Poincare Journal of Analysis & Applications
Vol. 9, No. 2 (2022), 369-376

CERTAIN PROPERTIES ASSOCIATED WITH SCHAUDER FRAMES IN BANACH SPACES

JHAVI LAL GHIMIRE† AND NARAYAN PRASAD PAHARI

Date of Receiving	:	06. 11. 2022
Date of Revision	:	26. 12. 2022
Date of Acceptance	:	26. 12. 2022

Abstract. Schauder frames satisfying property (I) and property (II) has been defined and studied. It has been proved that an atomic decomposition satisfies property (I) if and only if it satisfies property (II). Also, Schauder frames satisfying property ($\mathcal{M}$) has been defined and it has been proved that, in a uniformly convex Banach space, if a Schauder frame satisfies property ($\mathcal{M}$), then it also satisfies property (II) (and hence property (I)). Further, we define atomic decompositions satisfying property ($\mathcal{B}$) and prove that property ($\mathcal{B}$) is a necessary condition for an atomic decomposition satisfying property (I). Finally, we define property ($\mathcal{SB}$) for a Schauder frame and gave a necessary condition for it.

1. Introduction

Frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [7] and reintroduced by Daubechies et al. [6]. Today, frames are main tools for use in signal and image processing, compression, sampling theory, optics, filter banks, signal detection etc.

Coifman and Weiss [5], introduced the notion of atomic decompositions for function spaces. Later, Feichtinger and Gröchenig [8] extended the notion of atomic decomposition to Banach spaces. Gröchenig [9] introduced a more general concept for Banach spaces called Banach frame. Retro Banach frames for dual Banach spaces were introduced and studied in [12] and the notion of Weak*-Schauder frames for dual Banach spaces were introduced and studied in [14]. In [3], Casazza, et al. gave various definitions of frames for Banach spaces including that of Schauder frame. Later, Han and Larson [10] defined Schauder frame for a Banach space. In 2008, Casazza, et al. [2] studied the coefficient quantization of Schauder frames in Banach spaces. Liu [15] gave the concepts of minimal and maximal associated bases with respect to Schauder frames and closely connected them with the duality theory. In [17], Liu and Zheng gave a characterization of Schauder frames which are near-Schauder bases. Infact, they

2000 *Mathematics Subject Classification.* 42C15, 42A38.
Key words and phrases. Schauder frames, Reconstruction property, and Banach frames.
Communicated by. Sumit Kumar Sharma
† *Corresponding author*

generalized some results due to Holub [11]. Beanland et al.[1], proved that the upper and lower estimates theorems for finite dimensional decompositions of Banach spaces can be extended and modified to Schauder frames, and gave a complete characterization of duality for Schauder frames. Φ-Schauder frames were introduced and studied by Vashisht [18]. Liu [16] associated an operator with a Schauder frame and called it Hilbert-Schauder frame operator.

In the present paper, we study Schauder frames in Banach spaces and discussed property (I) and property (II) associated with them and proved that the two properties are equivalent. Also, Schauder frames satisfying property $(\mathcal{M})$ has been defined and it has been proved that, in a uniformly convex Banach space, if a Schauder frame satisfies property $(\mathcal{M})$, then it also satisfies property (II) (and hence property (I)). Further, Schauder frames satisfying property $(\mathcal{B})$ has been defined and it has been proved that if a Schauder frame satisfies property (I) (or property (II)), then it also satisfies property $(\mathcal{B})$. Finally, we define Schauder frame satisfying property $(\mathcal{SB})$ and gave a necessary condition for a Schauder frame to satisfy property $(\mathcal{SB})$.

Throughout the paper, E will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, E^* and E^{**}, respectively, the first and second conjugate spaces of E, indexed by $\mathbb{N}$, $[f_n]$ the closed linear span of $\{f_n\}$ and $[\widetilde{f_n}]$ the closed linear span of $\{f_n\}$ in the $\sigma(\mathrm{E}^*, \mathrm{E})$-topology. A sequence $\{f_n\}$ in E^* is said to be complete if $[f_n] = \mathrm{E}^*$ and total if $\{x \in \mathrm{E} : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition 1.1 ([10])**.** Let E be a Banach space, $\{x_n\}$ be a sequence in E and $\{f_n\}$ be a sequence in E^*. Then, the pair $(\{f_n\}, \{x_n\})$ is called a *Schauder frame* for E if

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \ x \in \mathrm{E}.$$

The positive constants A, B are called *Schauder frame bounds* for the Schauder frame $(\{f_n\}, \{x_n\})$.

