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WOVEN K-FRAMES IN QUATERNIONIC HILBERT SPACES

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Abstract. In this paper, we have defined and studied woven K-frames in quaternionic Hilbert spaces. Also, we have studied conditions under which a woven frame(K-frame) form a woven K-frame. Further, we have studied conditions under which a bounded image of woven K-frame forms a woven K-frame. Finally, we prove a perturbation result for woven K-frames in quaternionic Hilbert spaces.

1. Introduction

Frames is an over-redundant family of vectors which was introduced and studied by Duffin and Schaffer [9] in the context of the non-harmonic Fourier series. In the year 1982, it was reintroduced by Daubechies, Grossman, and Meyer [5], and after that many generalizations of frames have been given. For details regarding frames and their applications one may refer to [8] and reference therein. Although, frames have many applications, there are some problems in sampling theory which can not be tackled by frames, which required an atomic system related to a bounded linear operator. In [6], *Găvruta* introduced and studied the K-frames which is a generalization of ordinary frames. Recently in [16], the concept of weaving frames has been introduced by Bemrose et al. , which is used in distributed signal processing. For details regarding weaving K-frames and their properties one may refer to [1, 2, 18, 17].

In 1843, Hamilton discovered the field of quaternions which is the largest non-commutative, associative real algebra generated by $1, i, j, k$. Quaternions are used in quantum mechanics, the theory of relativity, number theory and to study rotation in higher dimensional Euclidean spaces. For details regarding various operations, properties and applications of quaternions, one may refer to [10, 12]. In the case of finite-dimensional quaternionic Hilbert spaces, frame was defined by M. Khokulan, K. Thirulogasanthar and S. Srisatkunarajah [7], further Sharma and Goel [14], has studied frames in quaternionic Hilbert spaces. Recently H. Ellouz [3] has defined and studied the concept of K-frames in quaternionic Hilbert spaces. For details regarding K-frames

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in quaternionic Hilbert spaces, one may refer to [3, 4]. In paper [13], we have defined the woven frames in quaternionic Hilbert spaces and recently in [11] Ruchi et al. have defined OPV-frames in quaternionic Hilbert spaces.

2. Definitions

Quaternions are four dimensional non commutative real algebra generated by $1, i, j, k$ where i, j, k are called imaginary units. As we know that quaternions are an extension of the complex number($\mathbb{C}$) and operations on $\mathbb{C}$ are those of $\mathfrak{Q}$ restricted over $\mathbb{C}$, for operation and various properties of quaternions see [10], for further details.

In [14], authors introduced frames in separable right quaternionic Hilbert spaces.

Definition 2.1 ([14])**.** Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathcal{I}}$ be a sequence in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathcal{I}}$ is said to be a *frame* of $\mathbb{H}^R(\mathfrak{Q})$, if there exist two finite real constants with $0 < r_1 \leq r_2$ such that

$$r_1\|u\|^2 \leq \sum_{i\in\mathcal{I}} |\langle u_i|u\rangle|^2 \leq r_2\|u\|^2, \text{ for all } u \in \mathbb{H}^R(\mathfrak{Q}).$$

The above inequality is called frame inequality and r_1, r_2 are called lower and upper frame bounds respectively. If only the upper bound condition holds then $\{u_i\}_{i\in\mathcal{I}}$ is said to form a Bessel sequence with Bessel bound r_2. For details regarding frame operators, perturbation and dual frames see [14].

Definition 2.2. [15] A family of elements $\{u_i\}_{i\in\mathcal{I}}$ of $\mathbb{H}^R(\mathfrak{Q})$ is called an atomic system for $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ if the following holds.

(1) The series $\sum\limits_{i\in\mathcal{I}} u_i q_i$ converges for all $\{q_i\}_{i\in\mathcal{I}} \in {}^2(\mathfrak{Q})$.

(2) There exist $C > 0$ such that for every $u \in \mathbb{H}^R(\mathfrak{Q})$ there exist $a_u = \{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q})$, such that $||a_u||_{l^2(\mathfrak{Q})} \leq C||u||$ and $K(u) = \sum\limits_{i\in\mathcal{I}} u_i q_i$.

