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APPROXIMATION OF FIXED POINTS FOR MULTI-VALUED G-NONEXPANSIVE MAPPINGS THROUGH ULLAH ITERATION SCHEME IN UNIFORMLY CONVEX BANACH SPACES ENDOWED WITH GRAPH

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Abstract. In this paper, we deal with the approximation of fixed points for multi-valued $G-$nonexpansive mappings through Ullah iteration process in uniformly convex Banach space. With the help of Matlab Software Program, we compare different established iterative schemes for multi-valued nonexpansive mapping.

1. Introduction

The theory of fixed points of single-valued as well as multi-valued mappings have a great implications in existence and approximate solutions of optimization problems, convex feasibility problems, integral equations, differential equations etc. In fact the technique of fixed point also have been applied in different fields such as biology, physics, engineering, chemistry, game theory, economics, computer science etc.

Fixed point theorems are developed for single-valued as well as multi-valued functions over different spaces. Banach contraction principle [3] is one of the pioneering work in the field of fixed point theory and widely used to find out solutions of different problems in the field of analysis. The interest in the study of nonexpansive mapping was basically motivated by Browder's [4] work on relationship between monotone operators and nonexpansive mappings and the significance of the geometric properties of the norm for the existence of fixed point for nonexpansive mapping given by Kirk [12].

A mapping $T : X \to X$ is called nonexpansive mapping if $||Tx - Ty|| \leq ||x - y||$, for all $x, y \in X$. Nonexpansive mappings appear in many nonlinear equations for which the

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Banach principle for contraction mappings fails.

For single-valued nonexpansive mappings, the fixed point theory is well developed in the literature. The study on multi-valued nonexpansive mapping was started from the extension of Leray Schauder topological degree to some classes of multi-functions by Petryshine in 1970s. The study on multi-valued functions started rapidly over few years due to its applications in control theory, convex optimization, differential equation etc.

Markin [14] introduced the concept of Hausdorff metric to approximate fixed points of multi-valued nonexpansive mappings. Let $CB(K)$ and $P(K)$ be the collection of all non-empty closed bounded subsets and the collection of all non-empty proximal bounded closed subsets of K, respectively. Let $H(.,.)$ be the Hausdorff distance on $CB(K)$ which is defined by

$$H(A,B) = \max\{\sup d(x,B), \sup d(y,A)\},$$

where $A, B \in CB(K)$, $d(x,A) = \inf_{a\in A} ||x-a||$.
According to [11], a multi-valued mapping $T : K \to CB(K)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq ||x-y||,$$

for all $x, y \in K$.

Some classical fixed-point theorems for single-valued nonexpansive mappings have been extended to multi-valued mappings. The multi-valued version of Banach contraction principle was given by Nadler [15] in 1969. Sastry and Babu [17] introduced multi-valued version of Mann [13] and Ishikawa [9] iteration and proved convergence theorems for nonexpansive mappings in Hilbert space. In 2016 Kim et al. [11] introduced multi-valued version of Thakur iteration [23] and proved convergence results in uniformly convex Banach space.

Different iterative schemes have been used to approximate fixed points of multi-valued nonexpansive mappings. Recently Ullah et al. [24] introduced the following iteration scheme-
Let K be a convex subset of a normed space X and $T : K \to K$ be a nonlinear mapping. For $x_1 \in K$, the sequence $\{x_k\}$ in K is defined by

$$\begin{cases} z_k = (1-\alpha_k)x_k + \alpha_k T x_k, \\ y_k = T z_k, \\ x_{k+1} = T y_k, \ \ k \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$. Ullah proved that their iterative scheme converges faster than the iterative scheme given by Thakur [23].

Following is the multi-valued version of Ullah iteration (1.1) [25]. Let K be a non-empty closed convex subset of a uniformly convex Banach space X and $T : K \to P(K)$ be a multi-valued mapping. For $x_1 \in K$, the sequence $\{x_k\}$ in K is defined by

$$\begin{cases} z_k = (1-\alpha_k)x_k + \alpha_k u_k, \\ y_k = w_k, \\ x_{k+1} = v_k, \ k \geq 1, \end{cases} \tag{1.2}$$

where $P_T : K \to 2^K$ be a multi-valued mapping, $u_k \in P_T(x_k)$, $v_k \in P_T(y_k)$, $w_k \in P_T(z_k)$, and $\{\alpha_k\}$ is a sequence in $(0,1)$. Here $P_T(x) = \{y \in Tx : d(x, Tx) = ||x - y||\}$ (introduced in [21]).