2. Main Results

Definition 2.1. Let E be a Banach space, $(\{f_n\}, \{x_n\})(\{f_n\} \subset \mathrm{E}^*, \{x_n\} \subset \mathrm{E})$ be a Schauder frame for E. Then,

(a) $(\{f_n\}, \{x_n\})$ is said to satisfy *property (I)*, if for each $c > 0$ there exists a number $\gamma_c > 0$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i f_i(x)x_i \right\| = 1, \ \left\| \sum_{i=n+1}^{\infty} \lambda_i f_i(x)x_i \right\| \geq c \ \Rightarrow \ \left\| \sum_{i=1}^{\infty} \lambda_i f_i(x)x_i \right\| \geq 1 + \gamma_c,$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is any family of weights, i.e., $\lambda_i > 0$ for all $i \in \mathbb{N}$ and $x \in \mathrm{E}$.

(b) $(\{f_n\}, \{x_n\})$ is said to satisfy *property (II)*, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i f_i(x)x_i \right\| > 1 - \delta, \ \left\| \sum_{i=1}^{\infty} \lambda_i f_i(x)x_i \right\| = 1 \ \Rightarrow \ \left\| \sum_{i=n+1}^{\infty} \lambda_i f_i(x)x_i \right\| \leq \varepsilon,$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is any family of weights and $x \in \mathrm{E}$.

Regarding the existence of property (I), one may observe that if $\{e_n\}$ is the unit vector sequence of $\mathrm{E} = \ell^1$ and $\{f_n\}$ is the unit vector sequence in E^*, then $(\{f_n\}, \{e_n\})$ is a Schauder frame for E satisfying property (I). Also, the Schauder frame defined in the Example 2.2 does not satisfy property (I).

Example 2.2. Let $\mathrm{E} = l^1$. Let $\{e_n\}$ be a sequence of unit vectors in E and $\{h_n\}$ be a sequence of unit vectors in E^*. Define $\{x_n\} \subset \mathrm{E}$ and $\{f_n\} \subset \mathrm{E}^*$ by

$$x_1 = \frac{1}{2}(e_1 + e_2), \quad x_2 = \frac{1}{2}(-e_1 + e_2), \quad x_n = e_n, \quad n \geq 3 \quad \text{and}$$
$$f_1 = h_1 + h_2, \quad f_2 = -h_1 + h_2, \quad f_n = h_n, \quad n \geq 3.$$

However, it does not satisfy property (I). Indeed, for $x = (0, 1, 0, 0, 0 \ldots) \in \mathrm{E}$ and for any sequence of weights $\{\lambda_i\}_{i=i}^{\infty}$ with $\lambda_1 = \lambda_2 = 1$, we have

$$\|\lambda_1 f_1(x) x_1\| = 1, \quad \left\| \sum_{i=2}^{\infty} \lambda_i f_i(x) x_i \right\| = 1 \quad \text{but} \quad \left\| \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i \right\| = 1.$$

In the following result we prove that the two properties are equivalent:

Theorem 2.3. *For a Schauder frame* $(\{f_n\}, \{x_n\})$ $(\{f_n\} \subset \mathrm{E}^*, \{x_n\} \subset \mathrm{E})$ *in a Banach space* E *the property (I) and property (II) are equivalent.*

Proof. Suppose that $(\{f_n\}, \{x_n\})$ is a Schauder frame satisfying property (I) but not property (II). Then, there exists an $\varepsilon > 0$, $x \in \mathrm{E}$, family of weights $\{\lambda_i\}_{i=1}^{\infty}$ and $n \in \mathbb{N}$ such that for each $\delta > 0$

$$\left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| > 1 - \delta, \quad \left\| \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i \right\| = 1 \quad \text{but} \quad \left\| \sum_{i=n+1}^{\infty} \lambda_i f_i(x) x_i \right\| > \varepsilon.$$

Let $\gamma_c > 0$ be any number. Choose

$$\delta = \gamma_c (1 + \gamma_c)^{-1} \quad \text{and } \mu_i = \lambda_i \left(\left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| \right)^{-1}.$$

Then, $\left\| \sum\limits_{i=1}^{n} \mu_i f_i(x) x_i \right\| = 1$. So

$$\begin{aligned} \left\| \sum_{i=n+1}^{\infty} \mu_i f_i(x) x_i \right\| &> \varepsilon \left(\left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| \right)^{-1} \\ &\geq \varepsilon \left(\sup_n \left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| \right)^{-1} \\ &= c > 0, \ \text{where} \ \ c = \varepsilon \left(\sup_n \left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| \right)^{-1}. \end{aligned}$$

But, $\left\| \sum\limits_{i=1}^{\infty} \mu_i f_i(x) x_i \right\| < 1 + \gamma_c$. This is a contradiction.