Definition 2.3. [15] A family $\{u_i\}_{i\in\mathcal{I}}$ is said to be K-frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper K-bounds r_1 and r_2, if $\{u_i\}_{i\in\mathcal{I}}$ satisfies (2) of Proposition 3.2.

By Proposition 3.2, if $\{u_i\}_{i\in\mathcal{I}}$ satisfies any of the three conditions then it form a K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Definition 2.4. [13] Let $[m]$ be the set of first m natural numbers, $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\mathfrak{F} = \{\{u_{ij}\}_{j\in\mathbb{N}} : \ i \in [m]\}$ be a family of frames for $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathfrak{F}$ is said to be *woven* if there are universal positive real numbers r_1 and r_2 so that for every partition $P = \{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\mathfrak{F}_P = \{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper frame bounds r_1 and r_2, respectively. Each family $\mathfrak{F}_P$ is called weaving. If every weaving is a Bessel sequence, then $\mathfrak{F}$ is called a woven Bessel sequence of $\mathbb{H}^R(\mathfrak{Q})$.

3. Preliminaries

Proposition 3.1. [13] Consider a family of Bessel sequences $\mathfrak{F} = \{\{u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_{2i}. Then for any partition $P = \{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a Bessel sequence with bound $\sum\limits_{i\in[m]} r_{2i}$.

Proposition 3.2. [15] Let $\{u_i\}_{i\in\mathcal{I}} \subset \mathbb{H}^R(\mathfrak{Q})$. Then, the following are equivalent:

(1) $\{u_i\}_{i\in\mathcal{I}}$ is an atomic system for $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$.
(2) there exist $r_1, r_2 > 0$ such that $r_1||K^*u||^2 \leq \sum\limits_{i\in\mathcal{I}} |\langle u_i|u\rangle|^2 \leq r_2||u||^2$, $\forall\, u \in \mathbb{H}^R(\mathfrak{Q})$.
(3) $\{u_i\}_{i\in\mathcal{I}}$ is a Bessel sequence and there exist a Bessel sequence $\{p_i\}_{i\in\mathcal{I}}$ such that $K(u) = \sum\limits_{i\in\mathcal{I}} u_i\langle p_i|u\rangle$, $\forall\, u \in \mathbb{H}^R(\mathfrak{Q})$.

Lemma 3.3. [3] *Let $\mathbb{H}^R(\mathfrak{Q})$ and $\mathbb{H}_1^R(\mathfrak{Q})$ be two right quaternionic Hilbert spaces and $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$ be a bounded right linear operator with closed range. Then there exists a right linear operator $T^\dagger \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}^R(\mathfrak{Q}))$ such that*

$$||T^\dagger||^{-1}||u|| \leq ||T^*u|| \leq ||T||\ ||u||,\ \forall\ u \in R(T).$$

Lemma 3.4. [4] *Let $L_1 \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}^R(\mathfrak{Q}))$, $L_2 \in \mathfrak{B}(\mathbb{H}_2^R(\mathfrak{Q}), \mathbb{H}^R(\mathfrak{Q}))$ be two bounded right linear operators. Then, the following are equivalent:*

(1) $R(L_1) \subset R(L_2)$.
(2) $L_1L_1^* \leq \lambda^2 L_2L_2^*$, for some $\lambda \geq 0$.
(3) there exists a bounded right linear operator $M \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}_2^R(\mathfrak{Q}))$, such that $L_1 = L_2M$.

4. Main Results

In this section, we have defined and studied the woven K-frame in $\mathbb{H}^R(\mathfrak{Q})$. We have studied conditions under which a bounded image of woven K-frames is a woven K-frame, also we have studied the conditions under which a family of K-frame of $\mathbb{H}^R(\mathfrak{Q})$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Definition 4.1. A family of K-frame $\mathfrak{F} = \{\{u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ of $\mathbb{H}^R(\mathfrak{Q})$ is said to be a woven K-frame with universal woven K-bounds r_1 and r_2, respectively, if for any partition $P = \{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ forms a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1 and r_2, respectively.

Lemma 4.2. *Consider a family of K-frames $\mathfrak{F} = \{\{u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ of $\mathbb{H}^R(\mathfrak{Q})$ with upper bounds r_{2i}. Then for any partition $P = \{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a Bessel sequence with bound $\sum\limits_{i\in[m]} r_{2i}$.*

Proof. Every K-frame is a Bessel sequence, hence this result will follow from Proposition 3.1 $\square$.

