

LOW SEPARATION AXIOMS ASSOCIATED WITH $\hat{g}^*$ -CLOSED SETS

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Abstract. In this paper, we introduce $\kappa T_{1/2}$ -spaces, ${}^*T_{1/2}$ -spaces, κT_b -spaces, κT_c -spaces, κT_d -spaces, κT_f -spaces, $\kappa T_{\hat{g}^*}$ -spaces and T_b^κ -spaces and investigate their characterizations.

1. Introduction

Levine [7], Mashhour et al. [13] and Njastad [14] have introduced the concept of semi-open sets, preopen sets and α -open sets, respectively. Levine [8] introduced generalised closed sets and studied their properties. Bhattacharya and Lahiri [3], Arya and Nour [2], Maki et al. [10, 9] introduced semi-generalised closed sets, generalised semi-closed sets and α -generalised closed sets and generalised α -closed sets, respectively. Veerakumar [21] defined $\hat{g}$ -closed sets. Chandrasekararao and Narasimhan [4] introduced and studied g^* -closed sets. Pious Missier and Anto [16] introduced $\hat{g}^*$ -closed sets in topological spaces.

The rest of this paper is organized as follows. In Section 2, we present the fundamental concepts and sets in topological spaces. In Section 3, we introduce $\kappa T_{1/2}$ -spaces, ${}^*T_{1/2}$ -spaces, κT_b -spaces, κT_c -spaces, κT_d -spaces, κT_f -spaces, $\kappa T_{\hat{g}^*}$ -spaces and T_b^κ -spaces and investigate their characterizations.

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2. Preliminaries

Throughout this paper, (X, τ) represents a non-empty topological spaces. For a subset A of a topological space (X, τ) , $cl(A)$, $int(A)$ and A^c or $X - A$ denote the closure, the interior and the complement of A , respectively. The power set of A is denoted by $P(A)$.

Definition 2.1. A subset A of a space (X, τ) is called

- (i) semi-open [7] if $A \subseteq cl(int(A))$ and semi-closed if $int(cl(A)) \subseteq A$,
- (ii) pre-open [13] if $A \subseteq int(cl(A))$ and pre-closed if $cl(int(A)) \subseteq A$,
- (iii) α -open [14] if $A \subseteq int(cl(int(A)))$ and α -closed if $cl(int(cl(A))) \subseteq A$.

Definition 2.2. [21] A subset A of a topological space (X, τ) is called $\hat{g}$ -closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X, τ) .

Definition 2.3. [8] Let A be a subset of a topological space. Then A is called g -closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 2.4. [15] A subset A of a topological space (X, τ) is called sg -closed if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X, τ) .

Definition 2.5. [17] Let A be a subset of a topological space. Then A is called g^* -closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open.

Definition 2.6. [6] Let A be a subset of a topological space. Then A is called g^*s -closed if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open.

The semi-closure (resp. pre closure, α -closure and g^*s closure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. pre closed, α -closed and g^*s closed) sets containing A and is denoted by $scl(A)$ (resp. $pcl(A)$, $\alpha-cl(A)$ and $g^*scl(A)$).

Definition 2.7. [1] A subset A of a topological space (X, τ) is called gs -closed if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .

Definition 2.8. [19] A subset A of a topological space (X, τ) is called $\hat{g}^*$ -closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ -open.

Remark 2.9. Note that M. K. R. S. Veerakumar called this $\hat{g}^*$ -closed set as *g -closed in his paper [19].

Definition 2.10. [22] A subset A of a topological space (X, τ) is called $\hat{g}^*\alpha$ -closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ -open in (X, τ) .

Definition 2.11. [11] A subset A of a topological space (X, τ) is called $g\alpha$ -closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α -open in (X, τ) .

Definition 2.12. [12] A subset A of a topological space (X, τ) is called αg -closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .

Definition 2.13. [18] A subset A of a topological space (X, τ) is called a ψ -closed set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg -open.

Definition 2.14. [13] A topological space (X, τ) is called T_b if every gs -closed set in it is closed.

Definition 2.15. [17] A topological space (X, τ) is called T_c if every g -s-closed set is g^* -closed

Definition 2.16. [5] A topological space (X, τ) is called T_d if every g -s-closed set in it is g -closed.

Definition 2.17. [20] A topological space (X, τ) is called a T_f space if every g -closed set in it is $\hat{g}$ -closed.

Definition 2.18. [8] A topological space (X, τ) is called $T_{1/2}$ if every g -closed set in it is closed.

Definition 2.19. [17] A topological space (X, τ) is called $T_{1/2}^*$ if every g^* -closed set in it is closed.

Definition 2.20. [17] A topological space (X, τ) is called ${}^*T_{1/2}$ if every g -closed set in it is g^* -closed.

Definition 2.21. [6] A topological space (X, τ) is called a strongly semi- $T_{1/2}$ space if every g -s-closed set is g^* -s-closed.

Definition 2.22. [6] A topological space (X, τ) is called a semi- T_b space if every g -s-closed set is semi closed.

Definition 2.23. [16] A subset A of a topological space (X, τ) is called a $\hat{g}^*$ -s-closed set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ -open.

Definition 2.24. [12] A subset A of a topological space (X, τ) is called gp -closed if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
If A is gp -closed, then A^c is gp -open.

Definition 2.25. [6] A topological space (X, τ) is called a semi- T_p space if every g^* -s-closed set is closed.

Lemma 2.26. [19] Let a subset A of a topological space (X, τ) be $\hat{g}^*$ -closed. Then $cl(A) - A$ contains no non-empty $\hat{g}$ -closed set.

Proposition 2.27. [16] Let A be a $\hat{g}^*$ -s-closed subset of X . Then $scl(A) - A$ contains no non-empty $\hat{g}$ closed set in X .

Proposition 2.28. [16] Every semi closed set in X is $\hat{g}^*$ -s-closed in X .

