

SOME RESULTS ON DIFFERENTIAL-DIFFERENCE PAINLEVÉ EQUATIONS

B. NARASIMHA RAO[†], SHILPA N, AND SUBHAS S. BHOOSNURMATH

Date of Receiving : 26. 04. 2022
 Date of Revision : 16. 07. 2022
 Date of Acceptance : 30. 10. 2022

Abstract. This paper studies the properties of finite order transcendental meromorphic solutions to certain differential-difference Painlevé III equations. Also discussed the exponent of convergence of poles of divided difference $\Delta f/f$, existence of “Borel exceptional value” and rational solutions.

1. Introduction

A meromorphic function is defined in this paper for the entire complex plane $\mathbb{C}$. Standard notations can be found in [2, 5, 12] as well as the fundamentals of Nevanlinna theory. Let $\mathcal{F} = \{f : f \text{ is finite - order transcendental meromorphic function in } \mathbb{C}\}$. For $\beta \in \mathcal{F}$, the term “small function” of f means if $T(r, \beta) = S(r, f)$. Also, $S(r, f)$ denotes any quantity that can satisfy $S(r, f) = o(T(r, f))$, $r \notin E$, $r \rightarrow \infty$.

If all the solutions of ordinary differential equations are single-valued around all removable singularities, they possess Painlevé property (for more details see [11]). In 1895-1900, Painlevé [9], Fuchs [6] and Gambier [1] worked on second order differential equations with an interpretation of Painlevé property, proposed by Picard. Painlevé and his colleagues established six non-linear differential equations, which cannot be solved using known functions, are called Painlevé equations. In physics as well as mathematics, Painlevé equations play vital role due to their applications of plasma physics, non-linear optics and solid state physics etc.

2010 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Borel Exceptional value, Difference Painlevé equation, Entire Function, Exceptional Value and Meromorphic Function.

Communicated by. Sujoy Majumder

[†] *Corresponding author*

2. Definitions

Let f be a non-constant meromorphic function. The “order of f ” is given by:

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}.$$

Let f be a non-constant meromorphic function. The “exponent of convergence of zeros” of f and “exponent of convergence of poles” of f for the value point a are respectively defined as:

$$\lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r} \text{ and } \lambda\left(\frac{1}{f}\right) = \limsup_{r \rightarrow \infty} \frac{\log^+ N(r, f-a)}{\log r}.$$

Let f be a non-constant meromorphic function. The fixed points of f is given by

$$\tau(f) = \limsup_{r \rightarrow \infty} \frac{\log N(r, \frac{1}{f-z})}{\log r}.$$

Let f be a non-constant meromorphic function. The value point ‘ a ’ is called the “Borel exceptional Value (BeV)” of f if

$$\limsup_{r \rightarrow \infty} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r} < \rho(f).$$

For simplicity we denote:

$$\bar{f} \equiv f(z+1), \quad \underline{f} \equiv f(z-1), \quad w_1 \equiv \frac{\bar{f}}{f} \text{ and } w_2 \equiv \frac{f}{\underline{f}}. \quad (2.1)$$

Recently, an independent effort of Halburd and Korhonen [10] and Chiang and Feng [15] achieved an important result on the derivative of the logarithm, entitled “Difference analogue of the lemma on the logarithmic derivative”. After their fabulous work, many authors [3, 4, 8] have focused on the properties of meromorphic solutions of differential and difference equations. In particular, the following equations were studied by Y. Du *et al.* [14] and L. Liu *et al.* [7], determined the properties of meromorphic solutions.

$$\bar{f}\underline{f} = \frac{f^2 - \lambda f + \mu}{(f-1)^2} \quad (2.2)$$

where μ and λ are constants such that $\lambda\mu \neq 0$ and

$$\bar{f}\underline{f} = \frac{f^2 - \lambda f}{f^2 - 1} \quad (2.3)$$

where $\lambda \neq \pm 1$ is constant.

Theorem 2.1. [14] *If $f \in \mathcal{F}$ be a solution of (2.2), then the following holds:*

- (1) *for any $\eta \in \mathbb{C} \setminus \{0\}$, $\tau(f(z+\eta)) = \sigma(f)$;*
- (2) $\lambda\left(\frac{1}{\Delta f}\right) = \lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f).$

Theorem 2.2. [7] *If $f \in \mathcal{F}$ be a solution of (2.3), then the following holds:*

- (1) $\lambda\left(\frac{1}{f}\right) = \tau(f) = \sigma(f)$;
- (2) *If $\lambda \neq 0$, then $\lambda(f) = \sigma(f)$;*

(3) f has at most one non zero finite Borel exceptional.

Theorem 2.3. [10] Let $f(z)$ be a non-constant meromorphic solution of

$$f(z)^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in $f(z)$, and let $\delta < 1$ and $\varepsilon > 0$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is at most n , then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta}\right) + o(T(r, f))$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Theorem 2.4. [5] Let $f(z)$ be a non-constant meromorphic function, then every irreducible rational function in f ,

$$R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^p \alpha_i(z) f^i(z)}{\sum_{j=0}^q \beta_j(z) f^j(z)}$$

here $\alpha_i(z), \beta_j(z)$ are small functions of f , $i = 0, 1, \dots, p$ and $j = 0, 1, \dots, q$, satisfying $T(r, R(z, f)) = dT(r, f) + S(r, f)$, $d = \max\{p, q\}$.