Remark 4.3. A bounded image of K-frame may not be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Example 4.4. We will consider the K-frames of example 3.1 of [3] let $T = K$. Consider, $\{g_i\}_{i=1}^3 = \{Kf_i\}_{i=1}^3 = \{e_1, e_2, e_3\}$ is not a K-frame because for $u = e_3$, the lower bound condition does not hold.

In the next Theorem, we studied a condition under which a bounded image of K-frame form a K-frame.

Theorem 4.5. *Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1, r_2, respectively and let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be an invertible right linear operator with bounded inverse such that $TK = KT$. Then $\{Tu_i\}_{i\in\mathcal{I}}$ forms a K-frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For any $u \in \mathbb{H}^R(\mathfrak{Q})$,

$$r_1||T^{*-1}||^{-2}||K^*u||^2 \le r_1||T^*K^*u||^2 = r_1||K^*T^*u||^2 \le \sum_{i\in\mathcal{I}} |\langle Tu_i|u\rangle|^2 \le r_2||T||^2||u||^2.$$

□

In the next result, we studied a condition under which a bounded image of a woven K-frame is a woven K-frame.

Theorem 4.6. *Let $\{u_i\}_{i\in\mathcal{I}}$, $\{u_i^{'}\}_{i\in\mathcal{I}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ and let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be an invertible right linear operator with bounded inverse and $KT = TK$. Then $\{Tu_i\}_{i\in\mathcal{I}}$, $\{Tu_i^{'}\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. From Theorem 4.5, $\{Tu_i\}_{i\in\mathcal{I}}$, $\{Tu_i^{'}\}_{i\in\mathcal{I}}$ be K-frames of $\mathbb{H}^R(\mathfrak{Q})$. Let r_1, r_2 be the bounds for woven K-frames $\{u_i\}_{i\in\mathcal{I}}$, $\{u_i^{'}\}_{i\in\mathcal{I}}$. For any subset $\sigma \subset \mathcal{I}$, consider

$$\begin{aligned}\sum_{i\in\sigma} |\langle Tu_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle Tu_i^{'}|u\rangle|^2 &= \sum_{i\in\sigma} |\langle u_i|T^*u\rangle|^2 + \sum_{i\in\sigma^c} |\langle u_i^{'}|T^*u\rangle|^2 \\ &\le r_2||T^*||^2\,||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}),\end{aligned}$$

and

$$r_1||T^{*-1}||^{-2}||K^*u||^2 \le r_1||T^*K^*u||^2 \le \sum_{i\in\sigma} |\langle Tu_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle Tu_i^{'}|u\rangle|^2, u \in \mathbb{H}^R(\mathfrak{Q}).$$

□

In the next result, we have studied the bounded image of a woven K-frame.

Proposition 4.7. Let $\{\{u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ be a family of woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1, r_2 respectively and $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. Then the family $\{\{Tu_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ is a woven TK-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbb{N}$, for any $u \in \mathbb{H}^R(\mathfrak{Q})$

$$r_1||(TK)^*u||^2 = r_1||K^*T^*u||^2 \le \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle Tu_{ij}|u\rangle|^2 \le r_2||T||^2||u||^2.$$

□

Remark 4.8. If $U\Phi$, $U\Psi$ is a woven UK-frame for some $U \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. Then it is not necessary that Φ, Ψ form a woven K-frame.

Example 4.9. Let $\mathbb{H}^R(\mathfrak{Q})$ be two-dimensional quaternionic Hilbert space with orthonormal basis $\{e_1, e_2\}$. Let $U, K : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}^R(\mathfrak{Q})$ be two right linear operators defined as $K(e_1) = e_2$, $K(e_2) = 0$ and $U(e_1) = 0$,$U(e_2) = e_1$. Theirs corresponding adjoint operators be $K^*(e_1) = 0, K^*(e_2) = e_1$, and $U^*(e_1) = e_2, U^*(e_2) = 0$. Consider two families $\Phi = \{u_i\}_{i=1}^3 = \{e_1, e_1, e_2\}$ and $\Psi = \{p_i\}_{i=1}^3 = \{e_2, e_1, e_1\}$ of K-frames, these two families do not form a woven K-frame. But $U\Phi = \{0, 0, e_1\}$ and $U\Psi = \{e_1, e_1, 0\}$ form a woven UK-frame of $\mathbb{H}^R(\mathfrak{Q})$.