The use of graph theory and the fixed point theory has increased rapidly in few years. The study of fixed point theory by combining graph theory was started by Echenique [6] in 2005. After that Kirk and Espinola [7] established some fixed point results in graph theory. Alfuraidan [2] studied some fixed point results for nonexpansive type mappings in hyperbolic metric space endowed with graph. Tiammee [22] proved Browder's convergence theorem for nonexpansive type mappings in Hilbert space endowed with directed graph.

In the case of multi-valued nonexpansive type mappings, to find existence of the fixed points of such types of mappings are general problem . Also most of the fixed point results have been established for $G-$nonexpansive mappings, but not in the case of multi-valued $G-$nonexpansive mapping by using graph. So our aim in this paper is to establish fixed point results for multi-valued $G-$nonexpansive type mappings by using graph. We prove strong convergence of the iteration scheme defined by (1.2) to approximate fixed points for the multi-valued $G-$nonexpansive mappings in uniformly convex Banach space endowed with directed graph. We also put some numerical examples in support of our main results and compare the iteration scheme (1.2) with multi-valued version of some well known iteration schemes given in [26].

2. Preliminaries

This section contains some preliminaries, which are require for convergence analysis.

Definition 2.1. [10] A Banach space X is called uniformly convex Banach space if and only if for every choice of $r \in (0, 2]$, one has a $s > 0$ such that for every $p, q \in X$, $\frac{1}{2}||p + q|| \leq (1 - s)$, whenever $||p|| \leq 1$, $||q|| \leq 1$ and $||p - q|| > r$.

Now we discuss some concepts related to graph. A graph $G = (V(G), E(G))$ contains two non-empty set, set of vertices and set of edges respectively. If x and y are vertices in G, then a path in G from x to y of the length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n+1$ vertices such that $x_0 = x, x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, ..., n$. A graph is called connected graph if there is an edge(path) between any two vertices of G. A directed graph, also called a digraph, is a graph in which the edges have a direction. For two vertices u and v in a graph G, the distance $d(u, v)$ from u to v is the length of a shortest path from u to v.

Definition 2.2. Let K be a non-empty convex subset of a uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$ and $E(G)$

contains all the loops, i.e., $(x,x) \in E(G)$ for any $x \in V(G)$. A mapping $T : K \to K$ is called an edge-preserving mapping (or $G-$ edge preserving mapping) if

$$for\ all\ x, y \in K, (x,y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G).$$

Definition 2.3. [22] Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that $(x,y) \in E(G)$ and $(y,z) \in E(G)$, then $(x,z) \in E(G)$.

Suppose that X is a uniformly convex Banach space. We can construct a graph in X by taking $V(G) = X$ or $V(G) =$ any subset of X, and $E(G) \supseteq \{(x,x) : x \in V(G)\}$, i.e., $E(G)$ contains all the loops. (For constructing graph in an arbitrary space, see [2, 20, 22]).

Definition 2.4. [22] Let K be non-empty convex subset of Banach space X, $G = (V(G), E(G))$ be a graph such that $V(G) = K$ and $T : K \to K$. Then T is said to be $G-$nonexpansive if the following conditions hold:

(i) T is edge-preserving;
(ii) $||Tx - Ty|| \leq ||x - y||$, whenever $(x,y) \in E(G)$ for any $x, y \in K$.

Next, we define multi-valued version of $G-$nonexpansive mapping.

Definition 2.5. Let K be a non-empty convex subset of uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a graph such that $V(G) = K$ and $T : K \to CB(K)$ is a multi-valued mapping. Then T is said to be $G-$nonexpansive if the following conditions hold:

(i) T is edge-preserving;
(ii) T is multi-valued nonexpansive mapping, i.e.,

$$H(Tx, Ty) \leq ||x - y||,$$

whenever $(x,y) \in E(G)$ for any $x, y \in K$.