3. Preliminaries

Lemma 3.1. [15] Let $f(z)$ be a meromorphic function of finite order σ and let η be a non-zero complex number. Then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 3.2. [2] Let f_j be a meromorphic functions and g_j be a entire functions, for $j = 1, \dots, n$ ($n \geq 2$) satisfying

- (i) $\sum_{j=1}^n f_j(z) e^{g_j} \equiv 0$;
- (ii) when $1 \leq j < k \leq n$, then $g_j - g_k$;
- (iii) when $1 \leq j \leq n, 1 \leq h < k \leq n$, then

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \rightarrow \infty, r \notin E),$$

where $E \subset (1, \infty)$ either logarithmic or finite linear measure. Then, $f_j \equiv 0$ ($j = 1, \dots, n$).

Lemma 3.3. [2] Let f be a meromorphic function in $|z| < R \leq \infty$, for every $b \in \mathbb{C}$ and $r \in (0, R)$,

$$T\left(r, \frac{1}{f-b}\right) = T(r, f) + S(r, f).$$

Lemma 3.4. [2] Suppose $f(z)$ is a non-constant meromorphic function in the complex plane and k is a positive integer, and let

$$\Psi(z) = \sum_{i=0}^k a_i(z) f^i(z),$$

where $a_1(z), a_2(z), \dots, a_k(z)$ are small functions of $f(z)$. Then

$$m\left(r, \frac{\Psi}{f}\right) = S(r, f)$$

and $T(r, \Psi) \leq (k+1)T(r, f) + S(r, f)$.

Recently, Y. Du *et al.* [13] investigated differential-difference Painlevé III equations

$$(\overline{f}f)' = \frac{\alpha(f-\beta)}{f-1}; \quad \alpha \neq 0, \beta \neq 1 \quad (3.1)$$

$$\overline{f}'f' = \frac{f^2 - \beta f}{f-1}; \quad \beta \neq 1 \quad (3.2)$$

and obtained relation between growth order, exponent convergence of zeros, fixed points and rational solutions.

Motivated by [13], we investigate the following differential-difference Painlevé III equations, obtain the properties of meromorphic solutions and existence of rational solutions.

$$(\overline{f}f)' = \frac{f^2 - \lambda f + \mu}{(f-1)^2}. \quad (3.3)$$

$$(\overline{f}f)' = \frac{f^2 - \lambda f}{f^2 - 1}. \quad (3.4)$$

$$\overline{f}'f' = \frac{(f-\lambda)^2}{(f-1)^2}. \quad (3.5)$$

4. Main Results

Theorem 4.1. *If $f \in \mathcal{F}$ is a solution of the equation (3.3) and μ, λ are non-zero constants with $\mu\lambda \neq 0$, then*

- (1) $\tau(f) = \lambda(f) = \sigma(f)$;
- (2) $\lambda\left(\frac{1}{\frac{\lambda}{\Delta f}}\right) = \sigma(f)$;
- (3) *If $\lambda \neq a+b$, then f has at most one non zero finite Borel exceptional value.*

Proof:

(i) Let $g(z) = f(z) - z$. Since $f(z)$ is meromorphic, we can write $T(r, g) = T(r, f) + S(r, f)$. Replacing $f(z)$ by $g(z) + z$ in equation (3.3), we obtain

$$[(\overline{g} + z + 1)(\underline{g} + z - 1)]'(g + z - 1)^2 = (g + z)^2 - \lambda(g + z) + \mu.$$

Denote

$$Q_1(z, g) := ([(\overline{g} + z + 1)(\underline{g} + z - 1)]')(g + z - 1)^2 - (g + z)^2 - \lambda(g + z) + \mu. \quad (4.1)$$

It yields from (4.1) that $Q_1(z, 0) = 2z^3 - 5z^2 + (\lambda + 2)z - \mu \not\equiv 0$, combining this with Theorem 2.3, we get $m\left(r, \frac{1}{g}\right) = S(r, f)$.

$\therefore N\left(r, \frac{1}{f-z}\right) = T(r, f) + S(r, f)$, and hence $\tau(f) = \sigma(f)$.

If $\mu \neq 0$ then (3.3) yields, let $Q_2(z, f) := (\bar{f}f)'(f-1)^2 - (f^2 - \lambda f + \mu)$. Clearly $Q_2(z, 0) = -\mu \neq 0$, combining this with Theorem 2.3, we get

$$m\left(r, \frac{1}{f}\right) = S(r, f).$$

Therefore $N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f)$, and hence $\lambda(f) = \sigma(f)$

$$\therefore \tau(f) = \lambda(f) = \sigma(f).$$

(ii) We rewrite equation (3.3) as follows

$$\begin{aligned} \left(\frac{\bar{f}f}{f^2}\right)' + 2\frac{f'}{f}\left(\frac{\bar{f}f}{f^2}\right) &= \frac{f^2 - \lambda f + \mu}{f^2(f-1)^2}, \\ \left(\frac{1}{f}\right)\left(\frac{\bar{f}f}{f^2}\right)' - 2\left(\frac{1}{f}\right)' \left(\frac{\bar{f}f}{f^2}\right) &= \frac{f^2 - \lambda f + \mu}{f^3(f-1)^2}. \end{aligned} \quad (4.2)$$