3. Low separation axioms associated with $\hat{g}^*$ -s-closed sets

Definition 3.1. A topological space (X, τ) is called ${}_{\kappa}T_{1/2}$ if every $\hat{g}^*$ -s-closed set in it is semi closed.

Definition 3.2. A topological space (X, τ) is called ${}_{\kappa}T_{1/2}^*$ if every $\hat{g}^*$ -s-closed set in it is g^* -s-closed.

Definition 3.3. A topological space (X, τ) is called ${}_{\kappa}T_b$ if every $\hat{g}^*$ -s-closed set in it is closed.

Definition 3.4. A topological space (X, τ) is called ${}_{\kappa}T_c$ if every $\hat{g}^*$ -s-closed set in it is g^* -closed.

Definition 3.5. A topological space (X, τ) is called ${}_{\kappa}T_d$ if every $\hat{g}^*$ -closed set in it is g -closed.

Definition 3.6. A topological space (X, τ) is called ${}_{\kappa}T_f$ if every $\hat{g}^*$ -closed set in it is $\hat{g}$ -closed.

Definition 3.7. A topological space (X, τ) is called ${}_{\kappa}T_{\hat{g}^*}$ if every $\hat{g}^*$ -closed set in it is $\hat{g}^*$ -closed.

Definition 3.8. A topological space (X, τ) is called T_b^{κ} if every gs -closed set in it is $\hat{g}^*$ -closed.

Proposition 3.9. Every strongly semi- $T_{1/2}$ space is a T_b^{κ} space.

Proof. Suppose (X, τ) is strongly semi- $T_{1/2}$. Let $A \subseteq X$ be gs -closed. Since (X, τ) is strongly semi- $T_{1/2}$, A is g^* -closed. But every g^* -closed set is $\hat{g}^*$ -closed. Therefore A is $\hat{g}^*$ -closed. Hence (X, τ) is T_b^{κ} . $\square$

Remark 3.10. The converse of the above Proposition is not true as seen from the following example.

Example 3.11. Let (X, τ) be a topological space, where $X = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a\}\}$ and $\tau^c = \{\phi, X, \{b, c, d\}\}$. We have $g^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{d\}, \{c\}, \{b\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, c\}, \{a, b\}, \{d\}, \{c\}, \{b\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, c\}, \{a, b\}, \{d\}, \{c\}, \{b\}\}$. Since gs -closed sets are all $\hat{g}^*$ -closed, (X, τ) is T_b^{κ} . As, gs -closed sets are all not g^* -closed, (X, τ) is not strongly semi- $T_{1/2}$.

Proposition 3.12. Every strongly semi- $T_{1/2}$ space is a ${}^*T_{1/2}$ space.

Proof. Suppose (X, τ) be strongly semi- $T_{1/2}$. Let $A \subseteq X$ be $\hat{g}^*$ -closed. But every $\hat{g}^*$ -closed set is gs -closed. Thus A is gs -closed. Since (X, τ) is strongly semi- $T_{1/2}$, A is g^* -closed. Hence (X, τ) is ${}^*T_{1/2}$. $\square$

Remark 3.13. The converse of the above Proposition is not true as seen from the following example.

Example 3.14. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $g^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. Since $\hat{g}^*$ -closed sets are all g^* -closed, (X, τ) is ${}^*T_{1/2}$. As, gs -closed sets are all not g^* -closed, (X, τ) is not strongly semi- $T_{1/2}$.

Proposition 3.15. A topological space (X, τ) is strongly semi- $T_{1/2}$ iff it is both ${}^*T_{1/2}$ and T_b^{κ} .

Proof. Suppose (X, τ) is strongly semi- $T_{1/2}$. Then by Proposition 3.12, (X, τ) is ${}^*T_{1/2}$. And by Proposition 3.9, (X, τ) is T_b^{κ} . Conversely, let (X, τ) be both ${}^*T_{1/2}$ and T_b^{κ} . Let A be gs -closed in (X, τ) . Since (X, τ) is T_b^{κ} , by Definition 3.8, A is $\hat{g}^*$ -closed. Since (X, τ) is ${}^*T_{1/2}$, by Definition 3.2, A is g^* -closed. Therefore (X, τ) is strongly semi- $T_{1/2}$. $\square$

Proposition 3.16. Every semi- T_b space is a T_b^κ space.

Proof. Let (X, τ) be semi- T_b and $A \subseteq X$, gs -closed. Since (X, τ) is semi- T_b , A is semi closed. But every semi closed set is $\hat{g}^*$ s-closed. Therefore A is $\hat{g}^*$ s-closed. Hence (X, τ) is T_b^κ . $\square$

Remark 3.17. The converse of the above Proposition is not true as seen from the following example.

Example 3.18. Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\tau^c = \{\phi, X, \{c, d\}, \{d\}, \{c\}\}$. We have $SC(X, \tau) = \{\phi, X, \{c, d\}, \{d\}, \{c\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}, \{c\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}, \{c\}\}$. Since gs -closed sets are all $\hat{g}^*$ s-closed, (X, τ) is T_b^κ . As, gs -closed sets are all not semi-closed, (X, τ) is not semi- T_b .

Proposition 3.19. Every semi- T_b space is a $\kappa T_{1/2}$ space.

Proof. Suppose (X, τ) be semi- T_b and let $A \subseteq X$ be $\hat{g}^*$ s-closed. But every $\hat{g}^*$ s-closed set is gs -closed. Therefore A is gs -closed. Since (X, τ) is semi- T_b , A is semi closed. Therefore (X, τ) is $\kappa T_{1/2}$. $\square$

Remark 3.20. The converse of the above Proposition is not true as seen from the following example.

Example 3.21. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $SC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. Since $\hat{g}^*$ s-closed sets are all semi closed, (X, τ) is $\kappa T_{1/2}$. As, gs -closed sets are all not semi closed, (X, τ) is not semi- T_b .

Proposition 3.22. A topological space (X, τ) is semi- T_b iff it is both $\kappa T_{1/2}$ and T_b^κ .

