac{2k+2}{\alpha}}. \end{aligned}$$

In addition, $\sum_{i=0}^{\infty}(2^i)^{\beta-\frac{2k+2}{\alpha}} < \infty$. So

$$|B(x)| \leq C r^{\beta-\frac{2k+2}{\alpha}}\|f\|_{q,\infty,\alpha}. \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$|I_\beta f(x)| \leq C(r^\beta Mf(x) + r^{\beta-\frac{2k+2}{\alpha}}\|f\|_{q,\infty,\alpha})$$

for all $r>0$. Thus, taking $r = \left(\frac{\|f\|_{q,\infty,\alpha}}{Mf(x)}\right)^{\frac{\alpha}{2k+2}}$, we get

$$|I_\beta f(x)| \leq C(Mf(x))^{1-\frac{\alpha\beta}{2k+2}}\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}} \tag{5.3}$$

for all $x \in \mathbb{R}$.
From (5.3), we have

$$\begin{aligned} {}_r\|I_\beta f\|_{\bar{q},\bar{p}} &\leq C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\, {}_r\|(Mf)^{1-\frac{\alpha\beta}{2k+2}}\|_{\bar{q},\bar{p}} \\ &\leq C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\, {}_r\|Mf\|_{\bar{q}(1-\frac{\alpha\beta}{2k+2}),\bar{p}(1-\frac{\alpha\beta}{2k+2})}^{1-\frac{\alpha\beta}{2k+2}} \\ &\leq C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\, {}_r\|Mf\|_{q,p}^{1-\frac{\alpha\beta}{2k+2}}. \end{aligned}$$

Then, multiplying both sides of the previous inequality by $\mu(B(0,r))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}}$, we get

$$\begin{aligned} &\mu(B(0,r))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}}\, {}_r\|I_\beta f\|_{\bar{q},\bar{p}} \\ \leq\;& C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}(\mu(B(0,r)))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}}\, {}_r\|Mf\|_{q,p}^{1-\frac{\alpha\beta}{2k+2}} \\ \leq\;& C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\mu(B(0,r))^{(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})(1-\frac{\alpha\beta}{2k+2})}\, {}_r\|Mf\|_{q,p}^{1-\frac{\alpha\beta}{2k+2}} \\ \leq\;& C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}[\mu(B(0,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\, {}_r\|Mf\|_{q,p}]^{1-\frac{\alpha\beta}{2k+2}}. \end{aligned}$$

Hence,

$$\|I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\|Mf\|_{q,p,\alpha}^{1-\frac{\alpha\beta}{2k+2}}$$

and by Proposition 5.1,

$$\|I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\|f\|_{q,p,\alpha}^{1-\frac{\alpha\beta}{2k+2}}. \tag{5.4}$$

We end the proof by combining Inequality (5.4) with the fact that

$$\|f\|_{q,\infty,\alpha} \leq C\|f\|_{q,p,\alpha},$$

where C is a constant not depending on f. □

Proof of Theorem 1.2. Let $1 < \alpha < p < \infty$, $0 < \beta < \frac{2k+2}{\alpha}$ and $f \in (L^1, L^p)^\alpha(\mu)$. We assume that $f \neq 0$ μ-almost everywhere, otherwise the result is obvious. From (5.3) we have

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{1,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}\|(Mf)^{1-\frac{\alpha\beta}{2k+2}}\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu)}.$$

Now, let $r > 0$. Then

$$\begin{aligned} &\|\|(Mf)^{1-\frac{\alpha\beta}{2k+2}}\tau_{(-\cdot)}^{\frac{1}{\bar{q}}}\chi_{B_r}\|_{L^{\bar{q},\infty}(\mu)}\|_{\bar{p}} \\ =\;& \left\|\left\|\left[(Mf)\tau_{(-\cdot)}^{\frac{1}{\bar{q}(1-\frac{\alpha\beta}{2k+2})}}\chi_{B_r}\right]^{1-\frac{\alpha\beta}{2k+2}}\right\|_{L^{\bar{q},\infty}(\mu)}\right\|_{\bar{p}} \\ =\;& \left\|\left\|(Mf)\tau_{(-\cdot)}^{\frac{1}{\bar{q}(1-\frac{\alpha\beta}{2k+2})}}\chi_{B_r}\right\|_{L^{\bar{q}(1-\frac{\alpha\beta}{2k+2}),\infty}(\mu)}^{1-\frac{\alpha\beta}{2k+2}}\right\|_{\bar{p}} \\ =\;& \left\|\left\|(Mf)\tau_{(-\cdot)}\chi_{B_r}\right\|_{L^{1,\infty}(\mu)}\right\|_{\bar{p}(1-\frac{\alpha\beta}{2k+2})}^{1-\frac{\alpha\beta}{2k+2}} \\ =\;& \left\|\left\|(Mf)\tau_{(-\cdot)}\chi_{B_r}\right\|_{L^{1,\infty}(\mu)}\right\|_{p}^{1-\frac{\alpha\beta}{2k+2}}. \end{aligned}$$