Equation (4.2) and (2.1), yield

$$\left(\frac{1}{f}\right)(w_1 w_2)' - 2\left(\frac{1}{f}\right)'(w_1 w_2) = \frac{f^2 - \lambda f + \mu}{f^3(f-1)^2} \quad (4.3)$$

and we have from (2.1) and Lemma 3.1, that

$$T(r, w_1) \leq T\left(r, \frac{\Delta f}{f}\right) + S(r, f). \quad (4.4)$$

$$T(r, w_2) \leq T(r, w_1) + S\left(r, \frac{f(z+1)}{f(z)}\right) + O(1)$$

$$T(r, w_2) \leq T\left(r, \frac{\Delta f}{f}\right) + S(r, f). \quad (4.5)$$

$$T(r, w_1 w_2) \leq 2T\left(r, \frac{\Delta f}{f}\right) + S(r, f). \quad (4.6)$$

In order to prove $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f)$, we take up the following three cases:

Case a: $\lambda - \mu \neq 1$.

Since $\lambda\mu \neq 0$ and in the view of equation (4.3), (4.4) to (4.6), Theorem 2.4, Lemma 3.3

and Lemma 3.4, we get

$$\begin{aligned}
5T(r, f) &= T\left(r, \frac{f^2 - \lambda f + \mu}{f^3(f-1)^2}\right) + O(1) \\
&= T\left(r, \left(\frac{1}{f}\right)(w_1 w_2)' - 2\left(\frac{1}{f}\right)'(w_1 w_2)\right) + O(1) \\
&\leq T\left(r, \frac{1}{f}\right) + T(r, (w_1 w_2)') + T(r, w_1 w_2) + T\left(r, \left(\frac{1}{f}\right)'\right) + O(1) \\
&\leq 3T(r, w_1 w_2) + 3T(r, f) + O(1) \\
2T(r, f) &\leq 6T\left(r, \frac{\Delta f}{f}\right) + S(r, f).
\end{aligned}$$

Therefore we have

$$\frac{1}{3}T(r, f) \leq T\left(r, \frac{\Delta f}{f}\right) + S(r, f). \quad (4.7)$$

From equation (4.7) and Lemma 3.1, we see that

$$\begin{aligned}
N\left(r, \frac{\Delta f}{f}\right) &= T\left(r, \frac{\Delta f}{f}\right) - m\left(r, \frac{\Delta f}{f}\right) \\
&\geq \frac{1}{3}T(r, f) + S(r, f),
\end{aligned}$$

which implies $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \geq \sigma(f)$ and by the fact we have $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \leq \sigma(f)$ hence $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f)$.

Case b: $\mu = 1, \lambda - \mu = 1$, equation (4.2) becomes

$$\begin{aligned}
\frac{1}{f^2} &= \left(\frac{\bar{f}f}{f^2}\right)' + 2\frac{f'}{f}\left(\frac{\bar{f}f}{f^2}\right) \\
\frac{1}{f^2 f'} &= \frac{1}{f'}\left(\frac{\bar{f}f}{f^2}\right)' + 2\frac{1}{f}\left(\frac{\bar{f}f}{f^2}\right) + O(1).
\end{aligned} \quad (4.8)$$

Equation (4.8) and (2.1), yield

$$\frac{1}{f^2 f'} = \frac{1}{f'}(w_1 w_2)' + 2\frac{1}{f}(w_1 w_2) + O(1). \quad (4.9)$$

Taking LHS of (4.9), Lemma 3.3 and Lemma 3.4, we get

$$\begin{aligned}
T\left(r, \frac{1}{f^2 f'}\right) &= T(r, f^2 f') + O(1) \\
&\leq T(r, f') + T(r, f^2) + S(r, f) \\
T\left(r, \frac{1}{f^2 f'}\right) &\leq 4T(r, f) + S(r, f).
\end{aligned} \quad (4.10)$$

On the other hand, we have

$$\begin{aligned}
 T\left(r, \frac{1}{f^2 f'}\right) &= T(r, f^2 f') - T(r, f') + T(r, f') + S(r, f) \\
 &= T(r, f^2 f') - T\left(r, \frac{1}{f'}\right) + T(r, f') + S(r, f) \\
 &\geq T\left(r, \frac{f^2 f'}{f'}\right) - (-T(r, f')) + S(r, f) \\
 &\geq 2T(r, f) - (-2T(r, f)) + S(r, f) \\
 T\left(r, \frac{1}{f^2 f'}\right) &\geq 4T(r, f) + S(r, f).
 \end{aligned} \tag{4.11}$$