Proof. Suppose (X, τ) is semi- T_b . Then by Proposition 3.19, (X, τ) is $\kappa T_{1/2}$. Thus by Proposition 3.16, (X, τ) is T_b^κ .

Conversely, let (X, τ) be both $\kappa T_{1/2}$ and T_b^κ . Let also A be gs -closed in (X, τ) . Since (X, τ) is T_b^κ , by Definition 3.8, A is $\hat{g}^*$ s-closed. Since (X, τ) is $\kappa T_{1/2}$, by Definition 3.1, A is semi closed. Therefore (X, τ) is semi- T_b . $\square$

Proposition 3.23. Every κT_b space is a semi- T_p space.

Proof. Let (X, τ) be a κT_b space. Let $A \subseteq X$ be g^* s-closed. But every g^* s-closed set is a $\hat{g}^*$ s-closed set. Therefore A is $\hat{g}^*$ s-closed in X . Since X is κT_b space, A is closed. Hence (X, τ) is a semi- T_p space. $\square$

Remark 3.24. The converse of the above Proposition is not true as seen from the following example.

Example 3.25. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $SC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. Since $\hat{g}^*$ s-closed sets are all semi closed, (X, τ) is $\kappa T_{1/2}$. As, gs -closed sets are all not semi closed, (X, τ) is not semi- T_p .

Proposition 3.26. Every T_b space is a ${}_{\kappa}T_b$ space.

Proof. Let (X, τ) be a T_b space. Let $A \subseteq X$ be $\hat{g}^*$ -s-closed. But every $\hat{g}^*$ -s-closed set is gs -closed. Therefore A is gs -closed in X . Since (X, τ) is T_b , A is closed. Therefore X is ${}_{\kappa}T_b$. $\square$

Remark 3.27. The converse is not true as seen from the following example.

Example 3.28. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. Since $\hat{g}^*$ -s-closed sets are all closed, (X, τ) is ${}_{\kappa}T_b$. Since gs -closed sets are all not closed, (X, τ) is not T_b .

Proposition 3.29. Every ${}_{\kappa}T_b$ space is a ${}_{\kappa}T_{1/2}$ space.

Proof. Let (X, τ) be a ${}_{\kappa}T_b$ space. Let $A \subseteq X$ be $\hat{g}^*$ -s-closed. Since X is ${}_{\kappa}T_b$, A is closed. Since X is closed, A is semi closed. Therefore X is ${}_{\kappa}T_{1/2}$. $\square$

Remark 3.30. The converse is not true as seen by the following example.

Example 3.31. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $SC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all $\hat{g}^*$ -s-closed sets are semi closed, (X, τ) is ${}_{\kappa}T_{1/2}$. As $\hat{g}^*$ -s-closed sets are all not closed, (X, τ) is not ${}_{\kappa}T_b$.

Proposition 3.32. Every ${}_{\kappa}T_{1/2}$ space is a ${}^*T_{1/2}$ space.

Proof. Let (X, τ) be a ${}_{\kappa}T_{1/2}$ space. Let $A \subseteq X$ be $\hat{g}^*$ -s-closed. Since X is ${}_{\kappa}T_{1/2}$, A is semi closed. But every semi closed set is g^* -s-closed. Therefore X is ${}^*T_{1/2}$ space. $\square$

Remark 3.33. The converse is not true as seen from the following example.

Example 3.34. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{c\}\}$. We have $SC(X, \tau) = \{\phi, X, \{c\}\}$; $g^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. Since every $\hat{g}^*$ -s-closed set is g^* -s-closed, (X, τ) is a ${}^*T_{1/2}$. By the fact that every $\hat{g}^*$ -s-closed set is not semi closed, (X, τ) is not ${}_{\kappa}T_{1/2}$.

Proposition 3.35. Every ${}_{\kappa}T_b$ space is a ${}^*T_{1/2}$ space.

Proof. By Proposition 3.29, every ${}_{\kappa}T_b$ space is a ${}_{\kappa}T_{1/2}$ space. By Proposition 3.32, every ${}_{\kappa}T_{1/2}$ space is a ${}^*T_{1/2}$ space. Therefore, every ${}_{\kappa}T_b$ space is a ${}^*T_{1/2}$ space. $\square$

It is worth mentioning that the converse of the above proposition is not true as seen by the following example.

Example 3.36. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $g^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all $\hat{g}^*$ -s-closed sets are g^* -s-closed, (X, τ) is ${}^*T_{1/2}$. As $\hat{g}^*$ -s-closed sets are all not closed, (X, τ) is not ${}_{\kappa}T_b$.

Proposition 3.37. A topological space (X, τ) is ${}_{\kappa}T_b$ iff it is both ${}^*T_{1/2}$ and semi- T_p .

Proof. Suppose (X, τ) is ${}_{\kappa}T_b$. Then by Proposition 3.35, (X, τ) is ${}^*T_{1/2}$. Due to Proposition 3.23, (X, τ) is semi- T_p .

Conversely, let (X, τ) be both ${}^*T_{1/2}$ and semi- T_p . Let A be $\hat{g}^*$ -closed in (X, τ) . Since (X, τ) is ${}^*T_{1/2}$, by Definition 3.2, A is g^* -closed. By the fact that (X, τ) is semi- T_p , A is closed. $\square$

Proposition 3.38. Every T_c space is a ${}_{\kappa}T_c$ space.

Proof. Let (X, τ) be a T_c space. Let $A \subseteq X$ be $\hat{g}^*$ -closed. But every $\hat{g}^*$ -closed set is gs -closed. Therefore A is gs -closed in X . Since X is T_c , A is g^* -closed. Therefore X is ${}_{\kappa}T_c$. $\square$

Remark 3.39. The converse is not true as seen from the following example.

Example 3.40. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. Since $\hat{g}^*$ -closed sets are all g^* -closed, (X, τ) is ${}_{\kappa}T_c$. As, gs -closed sets are all not g^* -closed, (X, τ) is not T_c .