Here is H is Hausdorff metric. (refer [14]).

Definition 2.6. Let K be a non-empty subset of a uniformly convex Banach space X. Let $T : K \to 2^K$ be a multi-valued mapping. An element $x \in K$ is said to be fixed points of T, if $x \in Tx$. Here, we denote the set of the fixed points of T by $F(T)$.

Definition 2.7. [11] Let K be a non-empty subset of a uniformly convex Banach space X. Then K is said to be proximal if for each $x \in X$, there exists an element $y \in K$ such that

$$||x - y|| = d(x, K) = \inf\{||x - z|| : z \in K\}.$$

Definition 2.8. [11] Let K be a non-empty subset of a uniformly convex Banach space X. A sequence $\{x_k\}$ in X is said to be Fejer monotone with respect to subset K, if

$$||x_{k+1} - p|| \leq ||x_k - p||,$$

for all $p \in K$, $k \geq 1$.

Proposition 2.9. [11] Let K be a non-empty subset of a uniformly convex Banach space X. Suppose that $\{x_k\}$ is Fejer monotone sequence with respect to K. Then the followings are hold:

(a) Sequence $\{x_k\}$ is bounded.

(b) For every $x \in K$, $\{||x_k - x||\}$ converges.

Note that the concept of Condition (I) in Banach space was given by Dotson and Senter [19]. Following is the multi-valued version of Condition (I).

Definition 2.10. Let K be a non-empty subset of a uniformly convex Banach space X. A multi-valued nonexpansive mapping $T : K \to CB(K)$ is said to satisfy Condition (I), if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $||x - Tx|| \geq f(d(x, F(T)))$ for all $x \in K$.

Lemma 2.11. [18] *Let X be a uniformly convex Banach space, and $\{\alpha_k\}$ be a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that $\{x_k\}$ and $\{y_k\}$ are in X such that* $\limsup_{k\to\infty} ||x_k|| \leq c$, $\limsup_{k\to\infty} ||y_k|| \leq c$, *and* $\limsup_{k\to\infty} ||\alpha_k x_k + (1 - \alpha_k)y_k|| = c$ *for some $c \geq 0$. Then* $\lim_{k\to\infty} ||x_k - y_k|| = 0$.

Lemma 2.12. [5] *Let $T : K \to P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$ and let $P_T : K \to 2^K$ be a multi-valued mapping defined by*

$$P_T(x) = \{y \in Tx : ||x - y|| = d(x, Tx)\}, x \in K.$$

Then the following conclusion holds:

(a) *P_T is multi-valued mapping from K to $P(K)$.*
(b) *$F(T) = F(P_T)$.*
(c) *$P_T(p) = \{p\}$, for each $p \in F(T)$.*
(d) *For each $x \in K$, $P_T(x)$ is a closed subset of Tx and so it is compact.*
(e) *$d(x, Tx) = d(x, P_T(x))$ for each $x \in K$.*

3. Main Results

First we prove the following lemmas.

Lemma 3.1. *Let X be a uniformly convex Banach space and K be a non-empty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to P(V(G))$ be a multi-valued mapping and $P_T : V(G) \to 2^{V(G)}$ be multi-valued $G-$nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, P_T(x_1)) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.2). Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Then we have the following:*

(a) *$(x_k, x_{k+1}) \in E(G)$ for any $k \geq 1$.*
(b) *$(x_k, P_T(x_k)) \in E(G)$ for any $k \geq 1$.*
(c) *$(x_{k+1}, P_T(x_k)) \in E(G)$ for any $k \geq 1$.*