From equation (4.10) and (4.11), we get

$$4T(r, f) = T\left(r, \frac{1}{f^2 f'}\right) + S(r, f). \tag{4.12}$$

From (4.4) to (4.6), (4.9), (4.12), Lemma 3.3 and Lemma 3.4, we have

$$4T(r, f) = T\left(r, \frac{1}{f^2 f'}\right) + O(1) \tag{4.13}$$

$$= T\left(r, \frac{1}{f'} (w_1 w_2)' + \frac{2}{f} (w_1 w_2)\right) + O(1) \tag{4.14}$$

$$\begin{aligned}
 &\leq T\left(r, \frac{1}{f'}\right) + T(r, (w_1 w_2)') + T\left(r, \frac{1}{f}\right) + T(r, w_1 w_2) + S(r, f) \\
 &\leq 3T(r, f) + 3T(r, w_1 w_2) + S(r, f)
 \end{aligned}$$

$$T(r, f) \leq 6T\left(r, \frac{\Delta f}{f}\right) + S(r, f). \tag{4.15}$$

From (4.15) and Lemma 3.1, we get

$$N\left(r, \frac{1}{\frac{\Delta f}{f}}\right) \geq \frac{1}{6}T(r, f) + S(r, f),$$

which implies $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \geq \sigma(f)$ and by the fact we have $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \leq \sigma(f)$ hence

$$\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f).$$

Case c: $\mu \neq 1, \lambda - \mu = 1$, equation (4.2) becomes

$$\frac{f - \mu}{f^3(f - 1)} = \left(\frac{1}{f}\right) \left(\frac{\bar{f}f}{f^2}\right)' - 2\left(\frac{1}{f}\right)' \left(\frac{\bar{f}f}{f^2}\right). \tag{4.16}$$

From (4.16) and (2.1), yield

$$\frac{f - \mu}{f^3(f - 1)} = \left(\frac{1}{f}\right) \left(w_1 w_2\right)' - 2\left(\frac{1}{f}\right)' (w_1 w_2). \tag{4.17}$$

In the view of (4.17), Theorem 2.4, (4.4) to (4.6), Lemma 3.3 and Lemma 3.4, we get

$$\begin{aligned}
 4T(r, f) &= T\left(r, \frac{f - \mu}{f^3(f-1)}\right) + O(1) \\
 &= T\left(r, \left(\frac{1}{f}\right)(w_1 w_2)' - 2\left(\frac{1}{f}\right)'(w_1 w_2)\right) + O(1) \\
 &\leq T\left(r, \frac{1}{f}\right) + T\left(r, \left(\frac{\bar{f}f}{f^2}\right)'\right) + T\left(r, \frac{1}{f}\right)' + T\left(r, \left(\frac{\bar{f}f}{f^2}\right)\right) + O(1) \\
 &\leq 3T(r, f) + 6T\left(r, \frac{\Delta f}{f}\right) + S(r, f) \\
 T(r, f) &\leq 6T\left(r, \frac{\Delta f}{f}\right) + S(r, f).
 \end{aligned} \tag{4.18}$$

From equation (4.19) and Corollary 2.5 in [15], we get

$$\begin{aligned}
 N\left(r, \frac{\Delta f}{f}\right) &= T\left(r, \frac{\Delta f}{f}\right) - m\left(r, \frac{\Delta f}{f}\right) + S(r, f) \\
 &\geq \frac{1}{6}T(r, f) + S(r, f)
 \end{aligned}$$

which leads to $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \geq \sigma(f)$ and by the fact we have $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \leq \sigma(f)$ hence $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f)$.

(iii) Suppose a and $b (\neq a)$ be two non-zero finite Borel exceptional values of f . Set

$$\psi(z) = \frac{f(z) - a}{f(z) - b}. \tag{4.20}$$

Clearly

$$\sigma(\psi) = \sigma(f), \quad \lambda(\psi) = \lambda(f - a) \leq \sigma(f), \quad \lambda\left(\frac{1}{\psi}\right) = \lambda(f - b) \leq \sigma(f).$$

Since ψ is of finite order, we write ψ in the form

$$\psi(z) = h(z)e^{cz^n}, \tag{4.21}$$

where c is a non-zero constant, n be a nonnegative integer and $h(z)$ is a meromorphic function with $\sigma(h) \leq \sigma(\psi) = n$.

From equation (4.20) and (4.21), we get

$$f(z) = \frac{a - bhe^{cz^n}}{1 - he^{cz^n}} \tag{4.22}$$

and

$$\bar{f} = \frac{a - b\bar{h}h_1e^{cz^n}}{1 - \bar{h}h_1e^{cz^n}}, \quad \underline{f} = \frac{a - b\underline{h}h_2e^{cz^n}}{1 - \underline{h}h_2e^{cz^n}}, \tag{4.23}$$

where $h_1(z) = e^{(ncz^{n-1} + \dots + c)}$ and $h_2(z) = e^{(-ncz^{n-1} + \dots + (-1)^n c)}$.