In the next result, we have studied a necessary and sufficient condition for a family of K-frame to be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proposition 4.10. Let $\mathfrak{F} = \{\{u_{ij}\}_{j=1}^{\infty} : i \in [m]\}$ be a family of K-frames of $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathfrak{F} = \{\{u_{ij}\}_{j=1}^{\infty} : i \in [m]\}$ is a woven K-frame if and only if $\{\{U(u_{ij})\}_{j=1}^{\infty} : i \in [m]\}$ is a woven UK-frame, where $U \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ is any bounded right linear operator.

Proof. Let $\mathfrak{F} = \{\{u_{ij}\}_{j=1}^{\infty} : i \in [m]\}$ is a woven K-frame with bounds r_1 and r_2, respectively. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbb{N}$, for any $u \in \mathbb{H}^R(\mathfrak{Q})$

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle U(u_{ij})|u\rangle|^2 = \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|U^*u\rangle|^2 \leq r_2||U^*||^2||u||^2,$$

and for the lower bound, consider

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle U(u_{ij})|u\rangle|^2 = \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|U^*u\rangle|^2 \geq r_1||K^*U^*u||^2 = r_1||(UK)^*u||^2.$$

For converse part take $U = I$. □

In the next theorem, we have proved that the bounded image of a woven frame of $\mathbb{H}^R(\mathfrak{Q})$ is a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.11. *Let $\{\{u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ be a family of woven frames of $\mathbb{H}^R(\mathfrak{Q})$. Then for any $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$, $\{\{Ku_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbb{N}$, r_1 and r_2 be woven frame bounds for $\{\{u_{ij}\}_{j=1}^{\infty} : i \in [m]\}$. Then for any $u \in \mathbb{H}^R(\mathfrak{Q})$,

$$r_1||K^*u||^2 \leq \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle Ku_{ij}|u\rangle|^2 \leq r_2||K^*u||^2 \leq r_2||K^*||^2||u||^2.$$

□

The converse of Theorem 4.11 may not be true, consider the following example.

Example 4.12. Let $\mathbb{H}^R(\mathfrak{Q})$ be a three dimensional right quaternionic Hilbert space with an orthonormal basis $\{e_1, e_2, e_3\}$, let $\Phi = \{u_i\}_{i=1}^4 = \{e_1, e_1, e_2, e_3\}$ and $\Psi = \{p_i\}_{i=1}^4 = \{e_1, e_2, e_3, e_2\}$. Then $\{u_i\}_{i=1}^4$ and $\{p_i\}_{i=1}^4$ forms frames of $\mathbb{H}^R(\mathfrak{Q})$ but not forms a woven frame. As, for $\sigma = \{1, 2, 3\}$, $u = e_3$ the lower bound condition does not hold. Consider an operator K, $K(e_1) = e_1, K(e_2) = e_2$ and $K(e_3) = 0$, the adjoint of K is $K^*(e_1) = e_1, K^*(e_2) = e_2$ and $K^*(e_3) = 0$. Further, $\{Ku_i\}_{i=1}^4$ and $\{Kp_i\}_{i=1}^4$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Corollary 4.13. If $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{H}^R(\mathfrak{Q})$. Then for any $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$, $\{K(e_i)\}_{i\in\mathbb{N}}$ is a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

In the next result, we have obtained a new family of woven K-frame from a given family of woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ under certain conditions.

Theorem 4.14. *Let $\{\{u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ be a family of woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1 and r_2, respectively. If $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a right linear operator such that TKK^* is a positive right linear operator on $\mathbb{H}^R(\mathfrak{Q})$. Then $\{\{(I+T)u_{ij}\}_{j\in\mathbb{N}} : i \in [m]\}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbb{N}$, for any $u \in \mathbb{H}^R(\mathfrak{Q})$ and using Theorem 4.1 [3] we have

$$r_1||K^*u||^2 \leq r_1||K^*(I+T)^*u||^2 \leq \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle (I+T)u_{ij}|u\rangle|^2 \leq r_2||(I+T)||^2||u||^2.$$

□

In the next result, we have obtained a woven T-frame of $\mathbb{H}_1^R(\mathfrak{Q})$ from a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ under some conditions.