Proof. (a) By our assumption, $z \in F(T)$, so by Lemma 2.12, we have $z \in P_T(z) = \{z\}$. Also $(x_1, z) \in E(G)$, hence by edge preserving of P_T, we have $(P_T(x_1), z) \in E(G)$, i.e. $(u_1, z) \in E(G)$. Note that $(x_1, z) \in E(G)$, $(u_1, z) \in E(G)$, by the convexity of $E(G)$, we have

$$((1 - \alpha_1)x_1 + \alpha_1 u_1, z) \in E(G),$$

i.e., $(z_1, z) \in E(G)$. Due to edge preserving of P_T, we have $(P_T(z_1), z) \in E(G)$, i.e., $(w_1, z) \in E(G) \Rightarrow (y_1, z) \in E(G)$. Since P_T is an edge-preserving mapping, so we have $(P_T(y_1), z) \in E(G)$, i.e., $(v_1, z) \in E(G)$. It follows that $(x_2, z) \in E(G)$. As, we have $(x_1, z), (x_2, z) \in E(G)$, by transitivity of $E(G)$, we have $(x_1, x_2) \in E(G)$. Continuing this process, we get $(x_k, x_{k+1}) \in E(G)$ for any $k \geq 1$.

(b) To show that $(x_k, P_T(x_k)) \in E(G)$ for any $k \geq 1$. We proceed by induction on k. By our assumption $(x_1, P_T(x_1)) \in E(G)$, hence induction is true for $k = 1$. Now suppose that $(x_k, P_T(x_k)) \in E(G)$ for $k \geq 2$. Since $(x_k, x_{k+1}) \in E(G), (x_k, P_T(x_k)) \in E(G)$, we have $(x_{k+1}, P_T(x_k)) \in E(G)$, as G is transitive. Since $(x_k, x_{k+1}) \in E(G)$, we have $(P_T(x_k), P_T(x_{k+1})) \in E(G)$ due to edge-preserving of T. Again, $(x_{k+1}, P_T(x_k)) \in E(G)$, and $(P_T(x_k), P_T(x_{k+1})) \in E(G)$, which gives us that $(x_{k+1}, P_T(x_{k+1})) \in E(G)$.

(c) By part (b), we have $(x_{k+1}, P_T(x_k)) \in E(G)$ for any $k \geq 1$.

□

Lemma 3.2. *Let X be a uniformly convex Banach space and K be a non-empty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to P(V(G))$ be a multi-valued mapping and $P_T : V(G) \to 2^{V(G)}$ be multi-valued $G-$nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, P_T(x_1)) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.2). Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Then (x_k, z) and (z, x_k) are in $E(G)$ for $k \geq 2$.*

Proof. We proceed by induction on k. Since P_T is an edge-preserving and $(x_1, z) \in E(G)$, we have $(P_T(x_1), z) \in E(G)$. By the convexity of $E(G)$ and $(x_1, z), (P_T(x_1), z) \in E(G)$, we have

$$((1 - \alpha_1)x_1 + \alpha_1 u_1, z) \in E(G),$$

i.e, $(z_1, z) \in E(G)$, hence $(P_T(z_1), z) \in E(G)$, i.e., $(w_1, z) \in E(G)$, which implies that $(y_1, z) \in E(G)$. Since P_T is an edge-preserving mapping, so we have $(P_T(y_1), z) \in E(G)$, i.e., $(v_1, z) \in E(G)$. It follows that $(x_2, z) \in E(G)$. Next, we assume that $(x_k, z) \in E(G)$ for $k \geq 2$. Since P_T is an edge-preserving, we have $(P_T(x_k), z) \in E(G)$, i.e., $(u_k, z) \in E(G)$. Since $(x_k, z) \in E(G)$, $(u_k, z) \in E(G)$, By the convexity of $E(G)$, we have $((1 - \alpha_k)x_k + \alpha_k u_k, z) \in E(G)$, i.e., $(z_k, z) \in E(G)$. So, we have $(P_T(z_k), z) \in E(G)$, i.e., $(w_k, z) \in E(G) \Rightarrow (y_k, z) \in E(G)$. By edge preserving of P_T, we have $(P_T(y_k), z) \in E(G)$, i.e., $(v_k, z) \in E(G)$. It follows that $(x_{k+1}, z) \in E(G)$. Therefore, $(x_k, z) \in E(G)$ for $k \geq 2$. Using a similar argument, we can show that $(z, x_k) \in E(G)$. □

Lemma 3.3. *Let X be a uniformly convex Banach space and K be a non-empty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to P(V(G))$ be a multi-valued mapping and $P_T : V(G) \to 2^{V(G)}$ be multi-valued $G-$nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, P_T(x_1)) \in E(G)$. Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.2). Then $\lim_{k\to\infty} ||x_k - z||$ exists for all $z \in F(T)$.*

Proof. Since $z \in F(T)$ and by Lemma 3.2, we have $(x_k, z) \in E(G)$, so by Lemma 2.12, we have $z \in P_T(z) = \{z\}$.