Putting (4.22) and (4.23) in (3.3), we get

$$A(z) + B(z)e^{cz^n} + C(z)e^{2cz^n} + D(z)e^{3cz^n} + E(z)e^{4cz^n} + F(z)e^{5cz^n} + G(z)e^{6cz^n} = 0, \tag{4.24}$$

where

$$\begin{aligned}
A(z) &= a^2 - \lambda a + \mu, \\
B(z) &= h(\lambda(a+b) - 2(ab+\mu)) - 2A_1(a^2 - \lambda a + \mu) - (a-1)^2 a(a-b)B_1, \\
C(z) &= h^2(b^2 - \lambda b + \mu) - 2A_1 h(\lambda(a+b) - 2(ab+\mu)) + B_5(a^2 - \lambda a + \mu) \\
&\quad + 2B_1 h(a-1)(b-1)a(a-b) - B_2(a-1)^2(b^2 - a^2), \\
D(z) &= B_5 h(\lambda(a+b) - 2(ab+\mu)) - 2h^2 A_1(b^2 - \lambda b + \mu) - 2B_6(a^2 - \lambda a + \mu) \\
&\quad - B_1 h^2 a(a-b)(1-b)^2 + 2hB_2(1-a)(b-1)(b^2 - a^2) - B_7 b(b-a)(a-1)^2, \\
E(z) &= B_5 h^2(b^2 - \lambda b + \mu) - 2B_6 h(\lambda(a+b) - 2(ab+\mu)) + A_2^2(a^2 - \lambda a + \mu) \\
&\quad + 2bhB_7(a-1)(1-b)(b-a) - B_2 h^2(1-b)^2(b^2 - a^2), \\
F(z) &= A_2^2 h(\lambda(a+b) - 2(ab+\mu)) - 2h^2 B_6(b^2 - \lambda b + \mu) - B_7 h^2 b(a-b)(1-b)^2, \\
G(z) &= A_2^2 h^2(b^2 - \lambda b + \mu).
\end{aligned}$$

Here

$$\begin{aligned}
A_1 &= \bar{h}h_1 + \underline{h}h_2, \\
A_2 &= \bar{h}\underline{h}h_1h_2, \\
B_1 &= K_1 + K_2, \\
K_1 &= \bar{h}'h_1 + \bar{h}h_1' + \bar{h}h_1cnz^{n-1}, \\
K_2 &= \underline{h}'h_2 + \underline{h}h_2' + \underline{h}h_2cnz^{n-1}, \\
B_2 &= \underline{h}h_2K_1 + \bar{h}h_1K_2, \\
B_3 &= (\bar{h}h_1 + \underline{h}h_2)(\underline{h}h_2K_1 + \bar{h}h_1K_2), \\
B_4 &= (\bar{h}h_1\underline{h}h_2)(K_1 + K_2), \\
B_5 &= (\bar{h}h_1 + \underline{h}h_2)^2 + 2(\bar{h}h_1\underline{h}h_2), \\
B_6 &= (\bar{h}h_1 + \underline{h}h_2)(\bar{h}h_1\underline{h}h_2), \\
B_7 &= (\bar{h}h_1)^2K_2 + (\underline{h}h_2)^2K_1.
\end{aligned} \tag{4.25}$$

It follows from the equation (4.24) and Lemma 3.2, we get $A(z) \equiv B(z) \equiv C(z) \equiv D(z) \equiv E(z) \equiv F(z) \equiv G(z) \equiv 0$.

Since $A(z) \equiv 0$ and $G(z) \equiv 0$, we have $a^2 - \lambda a + \mu = 0$ and $b^2 - \lambda b + \mu = 0$, on simplifying, we get $(b-a)(b+a-\lambda) = 0$.

Since $\lambda \neq a+b$, which implies $a=b$, which is a contradiction.

Therefore f has one Borel exceptional value.

Theorem 4.2. *If $f \in \mathcal{F}$ is a solution of the equation (3.4) and let $\lambda \neq 1$ be a constant, then*

- (1) $\tau(f) = \sigma(f)$;
- (2) $\lambda \left(\frac{1}{\frac{\Delta f}{f}} \right) = \sigma(f)$;
- (3) If $\lambda \neq 0$, then f has at most one non zero finite Borel exceptional value.

Proof:

- (i) Similar to case(i) of *Theorem 4.1*, one can conclude $\tau(f) = \sigma(f)$.
(ii) To prove $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f)$. We rewrite equation (3.4) as follows

$$\frac{f^2 - \lambda f}{f^3(f^2 - 1)} = \left(\frac{1}{f}\right) \left(\frac{\bar{f}f}{f^2}\right)' - 2 \left(\frac{1}{f}\right)' \left(\frac{\bar{f}f}{f^2}\right). \quad (4.26)$$

Equation (4.26) and (2.1), yield

$$\frac{f^2 - \lambda f}{f^3(f^2 - 1)} = \left(\frac{1}{f}\right) (w_1 w_2)' - 2 \left(\frac{1}{f}\right)' (w_1 w_2). \quad (4.27)$$

From (4.4) to (4.6), (4.27), Theorem 2.4, Lemma 3.3 and Lemma 3.4, yield

$$\begin{aligned} 5T(r, f) &= T\left(r, \frac{f^2 - \lambda f}{f^3(f^2 - 1)}\right) + O(1) \\ &= T\left(r, \left(\frac{1}{f}\right) (w_1 w_2)' - 2 \left(\frac{1}{f}\right)' (w_1 w_2)\right) + O(1) \\ &\leq 3T(r, f) + 6T\left(r, \frac{\Delta f}{f}\right) + S(r, f) \\ \frac{1}{3}T(r, f) &\leq T\left(r, \frac{\Delta f}{f}\right) + S(r, f). \end{aligned} \quad (4.28)$$