Since $\frac{1}{\alpha^*} - \frac{1}{\bar{q}} - \frac{1}{\bar{p}} = (\frac{1}{\alpha} - 1 - \frac{1}{p})(1 - \frac{\alpha\beta}{2k+2})$ then

$$\mu(B(0,r))^{\frac{1}{\alpha^*} - \frac{1}{\bar{q}} - \frac{1}{\bar{p}}} \| \| (Mf)^{1-\frac{\alpha\beta}{2k+2}} \tau_{(-\cdot)}^{\frac{1}{\bar{q}}} \chi_{B_r} \|_{L^{\bar{q},\infty}(\mu)} \|_{\bar{p}}$$

$$= \mu(B(0,r))^{(\frac{1}{\alpha}-1-\frac{1}{p})(1-\frac{\alpha\beta}{2k+2})} \left\| \left\| (Mf)\tau_{(-\cdot)}\chi_{B_r} \right\|_{L^{1,\infty}(\mu)} \right\|_p^{1-\frac{\alpha\beta}{2k+2}}$$

$$= \left[\mu(B(0,r))^{\frac{1}{\alpha}-1-\frac{1}{p}} \left\| \left\| (Mf)\tau_{(-\cdot)}\chi_{B_r} \right\|_{L^{1,\infty}(\mu)} \right\|_p \right]^{1-\frac{\alpha\beta}{2k+2}}.$$

Thus,

$$\|(Mf)^{1-\frac{\alpha\beta}{2k+2}}\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu)} = \|Mf\|_{(L^{1,\infty},L^p)^{\alpha}(\mu)}^{1-\frac{\alpha\beta}{2k+2}}$$

and

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{1,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}} \|Mf\|_{(L^{1,\infty},L^p)^{\alpha}(\mu)}^{1-\frac{\alpha\beta}{2k+2}}.$$

By applying Proposition 5.2, the previous inequality becomes

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*}(\mu_k)} \leq C\|f\|_{1,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}} \|f\|_{1,p,\alpha}^{1-\frac{\alpha\beta}{2k+2}}. \tag{5.5}$$

We end the proof by combining Inequality (5.5) with the fact that

$$\|f\|_{1,\infty,\alpha} \leq C\|f\|_{1,p,\alpha},$$

where C is a constant not depending on f. □

Proof of Corollary 1.3. Let $1 < q \leq \alpha \leq p < \infty$, $0 < \beta < \frac{2k+2}{\alpha}$ and $f \in (L^q, L^p)^\alpha(\mu)$. Let $x \in \mathbb{R}$ and $r > 0$. We have

$$\begin{aligned}
& \frac{1}{\mu(B(0,r))^{1-\frac{\beta}{2k+2}}} \int_{B(0,r)} \tau_x|f|(z)d\mu(z) \\
= & \int_{B(0,r)} \tau_x|f|(z) \frac{1}{\mu(B(0,r))^{1-\frac{\beta}{2k+2}}} d\mu(z) \\
= & \int_{B(0,r)} \tau_x|f|(z) \frac{1}{(d_k r^{2k+2})^{1-\frac{\beta}{2k+2}}} d\mu(z) \\
= & d_k^{\frac{\beta}{2k+2}-1} \int_{B(0,r)} \tau_x|f|(z) \frac{1}{r^{2k+2-\beta}} d\mu(z) \\
\leq & d_k^{\frac{\beta}{2k+2}-1} \int_{B(0,r)} \tau_x|f|(z)|z|^{\beta-2k-2} d\mu(z) \\
\leq & d_k^{\frac{\beta}{2k+2}-1} \int_{\mathbb{R}} \tau_x|f|(z)|z|^{\beta-2k-2} d\mu(z).
\end{aligned}$$

Hence,

$$M_\beta f(x) \leq d_k^{\frac{\beta}{2k+2}-1} I_\beta|f|(x). \tag{5.6}$$

We end the proof by taking the norm $\|\cdot\|_{\bar{q},\bar{p},\alpha^*}$ on the both sides of (5.6) and by applying Theorem 1.1. □

Proof of Corollary 1.4. The result follows from Inequality (5.6) and Theorem 1.2. □

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