By (1.1) we have,

$$\begin{aligned} ||z_k - z|| &= ||(1-\alpha_k)x_k + \alpha_k u_k - z|| \\ &\le (1-\alpha_k)||x_k - z|| + \alpha_k||u_k - z|| \\ &\le (1-\alpha_k)||x_k - z|| + \alpha_k H(P_T(x_k), P_T(z)) \\ &\le (1-\alpha_k)||x_k - z|| + \alpha_k||x_k - z|| = ||x_k - z||, \end{aligned}$$

$$\begin{aligned} ||y_k - z|| &= ||w_k - z|| \\ &\le H(P_T(z_k), P_T(z)) \\ &\le ||z_k - z||, \end{aligned}$$

and

$$\begin{aligned} ||x_{k+1} - z|| &= ||v_k - z|| \\ &\le H(P_T(y_k), P_T(z)) \\ &\le ||y_k - z|| \\ &\le ||x_k - z||. \end{aligned}$$

It follows that the sequence $\{x_k\}$ is a Fejer monotone with respect to $F(T)$. Hence from the Proposition 2.9, sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty}||x_k - z||$ exists for all $z \in F(T)$. □

Lemma 3.4. *Let X be a uniformly convex Banach space and K be a non-empty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to P(V(G))$ be a multi-valued mapping and $P_T : V(G) \to 2^{V(G)}$ be multi-valued $G-$nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, P_T(x_1)) \in E(G)$. Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.2). Then $\lim_{k\to\infty} d(x_k, Tx_k) = 0$.*

Proof. By Lemma 3.3, we have $\lim_{k\to\infty}||x_k - z||$ exists for all $z \in F(T)$. Let $\lim_{k\to\infty}||x_k - z|| = a$. If $a = 0$, then

$$\begin{aligned} d(x_k, Tx_k) &\le ||x_k - u_k|| \\ &\le ||x_k - z|| + ||z - u_k|| \\ &\le ||x_k - z|| + H(P_T(x_k), P_T(z)) \\ &\le 2||x_k - z|| \\ &\to 0 \ as \ k \to \infty. \end{aligned}$$

Let $a > 0$. Since $\lim_{k\to\infty}||x_k - z|| = a$, we have $\limsup_{k\to\infty}||x_k - z|| \le a$. Also

$$||y_k - z|| \le ||x_k - z|| \Rightarrow \limsup_{k\to\infty}||y_k - z|| \le a.$$

In addition to,

$$\begin{aligned} \limsup_{k\to\infty}||u_k - z|| &\le \limsup_{k\to\infty} H(P_T(x_k), P_T(z)) \\ &\le \limsup_{k\to\infty}||x_k - z|| \\ &\le a. \end{aligned}$$

Also

$$\begin{aligned}\limsup_{k\to\infty}||v_k - z|| &\leq \limsup_{k\to\infty} H(P_T(y_k), P_T(z))\\ &\leq \limsup_{k\to\infty}||y_k - z||\\ &\leq a.\end{aligned}$$

Now for any sequence $\{\gamma_k\}$ in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, we have

$$\begin{aligned}&\limsup_{k\to\infty}||\gamma_k(x_k - z) + (1-\gamma_k)(u_k - z)||\\ &\quad\leq \gamma_k \limsup_{k\to\infty}||x_k - z|| + (1-\gamma_k)\limsup_{k\to\infty}||u_k - z||\\ &\quad\leq a.\end{aligned}$$

Hence from Lemma 2.11, we have $\lim_{k\to\infty}||(x_k - z) - (u_k - z)|| = 0$, i.e.,

$$\lim_{k\to\infty}||x_k - u_k|| = 0.$$

Since

$$d(x_k, Tx_k) \leq ||x_k - u_k||,$$

we have

$$\lim_{k\to\infty} d(x_k, Tx_k) = 0.$$

□

Now, we prove the strong convergence of the iteration scheme defined by (1.2).