From equation (4.28) and Corollary 2.5 in [15], we get

$$N\left(r, \frac{\Delta f}{f}\right) \geq \frac{1}{3}T(r, f) + S(r, f), \text{ which means } \lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \geq \sigma(f) \text{ and by the fact we have } \lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \leq \sigma(f).$$

$$\therefore \lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f).$$

- (iii) Suppose a and $b(\neq a)$ be two non-zero finite Borel exceptional values f .
Similar to case(iii) of *Theorem 4.1*, one can obtain (4.20) to (4.23). Substituting (4.22) and (4.23) in (3.4), we get

$$A(z) + B(z)e^{cz^n} + C(z)e^{2cz^n} + D(z)e^{3cz^n} + E(z)e^{4cz^n} + F(z)e^{5cz^n} + G(z)e^{6cz^n} = 0, \quad (4.29)$$

where

$$\begin{aligned}
A(z) &= a(a - \lambda), \\
B(z) &= B_1 a(b - a)(a^2 - 1) + (\lambda(a + b) - 2ab)h - 2A_1 a(a - \lambda), \\
C(z) &= B_5 a(a - \lambda) - 2A_1 h(\lambda(a + b) - 2ab) + b(b - \lambda)h^2 \\
&\quad - 2hB_1 a(a - b)(1 - ab) - B_2(a^2 - 1)(b^2 - a^2), \\
D(z) &= B_5 h(\lambda(a + b) - 2ab) - 2A_1 h^2 b(b - \lambda) - 2B_6 a(a - \lambda) \\
&\quad - B_1 h^2 a(a - b)(b^2 - 1) - 2hB_2(1 - ab)(b^2 - a^2) - B_7(a^2 - 1)b(a - b), \\
E(z) &= A_2^2 a(a - \lambda) - 2B_6 h(\lambda(a + b) - 2ab) + B_5 h^2 b(b - \lambda) \\
&\quad - 2hB_7 b(a - b)(1 - ab) - B_2 h^2(b^2 - 1)(b^2 - a^2), \\
F(z) &= A_2^2 h(\lambda(a + b) - 2ab) - 2B_6 h^2 b(b - \lambda) - B_7 h^2 b(a - b)(b^2 - 1), \\
G(z) &= A_2^2 h^2 b(b - \lambda).
\end{aligned}$$

where $A_1, A_2, K_1, K_2, B_1, \dots, B_7$ are defined in equation (4.25).

From Lemma 3.2 and (4.29), we have

$$A(z) \equiv B(z) \equiv C(z) \equiv D(z) \equiv E(z) \equiv F(z) \equiv G(z) \equiv 0.$$

Since $A(z) \equiv 0$ and $G(z) \equiv 0$, we get $a(a - \lambda) = 0$ and $b(b - \lambda) = 0$.

Since $a \neq 0$ and $b \neq 0$, implies $a = \lambda = b$, which is contradiction.

Hence f has one non zero finite Borel exceptional value.

Theorem 4.3. *If $f \in \mathcal{F}$ is a solution of (3.5) and let $\lambda \neq 1$ be a constant, then*

- (1) $\tau(f) = \lambda(f) = \sigma(f)$;
- (2) $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f)$;
- (3) *If $\lambda \neq 0$, then f has at most one non zero finite Borel exceptional value.*

Proof:

(i) Let $w(z) = f(z) - z$. Since f is transcendental, we conclude $T(r, w) = T(r, f) + S(r, f)$.

Replacing f by $w(z) + z$ in equation (3.5), denoting by

$$Q_3(z, w) := (\bar{w} + z + 1)'(\underline{w} + z - 1)'(w + z - 1)^2 - (w + z - \lambda)^2 \equiv 0.$$

$Q_3(z, 0) := 1 + 2(\lambda - 1) - \lambda^2 \neq 0$, clearly $Q_3(z, w) \neq 0$, combining this with Theorem 2.3, we get $m\left(r, \frac{1}{w}\right) = S(r, f)$.

$$\therefore N\left(r, \frac{1}{f-z}\right) = T(r, f) + S(r, f), \text{ and hence } \tau(f) = \sigma(f).$$

If $\lambda \neq 0$, then from (3.5), denoting by $Q_4(z, f) := (\bar{f}'f')(f^2 - 1) - (f - \lambda)^2$, $Q_4(z, 0) := -\lambda^2 \neq 0$, combining with Theorem 2.3, we get $m\left(r, \frac{1}{f}\right) = S(r, f)$,

which implies $T(r, f) = N\left(r, \frac{1}{f}\right) + S(r, f)$.

Hence $\lambda(f) \geq \sigma(f)$. And hence $\tau(f) = \lambda(f) = \sigma(f)$.