Conversely, let $(\{f_n\}, \{x_n\})$ satisfies property (II) but not property (I). Then, there exists a $c > 0$ such that for every $\gamma_c > 0$, there is a sequence of weights $\{\lambda_i\}_{i=1}^{\infty} \subset \mathbb{K}$,

$x \in \mathrm{E}$ and $n \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| = 1, \quad \left\| \sum_{i=n+1}^{\infty} \lambda_i f_i(x) x_i \right\| \geq c \quad \text{but} \quad \left\| \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i \right\| < \quad 1 + \gamma_c.$$

So, there is an $\varepsilon > 0$ such that for no $\delta > 0$, the relation in property (II) is satisfied. Indeed, let $0 < \eta < 1$ be arbitrary but fixed and $\delta > 0$ be arbitrary such that $\delta \leq \eta$. Choose

$$\gamma_c = \delta(1-\delta)^{-1} \text{ and } \mu_i = \lambda_i \left(\left\| \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i \right\| \right)^{-1}.$$

Then, $\left\| \sum_{i=1}^{n} \mu_i f_i(x) x_i \right\| > 1 - \delta$ and $\left\| \sum_{i=1}^{\infty} \mu_i f_i(x) x_i \right\| = 1$. But

$$\begin{aligned} \left\| \sum_{i=n+1}^{\infty} \mu_i f_i(x) x_i \right\| &\geq c \left(\left\| \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i \right\| \right)^{-1} \\ &> c(1-\delta) \\ &\geq c(1-\eta). \end{aligned}$$

By taking $\varepsilon = c(1-\eta)$ our assertion is established. □

Next, we give another type of property, called property $(\mathcal{M})$, for a Schauder frame.

Definition 2.4. Let E be a Banach space, $(\{f_n\}, \{x_n\})(\{f_n\} \subset \mathrm{E}^*, \{x_n\} \subset \mathrm{E})$ be a Schauder frame for E. Then, $(\{f_n\}, \{x_n\})$ $(\{f_n\} \subset \mathrm{E}^*, \{x_n\} \subset \mathrm{E})$ for E is said to satisfy *property* $(\mathcal{M})$ if for any finite family of weights $\{\lambda_i\}_{i=1}^{n+m}$ and $x \in \mathrm{E}$

$$\left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| \leq \left\| \sum_{i=1}^{n+m} \lambda_i f_i(x) x_i \right\|, \quad m, n \in \mathbb{N}. \tag{2.1}$$

If the inequality (2.1) is strict then the Schauder frame $(\{f_n\}, \{x_n\})$ is said to satisfy property $(\mathcal{M})$ strictly.

In order to show the existence of a Schauder frame satisfying property $(\mathcal{M})$, we give the following examples.

Example 2.5. Let $\mathrm{E} = l^p$ $(p \geq 1)$. Let $\{e_n\}$ be a sequence of unit vectors in E and $\{f_n\}$ be a sequence of unit vectors in E^*. Then $(\{f_n\}, \{e_n\})$ is a Schauder frame for E satisfying property $(\mathcal{M})$.

Example 2.6. Let $\mathrm{E} = c_0$. Let $\{e_n\}$ be a sequence of unit vectors in E and $\{f_n\}$ be a sequence of unit vectors in E^*. Then $(\{f_n\}, \{e_n\})$ is a Schauder frame for E which satisfies property $(\mathcal{M})$ but not strictly.

In the following result, we prove that property (II) (or property (I)) is a necessary condition for a Schauder frame satisfying property $(\mathcal{M})$ in a uniformly convex Banach space.

Theorem 2.7. *Let* E *be a uniformly convex Banach space. If* $(\{f_n\}, \{x_n\})$ *is a Schauder frame for* E *satisfying property* $(\mathcal{M})$*, then* $(\{f_n\}, \{x_n\})$ *satisfies property* (II) (*hence property* (I)).