Proposition 3.41. Every ${}_{\kappa}T_c$ space is a ${}^*T_{1/2}$ space.

Proof. Let (X, τ) be a ${}_{\kappa}T_c$ space. Let $A \subseteq X$ be $\hat{g}^*$ -closed. Since (X, τ) is ${}_{\kappa}T_c$, A is g^* -closed. But every g^* -closed set is g^* -closed. Therefore (X, τ) is a ${}^*T_{1/2}$ space. $\square$

Remark 3.42. The converse is not true as seen from the following example.

Example 3.43. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $g^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are g^* -closed, (X, τ) is ${}^*T_{1/2}$. Since all $\hat{g}^*$ -closed sets are not g^* -closed, (X, τ) is not ${}_{\kappa}T_c$.

Proposition 3.44. Every ${}_{\kappa}T_b$ space is a ${}_{\kappa}T_c$ space.

Proof. Let (X, τ) be a ${}_{\kappa}T_b$ space. Let $A \subseteq X$ be $\hat{g}^*$ -closed. Since (X, τ) is ${}_{\kappa}T_b$, A is closed. But every closed set is g^* -closed. Therefore A is g^* -closed. i.e., (X, τ) is a ${}_{\kappa}T_c$ space. $\square$

Remark 3.45. The converse is not true as seen from the following example.

Example 3.46. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. Since all $\hat{g}^*$ -closed sets are g^* -closed, (X, τ) is ${}_{\kappa}T_c$. Since all $\hat{g}^*$ -closed sets are not closed, (X, τ) is ${}_{\kappa}T_b$.

Proposition 3.47. Every T_c space is a T_b^{κ} space.

Proof. Let (X, τ) be a T_c space. Let $A \subseteq X$ be gs -closed. Since (X, τ) is T_c , A is g^* -closed. But every g^* -closed set is $\hat{g}^*$ -closed. Therefore A is $\hat{g}^*$ -closed. i.e., (X, τ) is a T_b^{κ} space. $\square$

Remark 3.48. The converse is not true as seen from the following example.

Example 3.49. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all gs -closed sets are $\hat{g}^*s$ -closed, (X, τ) is T_b^κ . Since all gs -closed sets are not g^* -closed, (X, τ) is not T_c .

Proposition 3.50. A topological space (X, τ) is T_c iff it is both κT_c and T_b^κ .

Proof. Suppose (X, τ) is T_c . Then by Proposition 3.38, (X, τ) is κT_c . And by Proposition 3.47, (X, τ) is T_b^κ .

Conversely, let (X, τ) be both κT_c and T_b^κ . Let A be gs -closed in (X, τ) . Since (X, τ) is T_b^κ , by Definition 3.8, A is $\hat{g}^*s$ -closed. Since (X, τ) is κT_c , by Definition 3.4, A is g^* -closed. Therefore (X, τ) is a T_c space. $\square$

Proposition 3.51. Every T_b space is T_b^κ .

Proof. Let (X, τ) be a T_b space. Let $A \subseteq X$ be gs -closed. Since (X, τ) is T_b , A is closed. But every closed set is $\hat{g}^*s$ -closed. Therefore A is $\hat{g}^*s$ -closed, i.e., (X, τ) is a T_b^κ space. $\square$

Remark 3.52. The converse is not true as seen from the following example.

Example 3.53. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all gs -closed sets are $\hat{g}^*s$ -closed, (X, τ) is T_b^κ . Since all gs -closed sets are not closed, (X, τ) is not T_b .

Proposition 3.54. A topological space (X, τ) is T_b iff it is both κT_b and T_b^κ .

Proof. Suppose (X, τ) is T_b . Then by Proposition 3.26, (X, τ) is κT_b . And also by Proposition 3.51, (X, τ) is T_b^κ .

Conversely, let (X, τ) be both κT_b and T_b^κ . Suppose A is gs -closed in (X, τ) . Since (X, τ) is T_b^κ , by Definition 3.8, A is $\hat{g}^*s$ -closed. Further, since (X, τ) is κT_b , by Definition 3.3 A is closed. Therefore (X, τ) is a T_b space. $\square$

Proposition 3.55. If a topological space (X, τ) is both κT_d and $T_{1/2}$, then it is κT_b .

Proof. Let (X, τ) be both κT_d and $T_{1/2}$. Let A be $\hat{g}^*s$ -closed in (X, τ) . Since (X, τ) is κT_d , by Definition 3.5, A is g -closed. Since (X, τ) is $T_{1/2}$, A is closed. Therefore (X, τ) is a κT_b space. $\square$

Proposition 3.56. If a topological space (X, τ) is both κT_d and $^*T_{1/2}$, then it is κT_c .

Proof. Let (X, τ) be both κT_d and $^*T_{1/2}$. Suppose that A is $\hat{g}^*s$ -closed in (X, τ) . Since (X, τ) is κT_d , by Definition 3.5, A is g -closed. But the space (X, τ) is $^*T_{1/2}$ and hence A is g^* -closed. Therefore (X, τ) is a κT_c space. $\square$

Proposition 3.57. Every κT_b space is a $T_{1/2}^*$ space.

Proof. Let (X, τ) be a κT_b space. Let $A \subseteq X$ be g^* -closed. But every g^* -closed set is $\hat{g}^*s$ -closed. Therefore A is $\hat{g}^*s$ -closed. Since (X, τ) is κT_b , A is closed. Therefore A is closed. i.e. (X, τ) is a $T_{1/2}^*$ space. $\square$

Remark 3.58. The converse is not true as seen from the following example.

Example 3.59. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all g^* -closed sets are closed, (X, τ) is $T_{1/2}^*$. By the fact that all $\hat{g}^*$ -closed sets are not closed, (X, τ) is not κT_b .

Proposition 3.60. A topological space (X, τ) is κT_b iff it is both κT_c and $T_{1/2}^*$.