(ii) To prove $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) = \sigma(f)$. We rewrite (3.5) as follows

$$\frac{\bar{f}'f'}{f^2} = \frac{(f - \lambda)^2}{f^2(f - 1)^2}. \quad (4.30)$$

On simple calculation, we get

$$\begin{aligned}\frac{\bar{f}'}{f} &= \left(\frac{\bar{f}}{f}\right)' + \frac{\bar{f}}{f} \frac{f'}{f} \\ &= -f \left[\frac{\bar{f}}{f} \left(\frac{1}{f}\right)' - \frac{1}{f} \left(\frac{\bar{f}}{f}\right)' \right] \\ \frac{\bar{f}'}{-f^2} &= \left[\frac{\left(\frac{1}{f}\right)'}{\left(\frac{\bar{f}}{f}\right)} \right] \left(\frac{\bar{f}}{f}\right)^2.\end{aligned}\quad (4.31)$$

From equation (2.1) and (4.31), we get

$$\frac{\bar{f}'}{-f^2} = \left[\frac{\left(\frac{1}{f}\right)'}{w_1} \right] (w_1)^2 \quad (4.32)$$

Similarly

$$\frac{f'}{-f^2} = \left[\frac{\left(\frac{1}{f}\right)'}{w_2} \right] (w_2)^2. \quad (4.33)$$

On multiplying (4.32) and (4.33) and combining with (4.30) we get

$$\frac{(f - \lambda)^2}{f^4(f - 1)^2} = \left[\frac{\left(\frac{1}{f}\right)'}{w_1} \right] (w_1)^2 \left[\frac{\left(\frac{1}{f}\right)'}{w_2} \right] (w_2)^2. \quad (4.34)$$

By using (4.34), Theorem 2.4, (4.4) to (4.6), Lemma 3.3 and Lemma 3.4, we get

$$\begin{aligned}6T(r, f) &= T\left(r, \frac{(f - \lambda)^2}{f^4(f - 1)^2}\right) + O(1) \\ &= T\left(r, \left[\frac{\left(\frac{1}{f}\right)'}{w_1} \right] (w_1)^2 \left[\frac{\left(\frac{1}{f}\right)'}{w_2} \right] (w_2)^2\right) + O(1) \\ &\leq T\left(r, \left[\frac{\left(\frac{1}{f}\right)'}{w_1} \right]'\right) + T(r, w_1^2) + T\left(r, \left[\frac{\left(\frac{1}{f}\right)'}{w_2} \right]'\right) + T(r, w_2^2) + O(1) \\ &\leq 2T\left(r, \frac{1}{f}\right) + 2T\left(r, \frac{1}{w_1}\right) + 2T(r, w_1) + 2T\left(r, \frac{1}{f}\right) + 2T\left(r, \frac{1}{w_2}\right) + 2T(r, w_2) + S(r, f) \\ &\quad (4.35)\end{aligned}$$

$$\leq 4T(r, f) + 8T\left(r, \frac{\Delta f}{f}\right) + O(1)$$

$$\frac{1}{4}T(r, f) \leq T\left(r, \frac{\Delta f}{f}\right) + O(1). \quad (4.36)$$

From equation (4.36) and Lemma 3.1, we get

$N\left(r, \frac{\Delta f}{f}\right) \geq \frac{1}{4}T(r, f) + S(r, f)$, which means $\lambda\left(\frac{1}{\frac{\Delta f}{f}}\right) \geq \sigma(f)$, by the fact we have

$$\lambda \left(\frac{1}{\frac{\Delta f}{f}} \right) \leq \sigma(f).$$

$$\therefore \lambda \left(\frac{1}{\frac{\Delta f}{f}} \right) = \sigma(f).$$

(iii) Suppose a and $b(\neq a)$ be two non-zero finite Borel exceptional values of f . Similar to case(iii) of *Theorem 4.1*, one can obtain (4.20) to (4.23). Substituting (4.22) and (4.23) in (3.5), we have

$$A(z) + B(z)e^{cz^n} + C(z)e^{2cz^n} + D(z)e^{3cz^n} + E(z)e^{4cz^n} + F(z)e^{5cz^n} + G(z)e^{6cz^n} = 0. \quad (4.37)$$

where

$$\begin{aligned} A(z) &= (a - \lambda)^2, \\ B(z) &= 2h(a - \lambda)(\lambda - b) - A_5(a - \lambda), \\ C(z) &= (a - \lambda)(A_6(a - \lambda) - 4(\lambda - b)hA_5) + (\lambda - b)^2h^2 - (a - b)^2(a - 1)^2A_3A_4, \\ D(z) &= 2(a - \lambda)(\lambda - b)hA_6 - 2A_5(\lambda - b)^2 - 2A_7(a - \lambda)^2 - 2hA_3A_4(a - b)^2 \\ &\quad (a - 1)(1 - b), \\ E(z) &= (a - \lambda)^2A_8 - 4A_7(a - \lambda)(\lambda - b)h + h^2A_6(\lambda - b)^2 - h^2(a - b)^2(1 - b)^2A_3A_4, \\ F(z) &= 2(a - \lambda)(\lambda - b)hA_8 - 2A_7(\lambda - b)^2h^2, \\ G(z) &= (\lambda - b)^2h^2A_8. \end{aligned}$$