Proof. Suppose on the contrary that Schauder frame $(\{f_n\}, \{x_n\})$ does not satisfy property (II). Then, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ one can find family of weights $\{\lambda_i^\delta\}_{i=1}^\infty$, $x \in$ E and $n \in \mathbb{N}$ with series $\sum_{i=1}^\infty \lambda_i^\delta f_i(x)x_i$ converging in E such that

$$\left\|\sum_{i=1}^n \lambda_i^\delta f_i(x)x_i\right\| > 1-\delta, \quad \left\|\sum_{i=1}^\infty \lambda_i^\delta f_i(x)x_i\right\| = 1 \text{ but } \left\|\sum_{i=n+1}^\infty \lambda_i^\delta f_i(x)x_i\right\| > \varepsilon.$$

Then, in view of property $(\mathcal{M})$ for the Schauder frame $(\{f_n\}, \{x_n\})$, we have

$$1-\delta < \left\|\sum_{i=1}^n \lambda_i^\delta f_i(x)x_i\right\| \le \left\|\sum_{i=1}^\infty \lambda_i^\delta f_i(x)x_i\right\| = 1.$$

Since, the Banach space E is uniformly convex, there is a δ_0 with $0 < \delta_0 < 1$, such that whenever $\|x\| \le 1, \|y\| \le 1$ and $\|x-y\| > \varepsilon$, we have $\left\|\frac{x+y}{2}\right\| \le 1-\delta_0$. Take $x_0 = \sum_{i=1}^n \lambda_i^\delta f_i(x)x_i$ and $y_0 = \sum_{i=1}^\infty \lambda_i^\delta f_i(x)x_i$ then, $\|x_0\| \le 1, \|y_0\| \le 1$ and $\|x_0 - y_0\| > \varepsilon$. Hence,

$$\begin{aligned} 1-\delta_0 &\le \left\|\frac{x_0+y_0}{2}\right\| \\ &= \left\|\sum_{i=1}^n \lambda_i^\delta f_i(x)x_i + \frac{1}{2}\sum_{i=n+1}^\infty \lambda_i^\delta f_i(x)x_i\right\| \\ &\ge \left\|\sum_{i=1}^n \lambda_i^\delta f_i(x)x_i\right\| \\ &> 1-\delta_0 \end{aligned}$$

which is absurd. □

Next, we define two more properties for a Schauder frame related to earlier properties.

Definition 2.8. Let E be a Banach space, $(\{f_n\}, \{x_n\})(\{f_n\} \subset \mathrm{E}^*, \{x_n\} \subset \mathrm{E})$ be a Schauder frame for E. Then, $(\{f_n\}, \{x_n\})$ $(\{f_n\} \subset \mathrm{E}^*, \{x_n\} \subset \mathrm{E})$ is said to satisfy

(a) *property* $(\mathcal{B})$, if

$$\sup_n \left\|\sum_{i=1}^n \lambda_i f_i(x)x_i\right\| < \infty \quad \Rightarrow \quad \sum_{i=1}^\infty \lambda_i f_i(x)x_i < \infty,$$

where $\{\lambda_i\}_{i=1}^\infty$ is any family of weights and $x \in$ E.

(b) *property* $(\mathcal{SB})$, if

$$\sup_n \left\|\sum_{i=1}^n \alpha_i f_i(x)x_i\right\| < \infty \quad \Rightarrow \quad \sum_{i=1}^\infty \alpha_i f_i(x)x_i < \infty,$$

where $\{\alpha_i\}_{i=1}^\infty$ is any family of scalars and $x \in$ E.

Regarding the existence of a Schauder frame satisfying property $(\mathcal{B})$ and property $(\mathcal{SB})$ one may observe that for E $= l^p (p \ge 1)$, if $\{e_n\}$ be a sequence of unit vectors in E and $\{f_n\}$ be a sequence of unit vectors in E^*. Then, $(\{f_n\},\{e_n\})$ is a Schauder frame for E which satisfies property $(\mathcal{SB})$ and hence property $(\mathcal{B})$.

In the following result, we prove that property $(\mathcal{B})$ is a necessary condition for a Schauder frame satisfying property (I) (or property (II)).