Proof. Suppose (X, τ) is κT_b . Then by Proposition 3.44, (X, τ) is κT_c . And by Proposition 3.57, (X, τ) is $T_{1/2}^*$.

Conversely, let (X, τ) be both κT_c and $T_{1/2}^*$. Let A be $\hat{g}^*$ -closed in (X, τ) . Since (X, τ) is κT_c , by Definition 3.4, A is g^* -closed. Since (X, τ) is $T_{1/2}^*$, A is closed. Therefore (X, τ) is a κT_b . $\square$

Remark 3.61. T_c space and κT_b space are independent of each other as seen from the following two examples.

Example 3.62. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are closed, (X, τ) is κT_b . Since all gs -closed sets are not g^* -closed, (X, τ) is not T_c .

Example 3.63. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. Since all gs -closed sets are g^* -closed, (X, τ) is T_c . But since all $\hat{g}^*$ -closed sets are not closed, the space (X, τ) is not κT_b .

It follows from Examples 3.62 and 3.63, κT_b space and T_c space are independent of each other.

Remark 3.64. $^*T_{1/2}$ space and κT_c are independent of each other as seen from the following two examples.

Example 3.65. Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a\}, \{a, b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c, d\}, \{d\}\}$. We have $gC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{a, d\}, \{d\}\}$; $g^*C(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{a, d\}, \{d\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{d\}, \{c\}, \{b\}\}$. Since all g -closed sets are g^* -closed, (X, τ) is $^*T_{1/2}$. Since all $\hat{g}^*$ -closed sets are not g^* -closed, (X, τ) is not κT_c .

Example 3.66. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $gC(X, \tau) = P(X)$; $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are g^* -closed, (X, τ) is κT_c . And Since all g -closed sets are not g^* -closed, (X, τ) is not $^*T_{1/2}$. It follows from Examples 3.65 and 3.66, $^*T_{1/2}$ space and κT_c space are independent of each other.

Remark 3.67. ${}^*T_{1/2}$ space and ${}_{\kappa}T_b$ space are independent of each other as seen from the following two examples.

Example 3.68. Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a\}, \{a, b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c, d\}, \{d\}\}$. We have $gC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{a, d\}, \{d\}\}$; $g^*C(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{a, d\}, \{d\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{d\}, \{c\}, \{b\}\}$. Since all g -closed sets are g^* -closed, (X, τ) is ${}^*T_{1/2}$. Since all $\hat{g}^*$ -closed sets are not closed, (X, τ) is not ${}_{\kappa}T_b$.

Example 3.69. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{a\}, \{b, c\}\}$. We have $gC(X, \tau) = P(X)$; $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are closed, (X, τ) is ${}_{\kappa}T_b$. It is obvious that all g -closed sets are not g^* -closed, thus (X, τ) is not ${}^*T_{1/2}$. Therefore from Examples 3.68 and 3.69, ${}^*T_{1/2}$ space and ${}_{\kappa}T_b$ space are independent of each other.

Remark 3.70. semi- $T_{1/2}$ space and ${}_{\kappa}T_{1/2}$ space are independent of each other as seen from the following two examples.

Example 3.71. Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a\}, \{a, b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c, d\}, \{d\}\}$. We have $SC(X, \tau) = \{\phi, X, \{b, c, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{d\}, \{c\}, \{b\}\}$; $sgC(X, \tau) = \{\phi, X, \{b, c, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{d\}, \{c\}, \{b\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{d\}, \{c\}, \{b\}\}$. Since all sg -closed sets are semi closed, (X, τ) is semi- $T_{1/2}$. Moreover, since all $\hat{g}^*$ -closed sets are not semi closed, (X, τ) is not ${}_{\kappa}T_{1/2}$.

Example 3.72. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $SC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $sgC(X, \tau) = P(X)$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are semi closed, (X, τ) is ${}_{\kappa}T_{1/2}$. Since all sg -closed sets are not semi closed, (X, τ) is not semi- $T_{1/2}$. Therefore from Examples 3.71 and 3.72, semi- $T_{1/2}$ space and ${}_{\kappa}T_{1/2}$ space are independent of each other.

Remark 3.73. ${}_{\kappa}T_{1/2}$ and ${}_{\kappa}T_c$ are independent of each other as seen from the following two examples.

Example 3.74. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $SC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$; $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are semi closed, (X, τ) is ${}_{\kappa}T_{1/2}$. Since all $\hat{g}^*$ -closed sets are not g^* -closed, (X, τ) is not ${}_{\kappa}T_c$.

Example 3.75. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{c\}\}$. We have $SC(X, \tau) = \{\phi, X, \{c\}\}$;

$g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. Since all $\hat{g}^*$ -closed sets are not semi closed, (X, τ) is not ${}_{\kappa}T_{1/2}$. Since all $\hat{g}^*$ -closed sets are g^* -closed, (X, τ) is ${}_{\kappa}T_c$.

It follows from Examples 3.74 and 3.75, ${}_{\kappa}T_{1/2}$ and ${}_{\kappa}T_c$ are independent of each other.

Remark 3.76. ${}_{\kappa}T_b$ and T_b^{κ} are independent of each other as seen from the following two examples.

Example 3.77. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $gsC(X, \tau) = P(X)$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$. Since all $\hat{g}^*$ -closed sets are closed, (X, τ) is ${}_{\kappa}T_b$. Since all gs -closed sets are not $\hat{g}^*$ -closed, (X, τ) is not T_b^{κ} .

Example 3.78. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{c\}\}$. We have $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. Since all gs -closed sets are $\hat{g}^*$ -closed, (X, τ) is T_b^{κ} . Since all $\hat{g}^*$ -closed sets are not closed, (X, τ) is ${}_{\kappa}T_b$.

It follows from Examples 3.77 and 3.78, T_b^{κ} and ${}_{\kappa}T_b$ are independent of each other.

Remark 3.79. T_b^{κ} and ${}_{\kappa}T_c$ are independent of each other as seen from the following two examples.

Example 3.80. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a\}\}$ and $gsC(X, \tau) = P(X)$.