Here

$$\begin{aligned} A_3 &= \overline{h}'h_1 + \overline{h}h_1' + \overline{h}h_1cnz^{n-1}, \\ A_4 &= \underline{h}'h_2 + \underline{h}h_2' + \underline{h}h_2cnz^{n-1}, \\ A_5 &= \overline{h}h_1 + \underline{h}h_2, \\ A_6 &= (\overline{h}h_1)^2 - 4\overline{h}\underline{h}h_1h_2 + (\underline{h}h_2)^2, \\ A_7 &= (\overline{h}h_1 + \underline{h}h_2)(\overline{h}\underline{h}h_1h_2), \\ A_8 &= (\overline{h}\underline{h}h_1h_2)^2. \end{aligned}$$

From equation (4.37) and Lemma 3.2, we get

$$A(z) \equiv B(z) \equiv C(z) \equiv D(z) \equiv E(z) \equiv F(z) \equiv G(z) \equiv 0.$$

Since $A(z) \equiv 0$ and $G(z) \equiv 0$, we get $(a - \lambda)^2 = 0$ and $(\lambda - b)^2 = 0$.

which implies that $a = \lambda = b$, which is a contradiction to that a and b are distinct.

$\therefore f$ has one non-zero finite Borel exceptional value.

Theorem 4.4. Let $\lambda \neq 1$ be a constant and f be a rational solution to the equation (3.4)

$$f(z) = \frac{R(z)}{Q(z)} = \frac{\alpha_0 + \alpha_1z + \cdots + \alpha_{m-1}z^{m-1} + \alpha_mz^m}{\beta_0 + \beta_1z + \cdots + \beta_{n-1}z^{n-1} + \beta_nz^n} \quad (4.38)$$

where R and Q are relatively prime polynomials, $\alpha_m \neq 0, \alpha_{m-1}, \dots, \alpha_0$ and $\beta_n \neq 0, \beta_{n-1}, \dots, \beta_0$ are constants, then $m = n$ and $\lambda = \frac{\alpha_m}{\beta_n}$.

Proof: Substituting $f(z) = \frac{R(z)}{Q(z)}$ in (4.38), we get

$$\left(\frac{\overline{R} R}{\overline{Q} Q}\right)' \left(\left(\frac{R}{Q}\right)^2 - 1\right) = \left(\frac{R}{Q}\right)^2 - \lambda \left(\frac{R}{Q}\right). \quad (4.39)$$

Let $l = m - n$, we see three cases which are as follows:

Case(i) If $l > 0$, then $\frac{R}{Q} = \left(\frac{\alpha_m}{\beta_n}\right) z^l (1 + O(1))$ as $r \rightarrow \infty$, putting in (4.39), we get

$$2l \left(\frac{\alpha_m}{\beta_n}\right)^2 z^{2l-1} \left(\left(\frac{\alpha_m}{\beta_n}\right)^2 z^{2l} (1 + O(1)) - 1\right) = \left(\left(\frac{\alpha_m}{\beta_n}\right)^2 z^{2l} - \lambda \left(\frac{\alpha_m}{\beta_n}\right) z^l\right) (1 + O(1)). \quad (4.40)$$

as $r \rightarrow \infty$. Suppose equation (4.40) holds then $l = \frac{1}{2}$, which is a contradiction, then (4.38) has no rational solution.

Case(ii) If $l < 0$, On simplification of equation (4.39), we have

$$((\overline{Q}R')\underline{R}Q + \overline{R}Q(Q\underline{R}' - \underline{R}Q'))(R^2 - Q^2) = (R^2 - \lambda RQ)(\overline{Q}Q). \quad (4.41)$$

Since the degrees of (4.41) are not equal, it is contradiction. Therefore if $m < n$, then (4.38) has no rational solution.

Case(iii) If $l = 0$, then $\frac{R}{Q} = \frac{\alpha_m}{\beta_n} + O(1)$ as $r \rightarrow \infty$, putting in (4.39), we get $\lambda = \frac{\alpha_m}{\beta_n}$.

Theorem 4.5. Let $\lambda \neq 1$ be a constant and $f(z)$ defined in (4.38) be a rational solution to equation (3.5), then either of the following holds:

- (1) $m = n + 1$ and $\alpha = \pm\beta$;
- (2) $m = n$ and $\lambda = \frac{\alpha_m}{\beta_n}$.

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(B. Narasimha Rao) DEPARTMENT OF MATHEMATICS, CMR INSTITUTE OF TECHNOLOGY, BENGALURU - 560037, INDIA.

E-mail address, Corresponding author: `narasimha.r@cmrit.ac.in`

(Shilpa N) DEPARTMENT OF MATHEMATICS, SCHOOL OF ENGINEERING, PRESIDENCY UNIVERSITY, BENGALURU - 560064, INDIA.

E-mail address: `shilpajaikumar@gmail.com`

(Subhas S. Bhoosnurmath) DEPARTMENT OF MATHEMATICS, KARNATAKA UNIVERSITY, DHARWAD - 580003, INDIA.

E-mail address: `ssbmath@gmail.com`