Theorem 3.5. *Let X be a uniformly convex Banach space and K be a non-empty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to P(V(G))$ be a multi-valued mapping such that it satisfy Condition (I) and $P_T : V(G) \to 2^{V(G)}$ be multi-valued $G-$nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, P_T(x_1)) \in E(G)$. Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Then the sequence $\{x_k\}$ defined by (1.2) converges strongly to a fixed point of T.*

Proof. From Lemma 3.3, we have

$$||x_{k+1} - z|| \leq ||x_k - z||,$$

it gives that

$$d(x_{k+1}, F(T)) \leq d(x_k, F(T)).$$

This implies that $\lim_{k\to\infty} d(x_k, F(T))$ exists. Since T satisfy Condition (I) and by Lemma 3.2, we have $\lim_{k\to\infty} d(x_k, Tx_k) = 0$, so we have $\lim_{k\to\infty} d(x_k, F(T)) = 0$.
Next we prove that $\{x_k\}$ is Cauchy sequence in K. As, we have $\lim_{k\to\infty} d(x_k, F(T)) = 0$, for $\varepsilon > 0$, there exists a constant k_0 such that for all $k \geq k_0$, we have

$$d(x_k, F(T)) < \frac{\varepsilon}{4}.$$

In particular, there must exists $p \in F(T)$ such that

$$||x_{k_0} - p|| < \frac{\varepsilon}{2}.$$

For $k, m \geq k_0$, we have

$$||x_{k+m} - x_k|| \leq ||x_{k+1} - p|| + ||p - x_k|| < \varepsilon.$$

It follows that $\{x_k\}$ is a Cauchy sequence in K. Since K is closed subset of uniformly convex Banach space X, it must converges in K i.e., there exists a point $x \in K$ such that $\lim_{k\to\infty} ||x_k - x|| = 0$. Now

$$\begin{aligned} 0 \leq d(x, P_T(x)) &\leq ||x_k - x|| + d(x_k, P_T(x_k)) + H(P_T(x_k), P_T(x)) \\ &\leq ||x_k - x|| + ||x_k - u_k|| + ||x_k - x|| \to 0 \ as \ k \to \infty. \end{aligned}$$

It follows that $d(x, P_T(x)) = 0$. By Lemma 2.12, $F(P_T)$ is closed, therefore $x \in F(P_T) = F(T)$. □

Now, we put a numerical example in support of the Theorem 3.5.

Example 3.6. Let X be a real uniformly convex Banach space, whose norm $||.||$ is induced by a metric d such that $d(x, y) = ||x - y||$, for all $x, y \in X$. Let $K = [0, 1]$ be a non-empty subset of X. Let $G = (V(G), E(G))$ be a graph in X such that $V(G) = K$. Let $x, y \in V(G)$ such that $||x|| = ||y|| \leq \frac{1}{2}$, and $(x, y) \in E(G)$ with $||x - y|| \leq 1$. Let $T : V(G) \to P(V(G))$ defined by

$$Tx = \begin{cases} [0, \frac{x+1}{2}], \ x \in [0, \frac{1}{2}), \\ \{0\}, \ x \in [\frac{1}{2}, 1). \end{cases}$$

It is clear that T is edge-preserving mapping. Consider the following cases:

Case I: when $x \in [\frac{1}{2}, 1)$. Then $F(T) = \{0\}$ and from Lemma 2.12, we have $F(T) = \{0\} = F(P_T)$ and $P_T(0) = \{0\}$.

Case II: when $x \in [0, \frac{1}{2})$. Then $F(T) = [0, \frac{x+1}{2}]$. Now

$$\begin{aligned} P_T(x) &= \{y \in Tx : d(x, Tx) = ||y - x||\} \\ P_T(x) &= \{y \in Tx : ||y - x|| = d(x, [0, \frac{x+1}{2}])\} \\ &= \{y \in Tx : ||y - x|| = ||x - \frac{x+1}{2}||\} \\ &= \{y \in Tx : ||y - x|| = ||\frac{x-1}{2}||\} \\ &= \{y = \frac{x+1}{2}\} \end{aligned}$$