Since all $\hat{g}^*$ -closed sets are g^* -closed, (X, τ) is ${}_{\kappa}T_c$. And since all gs -closed sets are not $\hat{g}^*$ -closed, (X, τ) is not T_b^{κ} .

Example 3.81. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}\}$ and $gsC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}\}$. Since all $\hat{g}^*$ -closed sets are not g^* -closed, (X, τ) is not ${}_{\kappa}T_c$. Since all gs -closed sets are $\hat{g}^*$ -closed, (X, τ) is T_b^{κ} .

Therefore from Examples 3.80 and 3.81, T_b^{κ} and ${}_{\kappa}T_c$ are independent of each other.

Remark 3.82. From the above Definitions and Remarks, we observe that semi- T_b implies not only strongly semi- $T_{1/2}$ but also T_b^k and ${}_kT_{1/2}$. Moreover, T_b implies not only semi- T_b and $T_{1/2}$ but also T_c , ${}_kT_b$ and T_b^k . Both semi- T_b and strongly semi- $T_{1/2}$ imply T_b^k . Also strongly semi- $T_{1/2}$ implies ${}^*{}_kT_{1/2}$. Further, T_c implies not only T_b^k and strongly semi- $T_{1/2}$ but also ${}_kT_c$ and ${}^*T_{1/2}$. We observe that ${}_kT_b$ implies ${}_kT_c$, ${}^*T_{1/2}$, semi- T_p and ${}_kT_{1/2}$. $T_{1/2}$ implies both ${}^*T_{1/2}$ and $T_{1/2}^*$. Also, Semi- T_p implies $T_{1/2}^*$.

Proposition 3.83. In a topological space (X, τ) which is both T_d and T_f , every $\hat{g}^*$ -closed set in it is $\hat{g}$ -closed.

Proof. Let (X, τ) be both T_d and T_f . Let $A \subseteq X$ be $\hat{g}^*$ -closed. But every $\hat{g}^*$ -closed set is gs -closed a set. Therefore A is gs -closed. Since X is T_d , A is g -closed. By the fact that X is T_d , then A is g -closed. $\square$

Proposition 3.84. Every $\hat{g}^*$ -closed set in a ${}_{\kappa}T_b$ space is

- (i) g -closed.

- (ii) $\hat{g}$ -closed.
- (iii) g^* -closed.
- (iv) $\hat{g}^*$ -closed.
- (v) semi closed.
- (vi) g^*s -closed.
- (vii) sg -closed.
- (viii) gs -closed.
- (ix) ψ -closed.
- (x) α -closed.
- (xi) αg -closed.
- (xii) $g\alpha$ -closed.
- (xiii) $\hat{g}^*\alpha$ -closed.
- (xiv) pre closed.
- (xv) (xv) gp closed.

Proof. (i) Let X be a κT_b space and A $\hat{g}^*s$ -closed in X . Consequently, by Definition 3.3, A is closed. But every closed set is g -closed. Therefore A is g -closed.

- (ii) Since every closed set is $\hat{g}$ -closed.
- (iii) Since every closed set is g^* -closed.
- (iv) Since every closed set is $\hat{g}^*$ -closed.
- (v) Since every closed set is semi closed.
- (vi) Since every semi closed set is g^*s -closed.
- (vii) Since every semi closed set is sg -closed.
- (viii) Since every semi closed set is gs -closed.
- (ix) Since every semi closed set is ψ -closed.
- (x) Since every closed set is α -closed.
- (xi) Since every α -closed set is $g\alpha$ -closed.
- (xii) Since every $g\alpha$ -closed set is αg -closed.
- (xiii) Since every α -closed set is $\hat{g}^*\alpha$ -closed.
- (xiv) Since every closed set is pre-closed.
- (xv) Since every pre closed set is gp -closed.

□

Proposition 3.85. Every gs -closed set in a T_b space is $\hat{g}^*s$ -closed.

Proof. Let X be a T_b space and A a gs -closed set in X .

Consequently, A is closed. But every closed set is semi closed and every semi closed set is $\hat{g}^*s$ -closed. Therefore A is $\hat{g}^*s$ -closed. □

Proposition 3.86. Every gs -closed set in a T_c space is $\hat{g}^*s$ -closed.

Proof. Let X be a T_c space and A a gs -closed set in X .

Consequently, A is g^* -closed. But every g^* -closed set is g^*s -closed and every g^*s -closed set is $\hat{g}^*s$ -closed. Therefore A is $\hat{g}^*s$ -closed. □

Proposition 3.87. Every $\hat{g}^*s$ -closed set in a $\kappa T_{1/2}$ space is

- (i) sg -closed.
- (ii) g^*s -closed.

(iii) ψ -closed.

Proof. (i) Let X be a $\kappa T_{1/2}$ space and A be $\hat{g}^*$ -closed in X . Consequently, A is semi closed. But every semi closed set is sg -closed. Hence A is sg -closed.

(ii) Since every semi closed set is g^* -closed.

(iii) Since every semi closed set is ψ -closed. □

Proposition 3.88. Every $\hat{g}^*$ -closed set in a κT_c space is

(i) $\hat{g}^*$ -closed and hence g -closed.

(ii) g^* -closed.

Proof. (i) Let X be a κT_c space and A be $\hat{g}^*$ -closed in X . Consequently, A is g^* -closed. But every g^* -closed set is $\hat{g}^*$ -closed and every $\hat{g}^*$ -closed set is g -closed. Hence (i) follows.

(ii) Since every g^* -closed set is g^* -closed. Therefore A is g^* -closed. □

Proposition 3.89. Every $\hat{g}^*$ -closed set in a $\kappa T_{\hat{g}^*}$ space is g -closed.

Proof. Let X be a $\kappa T_{\hat{g}^*}$ space and $A \subseteq X$ be a $\hat{g}^*$ -closed set. Since X is $\kappa T_{\hat{g}^*}$, A is $\hat{g}^*$ -closed. But every $\hat{g}^*$ -closed set is g -closed. Therefore, A is g -closed. □

Proposition 3.90. Let the topological space (X, τ) be a κT_c space. Then for each $x \in X$, $\{x\}$ is $\hat{g}$ -closed or g^* -open.