Theorem 2.9. *If* $(\{f_n\},\{x_n\})$ $(\{f_n\} \subset E^*, \{x_n\} \subset E)$ *is a Schauder frame for* E *satisfying property* (I) *then* $(\{f_n\},\{x_n\})$ *satisfies property* $(\mathcal{B})$.

Proof. Let $\sup_n \left\| \sum_{i=1}^{n} \lambda_i f_i(x) x_i \right\| < \infty$. Let $\{y_n\} \subset E$ be any bounded sequence such that

$$y_n = \sum_{i=1}^{n} \lambda_i f_i(x) x_i, \quad n \in \mathbb{N}.$$

To complete the proof, it is enough to establish that $\lim_{n\to\infty} y_n$ exists in E. Choose a sequence $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty} \|y_{n_k}\| = \overline{\lim_{n\to\infty}} \|y_n\| = B.$$

If $B = 0$, then $\lim_{n\to\infty} y_n = 0$ exists. If $B \neq 0$ then, we shall show that $\{y_{n_k}\}$ is a Cauchy sequence in E. Suppose on the contrary that $\{y_{n_k}\}$ is not a Cauchy sequence in E, then there exist $\delta > 0$ and subsequences $\{y_{n_{k_j}}\}$ and $\{y_{n_{p_j}}\}$ of $\{y_{n_k}\}$ with $n_{k_j} > n_{p_j}$, $j \in \mathbb{N}$ such that

$$\|y_{n_{k_j}} - y_{n_{p_j}}\| \geq \delta, \quad j \in \mathbb{N}.$$

Since $\left\| \dfrac{y_{n_{k_j}} - y_{n_{p_j}}}{\|y_{n_{p_j}}\|} \right\| \geq \dfrac{\delta}{A} = c > 0$, where $A = \sup_n \|y_n\| < \infty$, it follows, in view of property (I), that

$$\lim_{j\to\infty} \|y_{n_{k_j}}\| \geq \lim_{j\to\infty} \|y_{n_{p_j}}\|(1+\gamma_c).$$

This gives $B \geq B(1+\gamma_c)$, which is impossible. Hence, $\{y_{n_k}\}$ is a Cauchy sequence in E. Let $\lim_{k\to\infty} y_{n_k} = a$, where $a \in E$. Then

$$\begin{aligned} a &= \lim_{k\to\infty} \sum_{i=1}^{n_k} \lambda_i f_i(x) x_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i f_i(x) x_i = \lim_{n\to\infty} y_n. \end{aligned}$$ □

Corollary 2.10. If $(\{f_n\},\{x_n\})$ is a Schauder frame satisfying property $(\mathcal{M})$ in a uniformly convex Banach space E, then $(\{f_n\},\{x_n\})$ also satisfies property $(\mathcal{B})$.

Proof. The result follows in view of Theorem 2.7 and Theorem 2.9. □

Remark 2.11. The converse of Theorem 2.9 is not be true. In fact, in view of Example 2.2, one may verify that $(\{f_n\},\{x_n\})$ satisfies property $(\mathcal{B})$ but does not satisfies property (I).

Finally, we obtain a necessary condition for a Schauder frame satisfying property $(\mathcal{SB})$.

Theorem 2.12. *Let* $(\{f_n\},\{x_n\})(\{f_n\}\subset \mathrm{E}^*,\{x_n\}\subset \mathrm{E})$ *be a Schauder frame for a Banach space* E *satisfying property* $(\mathcal{SB})$. *Then, for every* $\phi\in \mathrm{E}^{**}$ *the series* $\sum\limits_{i=1}^{\infty}\phi(f_i)x_i$ *converges in* E.

Proof. Let $\phi\in \mathrm{E}^{**}$ be arbitrary. If $\phi\in[f_n]^{\perp}$ such that $\phi\neq 0$, then the series $\sum\limits_{i=1}^{\infty}\phi(f_i)x_i$ is convergent. Let $\phi\notin[f_n]^{\perp}$ and for each $n\in\mathbb{N}, S_n$ be the partial sum operator to $\sum\limits_{i=1}^{\infty}f_i(x)x_i$ with adjoint S_n^*, then

$$\begin{aligned}[S_n^*(g)](x) &= g\Big[\sum_{i=1}^{n}f_i(x)x_i\Big]\\ &= \Big[\sum_{i=1}^{n}g(x_i)f_i\Big](x),\quad g\in \mathrm{E}^*,\quad x\in \mathrm{E}.\end{aligned}$$