Therefore, we have $F(T) = F(P_T) = [0, \frac{x+1}{2}]$

Next, we show that a sequence $\{x_k\} \in K$ defined by (1.2) converges strongly to a point of $F(T)$.
Start with initial value $x_1 = \frac{1}{2}$ and choose $\alpha_k = \frac{2}{3}$, then we have

$$P_T(x_1) = \left\{\frac{x_1 + 1}{2}\right\} = \left\{\frac{3}{4}\right\}.$$

Choose $u_1 \in P_T(x_1)$, then $u_1 = \frac{3}{4}$. Now

$$\begin{aligned} z_1 &= (1-\alpha_1)x_1 + \alpha_1 u_1 \\ &= \frac{1}{6} + \frac{1}{2} \\ &= \frac{2}{3}. \end{aligned}$$

$$P_T(z_1) = \left\{\frac{z_1+1}{2}\right\} = \frac{5}{6}.$$

Choose $w_1 \in P_T(z_1)$, then $w_1 = \frac{5}{6}$. Hence $y_1 = w_1 = \frac{5}{6}$. Choose $v_1 \in P_T(y_1) = \{\frac{y_1+1}{2}\} = \frac{11}{12}$.

Choose $x_2 = v_1 = \frac{11}{12}$, then doing same procedure, we have $u_2 = \frac{23}{24}$, $z_2 = \frac{17}{18}$, $y_2 = w_2 = \frac{35}{36}$ and $x_3 = v_2 = \frac{71}{72}$. Continuing the process, we get that $x_1 < 1$, $x_2 < 1, ... x_n < 1, ...$ Hence, we conclude that sequence $\{x_k\} \in K$ defined by (1.2) converges strongly to a point of $F(T)$.

In the next theorem, we prove the strong convergence of the iterative scheme defined by (1.2), without using Condition (I).

Theorem 3.7. *Let X be a uniformly convex Banach space and K be a non-empty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to P(V(G))$ be a multi-valued mapping and $P_T : V(G) \to 2^{V(G)}$ be multi-valued $G-$nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, P_T(x_1)) \in E(G)$. Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Then the sequence $\{x_k\}$ defined by (1.2) converges strongly to a fixed point of T if and only in $\liminf_{k\to\infty} d(x_k, F(T)) = 0$.*

Proof. If $\liminf_{k\to\infty} d(x_k, F(T)) = 0$, then it is obvious that the sequence $\{x_k\}$ converges strongly to a fixed point of T.
For the converse part, suppose that $\liminf_{k\to\infty} d(x_k, F(T)) = 0$, then $\lim_{k\to\infty} d(x_k, F(T)) = 0$. Using similar argument of the proof as in Theorem 3.5, we obtain that $\{x_k\}$ is a Cauchy sequence in K. Let $\lim_{k\to\infty} x_k = q$. Then

$$\begin{aligned} 0 \leq d(q, P_T(q)) &\leq ||x_k - q|| + d(x_k, P_T(x_k)) + H(P_T(x_k), P_T(q)) \\ &\leq ||x_k - q|| + ||x_k - u_k|| + ||x_k - q|| \to 0 \ as \ k \to \infty. \end{aligned}$$

It follows that $d(q, P_T(x)) = 0$. Since $F(P_T)$ is closed, therefore $q \in F(P_T) = F(T)$. □

Now, we show the fastness of the iteration scheme (1.2) by comparing some well known iteration schemes with the help of following example.

Example 3.8. Let $X = R$ be a uniformly convex Banach space, whose norm $||.||$ is induced by a metric d such that $d(x, y) = ||x - y||$, for all $x, y \in X$. Let $K = [0, \infty)$ be a non-empty subset of X. Let $G = (V(G), E(G))$ be a graph in X such that $V(G) = K$. Let $x, y \in V(G)$ such that $(x, y) \in E(G)$ with $||x - y|| \leq 1$. Let $T : V(G) \to P(V(G))$ defined by

$$Tx = \begin{cases} \{0\}, \ x \in [0, \frac{1}{100}) = A, \\ [0, \frac{x}{5}], \ x \in [\frac{1}{100}, \infty) - \{\frac{8}{7}\} = B, \\ [0, \frac{9}{10}], \ x \in \{\frac{x}{7}\} = C. \end{cases}$$

It is clear that T is edge-preserving mapping. We now show that T is multi-valued nonexpansive mapping. Consider the following cases:

Case I: when $x, y \in A$ and $x, y \in C$, then it is clear that $H(Tx, Ty) \leq ||x - y||$.