Proof. Suppose $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Therefore X is the only $\hat{g}$ -open set containing $X - \{x\}$. Hence $scl(X - \{x\}) \subseteq X$.

Therefore $X - \{x\}$ is $\hat{g}^*$ -closed. But in a κT_c space, every $\hat{g}^*$ -closed set is g^* -closed. Thus, $X - \{x\}$ is g^* -closed. Hence $\{x\}$ is g^* -open. □

Remark 3.91. The converse need not be true as seen from the following example.

Example 3.92. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $g^*C(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$; $g^*O(X, \tau) = \{\phi, X, \{a, b\}, \{b\}, \{a\}\}$; $\hat{g}C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \}$. In this example, for every $x \in X$, $\{x\}$ is either $\hat{g}$ -closed or g^* -open. But, not every $\hat{g}^*$ -closed set is g^* -closed, i.e., (X, τ) is not κT_c .

Proposition 3.93. Let the topological space (X, τ) be a κT_b space. Then for each $x \in X$, $\{x\}$ is $\hat{g}$ -closed or open.

Proof. Suppose $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Therefore X is the only $\hat{g}$ -open set containing $X - \{x\}$. Thus $scl(X - \{x\}) \subseteq X$. Therefore $X - \{x\}$ is $\hat{g}^*$ -closed. But in a κT_b space, every $\hat{g}^*$ -closed set is closed. Hence $X - \{x\}$ is closed. It follows that $\{x\}$ is open. □

Remark 3.94. The converse of the above proposition need not be true as seen from the following example.

Example 3.95. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $\hat{g}C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \}$. In this example, for every $x \in X$, $\{x\}$ is either $\hat{g}$ -closed or open. But, not every $\hat{g}^*s$ -closed set is closed, i.e., (X, τ) is not κT_b .

Proposition 3.96. Let the topological space (X, τ) be a κT_d space. Then for each $x \in X$, $\{x\}$ is $\hat{g}$ -closed or g -open.

Proof. Suppose $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Then X is the only $\hat{g}$ -open set containing $X - \{x\}$. Therefore $scl(X - \{x\}) \subseteq X$.

Therefore $X - \{x\}$ is $\hat{g}^*s$ -closed. But in a κT_d space, every $\hat{g}^*s$ -closed set is g -closed. Hence $X - \{x\}$ is g -closed. Therefore $\{x\}$ is g -open. $\square$

Remark 3.97. The converse of the above proposition need not be true as seen from the following example.

Example 3.98. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $\hat{g}C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $gO(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$; $gC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \}$. From this example, we find out $\{x\}$ is either $\hat{g}$ -closed or g -open. But not every $\hat{g}^*s$ -closed set is g -closed. Therefore (X, τ) is not a κT_d space.

Proposition 3.99. Let the topological space (X, τ) be a κT_f space. Then for each $x \in X$, $\{x\}$ is $\hat{g}$ -closed or $\hat{g}$ -open.

Proof. Suppose $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Then X is the only $\hat{g}$ -open set containing $X - \{x\}$. Therefore $scl(X - \{x\}) \subseteq X$. This gives $X - \{x\}$ is $\hat{g}^*s$ -closed. But in a κT_f space, every $\hat{g}^*s$ -closed set is $\hat{g}$ -closed. Thus $X - \{x\}$ is $\hat{g}$ -closed. Hence $\{x\}$ is $\hat{g}$ -open. $\square$

Remark 3.100. The converse of the above proposition need not be true as seen from the following Example.

Example 3.101. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$. We have $\hat{g}C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $\hat{g}O(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \}$. In this example, for every $x \in X$, $\{x\}$ is either $\hat{g}$ -closed or $\hat{g}$ -open. But, not every $\hat{g}^*s$ -closed set is $\hat{g}$ -closed. This means that (X, τ) is not κT_f .

Proposition 3.102. Let the topological space (X, τ) be a $\kappa T_{\hat{g}^*}$ space. Then for each $x \in X$, $\{x\}$ is either $\hat{g}$ -closed or $\hat{g}^*$ -open.

Proof. Suppose that for some $x \in X$, $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Thus X is the only $\hat{g}$ -open set containing $X - \{x\}$. Therefore $scl(X - \{x\}) \subseteq X$. Hence $X - \{x\}$ is $\hat{g}^*s$ -closed. Since X is $\kappa T_{\hat{g}^*}$, $X - \{x\}$ is $\hat{g}^*$ -closed. It follows that $\{x\}$ is $\hat{g}^*$ -open. $\square$

Remark 3.103. The converse need not be true as seen from the following example.

Example 3.104. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^c = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$.

We have $\hat{g}C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$; $\hat{g}^*O(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$; $\hat{g}^*C(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}$ and $\hat{g}^*sC(X, \tau) = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. In this example, for every $x \in X$, $\{x\}$ is either $\hat{g}$ -closed or $\hat{g}^*$ -open. But, not every $\hat{g}^*$ -closed set is $\hat{g}^*$ -closed, i.e., (X, τ) is not $\kappa T_{\hat{g}^*}$.

Proposition 3.105. For a topological space (X, τ) , the following are equivalent.

- (i) Suppose (X, τ) is $\kappa T_{1/2}$.
- (ii) Every $\hat{g}^*$ -closed set in it is semi closed.
- (iii) Every singleton $\{x\}$ of X is either semi open or $\hat{g}$ -closed.

Proof. (i) $\Rightarrow$ (iii) Suppose that for some $x \in X$, $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Then X is the only $\hat{g}$ -open set containing $X - \{x\}$. Thus $scl(X - \{x\}) \subseteq X$. Therefore $X - \{x\}$ is $\hat{g}^*$ -closed. Since X is $\kappa T_{1/2}$, $X - \{x\}$ is semi closed. Hence $\{x\}$ is semi open.