This gives

$$[S_n^*(g)]=\sum_{i=1}^{n}g(x_i)f_i,\quad g\in \mathrm{E}^*,\ n=1,2,3\ldots.$$

Further, we have

$$[S_n^{**}(\phi)](f)=\phi\Big[\sum_{i=1}^{n}f(x_i)f_i\Big]=f\Big[\sum_{i=1}^{n}\phi(f_i)x_i\Big].$$

Therefore

$$[S_n^{**}(\phi)]=\pi\Big[\sum_{i=1}^{n}\phi(f_i)x_i\Big],$$

where π is the canonical mapping of E into E^{**}. Since π is an isometry, it follows that

$$\begin{aligned}\Big\|\sum_{i=1}^{n}\phi(f_i)x_i\Big\| &= \|S_n^{**}(\phi)\|\\ &\leq \|S_n\|\|\phi\|.\end{aligned}$$

Thus,

$$\sup_n\Big\|\sum_{i=1}^{n}\phi(f_i)x_i\Big\|<\infty.$$

Without loss of generality we may assume that $f_n\neq 0$ for all $n\in\mathbb{N}$. Then, there exists $0\neq x\in \mathrm{E}$ such that $f_n(x)\neq 0$ for all $n\in\mathbb{N}$. Choose $\{\alpha_i\}\subset\mathbb{K}$ such that $\phi(f_i)(x)=\alpha_i f_i(x),\ i\in\mathbb{N}$. Then $\sup\limits_n\Big\|\sum\limits_{i=1}^{n}\alpha_i f_i(x)x_i\Big\|<\infty$. Hence, $\sum\limits_{i=1}^{\infty}\alpha_i f_i(x)x_i$ converges.

References

[1] K. Beanland, D. Freeman and R. Liu, Upper and lower estimates for Schauder frames and atomic decompositions, eprint arXiv:1202.2492v1 [math.FA], Feb 2012.
[2] P. G. Casazza, S. J. Dilworth, E. Odell, TH. Schlumprecht and A. Zsak, Coefficient Quantization for Frames in Banach Spaces, J. Math. Anal. Appl., 348 (2008), 66-86.
[3] P. G. Casazza, D. Han and D. R. Larson, Frames for Banach spaces, Contemp. Math., 247(1999), 149-182.
[4] O. Christensen, *Frames and Bases*, Birkhäuser, 2008.
[5] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83(1977) 569-645.
[6] I. Daubechies, A. Grossmann and Y. Meyer, Painless non-orthogonal expansions, J. Math. Phys., 27(1986), 1271-1283.
[7] R. J. Duffin and A. C. Schaeffer, A class of non-harmonic Fourier series, Trans. Amer. Math. Soc., 72 (1952), 341-366.
[8] H. G. Feichtinger and K. Gröchenig, A unified approach to atomic decompositions via integrable group representations, In: Proc. Conf. Function Spaces and Applications, Lecture Notes Math. 1302, Berlin-Heidelberg-New York: Springer (1988), 52-73.
[9] K. Gröchenig, Describing functions: Atomic decompositions versus frames, Monatsh. Math., 112(1991), 1-41.
[10] D. Han and D. R. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc., 147(2000), no. 697, x+94 pp.
[11] J. R. Holub, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces, Proc. Amer. Math. Soc., 122(1994), no. 3, 779–785.
[12] S. K. Kaushik and S. K. Sharma, Atomic decompositions satisfying certain properties, J. Adv. Res. Pure Math., 3(2011), no. 2, 40–48.
[13] S. K. Kaushik, S. K. Sharma and K. T. Poumai, On Schauder frames in conjugate Banach spaces, J. Math., 2013, Art. ID 318659, 4 pp.
[14] R. Liu, On shrinking and boundedly complete Schauder frames of Banach spaces, J. Math. Anal. Appl. 365 (2010), 385-398.
[15] R. Liu, Hilbert-Schauder frame operators, Oper. Matrices, 7(2013), no. 1, 91–99.
[16] R. Liu and B. Zheng, A characterization of Schauder frames which are near-Schauder bases, J. Fourier Anal. Appl., 16(2010), no. 5, 791–803.

(Jhavi Lal Ghimire and Narayan Prasad Pahari) Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.

E-mail address, Jhavi Lal Ghimire: `jhavighimire@gmail.com`

E-mail address, Narayan Prasad Pahari: `nppahari@gmail.com`