Case II: when $x, y \in B$, then

$$\begin{aligned} H(Tx, Ty) &= H([0, \frac{x}{5}], [0, \frac{y}{5}]) \\ &= ||\frac{x}{5} - \frac{y}{5}|| \\ &= \frac{1}{5}||x - y|| < ||x - y||. \end{aligned}$$

Case III: when $x \in A$ and $y = \frac{8}{7}$. Then $H(Tx, Ty) = H(\{0\}, [0, \frac{9}{10}]) = \frac{9}{10} < 1 = ||x - y||$.

Case IV: when $x \in B$ and $y = \frac{8}{7}$. Then

$$\begin{aligned} ||x - y|| = d(x, y) &= d(x, \frac{8}{7}) \\ &= ||x - \frac{8}{7}|| \\ &\leq ||x|| + \frac{8}{7} \end{aligned}$$

and

$$\begin{aligned} H(Tx, Ty) &= H([0, \frac{x}{5}], [0, \frac{9}{10}]) \\ &= ||\frac{x}{5} - \frac{9}{10}|| \\ &\leq ||\frac{x}{5}|| + \frac{9}{10}. \end{aligned}$$

Clearly $H(Tx, Ty) \leq ||x - y||$.

Case V: when $x \in A$ and $y \in B$. Then

$$\begin{aligned} H(Tx, Ty) &= H(\{0\}, [0, \frac{y}{5}]) \\ &= ||\frac{y}{5}|| < ||y|| \\ &< ||x|| + ||y|| \\ &= ||x - y||. \end{aligned}$$

Hence, we conclude that T is multi-valued nonexpansive mapping.

Now with the help of Matlab software program, we compare iteration scheme (1.2) with multi-valued version of different iteration schemes given in [26].

Iteration	Ullah	Thakur	Abbas	Noor	Ishikawa	Picard S
0	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
1	-0.01200000	-0.04400000	- 0.02800000	-0.58800000	-0.62000000	-0.09999911
2	0.00009600	0.00140800	0.00078400	0.32222400	0.34720000	-0.02000009
3	-0.00000026	-0.00002378	-0.00001417	-0.11466401	-0.12190578	0.00240000
4	0.00000000	0.00000024	0.00000018	0.03009930	0.03047644	-0.00005333
5	0.00000000	-0.00000000	-0.00000000	-0.00623946	-0.00580272	0.00000213
6	0.00000000	0.00000000	0.00000000	0.00106579	0.00087685	-0.00000017
7	0.00000000	0.00000000	-0.00000000	-0.00015444	-0.00010809	0.00000001
8	0.00000000	0.00000000	0.00000000	-0.00001939	0.00001108	-0.00000000

Table 1. Strong convergence of multivalued version of Ullah (1.2), Abbas [1], Ishikawa [9], Noor [16], Picard S [8] and Thakur [11] iterations to the fixed point $x = 0$ of T in Example 3.8.

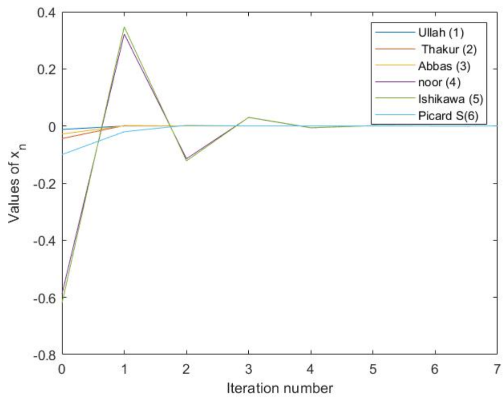

Figure 1. Behaviors of Ullah iteration (magenta), Thakur iteration (carrot orange), Abbas iteration (yellow), Noor iteration (purple), Ishikawa iteration (green), Picard S iteration (cyan) to the fixed point $x = 0$ of the mapping T.

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