(iii) $\Rightarrow$ (i)

Let A be $\hat{g}^*$ -closed in X . To show that A is semi closed in X , it means to show that $scl(A) = A$.

Let $x \in scl(A)$. Then $\{x\}$ is $\hat{g}$ -closed or semi open.

Case (i) : Let $\{x\}$ be $\hat{g}$ -closed.

Suppose $x \notin A$. Therefore $x \in scl(A) - A$. Thus $\{x\} \subseteq scl(A) - A$. Then by Proposition 2.27, A is not $\hat{g}^*$ -closed in X which is a contradiction. Therefore $x \in A$.

Case (ii) : Suppose $\{x\}$ is semi open. Then $\{x\} \cap A \neq \phi$. Therefore $x \in A$. Thus in both cases $x \in A$. Thus $scl(A) \subseteq A$. Hence A is semi closed in X . It follows that X is $\kappa T_{1/2}$.

(ii) $\Rightarrow$ (iii)

Let $x \in X$. Suppose that for some $x \in X$, $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. It follows that X is the only $\hat{g}$ -open set containing $X - \{x\}$. Thus $cl(X - \{x\}) \subseteq X$. Therefore $X - \{x\}$ is $\hat{g}^*$ -closed. Therefore, by assumption, $X - \{x\}$ is semi closed. Hence $\{x\}$ is semi open.

(iii) $\Rightarrow$ (ii)

Let A be $\hat{g}^*$ -closed in X . Showing that A is semi closed in X , it means that to show that $scl(A) = A$. Let $x \in scl(A)$. Then $\{x\}$ is $\hat{g}$ -closed or semi open.

Case (i) : Let $\{x\}$ be $\hat{g}$ -closed. Suppose $x \notin A$. Therefore $x \in scl(A) - A$. Thus $\{x\} \subseteq scl(A) - A$. Therefore $\{x\} \subseteq cl(A) - A$. It follows, by Lemma 2.26, A is not $\hat{g}^*$ -closed in X which is a contradiction. Therefore $x \in A$.

Case (ii) : Suppose $\{x\}$ is semi open. Then $\{x\} \cap A \neq \phi$. Therefore $x \in A$. Thus in both cases $x \in A$. This means that $scl(A) \subseteq A$. Therefore A is semi closed. $\square$

Proposition 3.106. For a topological space (X, τ) , the following are equivalent.

- (i) Suppose (X, τ) is $\kappa^*T_{1/2}$.
- (ii) Every $\hat{g}^*$ -closed set in it is g^* -closed.
- (iii) Every singleton $\{x\}$ of X is $\hat{g}$ -closed or g^* -open.

Proof. (i) $\Rightarrow$ (iii)

Suppose that for some $x \in X$, $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. Thus X is the only $\hat{g}$ -open set containing $X - \{x\}$. Therefore $scl(X - \{x\}) \subseteq X$. It follows

that $X - \{x\}$ is $\hat{g}^*s$ -closed. Since X is ${}^*_\kappa T_{1/2}$, $X - \{x\}$ is g^*s -closed. Therefore $\{x\}$ is g^*s -open.

(iii) $\Rightarrow$ (i)

Let A be $\hat{g}^*s$ -closed in X . To show that A is g^*s -closed in X . i.e. to show that $g^*scl(A) = A$. Let $x \in g^*scl(A)$. Then $\{x\}$ is $\hat{g}$ -closed or g^*s -open.

Case (i) : Let $\{x\}$ be $\hat{g}$ -closed. Since $x \in g^*scl(A)$, by Proposition 2.28, we have $x \in scl(A)$. Suppose $x \notin A$. Therefore $x \in scl(A) - A$. It follows that $\{x\} \subseteq scl(A) - A$. Then, by Proposition 2.27, A is not g^*s -closed which is a contradiction. Hence $x \in A$.

Case (ii) : Suppose $\{x\}$ is g^*s -open. Then $\{x\} \cap A \neq \emptyset$. Therefore $x \in A$. Thus in both cases $x \in A$. Thus $g^*scl(A) \subseteq A$. Therefore A is $\hat{g}^*s$ -closed in X . Hence X is ${}^*_\kappa T_{1/2}$.

(ii) $\Rightarrow$ (iii)

Suppose that for some $x \in X$, $\{x\}$ is not $\hat{g}$ -closed. Then $X - \{x\}$ is not $\hat{g}$ -open. This implies that X is the only $\hat{g}$ -open set containing $X - \{x\}$. Therefore $cl(X - \{x\}) \subseteq X$. Thus $X - \{x\}$ is g^* -closed. By our assumption, every $\hat{g}^*$ -closed set is g^*s -closed. Therefore $X - \{x\}$ is g^*s -closed. Hence $\{x\}$ is g^*s -open.

(iii) $\Rightarrow$ (ii)

Let A be $\hat{g}^*$ -closed in X . To show that A is g^*s -closed in X . i.e. to show that $g^*scl(A) = A$. Let $x \in g^*scl(A)$. Then $\{x\}$ is $\hat{g}$ -closed or g^*s -open.

Case (i) : Let $\{x\}$ be $\hat{g}$ -closed. Since, every closed set is g^*s -closed, we have $g^*scl(A) \subseteq cl(A)$, we have $x \in cl(A)$. Suppose $x \notin A$. Therefore $x \in cl(A) - A$. It follows that $\{x\} \subseteq cl(A) - A$. Thus, by Lemma 2.26, A is not $\hat{g}^*$ -closed in X which is a contradiction. Hence $x \in A$.

Case (ii) : Suppose $\{x\}$ is g^*s -open. Then $\{x\} \cap A \neq \emptyset$. Therefore $x \in A$. Thus in both cases $x \in A$. It follows that $g^*scl(A) \subseteq A$. Therefore A is g^*s -closed in X . Hence X is ${}^*_\kappa T_{1/2}$. $\square$

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