Proposition 4.15. Let $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$. Further, let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$ be a bounded right linear operator such that $R(T^*) \subset R(K)$. If $R(K)$ is closed. Then $\{Tu_j\}_{j\in\mathbb{N}}$ and $\{Tp_j\}_{j\in\mathbb{N}}$ form a woven T-frame of $\mathbb{H}_1^R(\mathfrak{Q})$.

Proof. Let r_1, r_2 be the woven K-frame bounds corresponding to woven K-frame $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$. For any subset $\sigma \subset \mathbb{N}$, any $u \in \mathbb{H}_1^R(\mathfrak{Q})$. We have

$$\sum_{i\in\sigma} |\langle Tu_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle Tp_i|u\rangle|^2 \leq r_2||T^*u||^2 \leq r_2||T||^2||u||^2.$$

Further, by Lemma 3.3

$$r_1||K^\dagger||^{-2}||T^*u||^2 \leq r_1||K^*T^*u||^2 \leq \sum_{i\in\sigma} |\langle Tu_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle Tp_i|u\rangle|^2, u \in \mathbb{H}_1^R(\mathfrak{Q}).$$

□

Theorem 4.16. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$. Then $\{Tu_j\}_{j\in\mathbb{N}}$ is a TKT^*-frame of $\mathbb{H}_1^R(\mathfrak{Q})$, where $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$.*

Proof. Let r_1, r_2 be the bounds corresponding to the given K-frame $\{u_j\}_{j\in\mathbb{N}}$. For any $u \in \mathbb{H}_1^R(\mathfrak{Q})$

$$\sum_{j=1}^{\infty} |\langle Tu_j|u\rangle|^2 \leq r_2||T||^2||u||^2$$

and for the lower bound, consider

$$\frac{r_1}{||T||^2}||(TKT^*)^*u||^2 \leq r_1||K^*T^*u||^2 \leq \sum_{j=1}^{\infty} |\langle Tu_j|u\rangle|^2.$$

□

Theorem 4.17. *Let $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$. Then $\{Tu_j\}_{j\in\mathbb{N}}$, $\{Tp_j\}_{j\in\mathbb{N}}$ forms a woven TKT^*-frame of $\mathbb{H}_1^R(\mathfrak{Q})$, where $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$.*

Proof. We can prove this theorem by using the step of Theorem 4.16. □

In the next theorem, we have studied the necessary and sufficient condition under which a bounded image of a woven frame forms a woven K-frame.

Theorem 4.18. *Let $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ be any two sequences in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ forms a woven frame of $R(K^*)$ if and only if $\{Ku_j\}_{j\in\mathbb{N}}$, $\{Kp_j\}_{j\in\mathbb{N}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ forms a woven frame of $R(K^*)$ with bounds r_1 and r_2, respectively. Then for any subset $\sigma \subset \mathbb{N}$,

$$r_1||u||^2 \leq \sum_{j\in\sigma} |\langle u_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle p_j|u\rangle|^2 \leq r_2||u||^2, u \in R(K^*).$$

Further, for any $u \in \mathbb{H}^R(\mathfrak{Q})$, $K^*(u) \in R(K^*)$

$$r_1||K^*u||^2 \leq \sum_{j\in\sigma} |\langle Ku_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle Kp_j|u\rangle|^2 \leq r_2||K^*u||^2 \leq r_2||K||^2||u||^2, u \in \mathbb{H}^R(\mathfrak{Q}).$$

Conversely, let $\{Ku_j\}_{j=1}^{\infty}$, $\{Kp_j\}_{j=1}^{\infty}$ form a woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1, r_2. For any $g \in R(K^*)$ there exist $u \in \mathbb{H}^R(\mathfrak{Q})$ such that $g = K^*(u)$, hence rest will follow immediately. □

In the next theorem, we have obtained a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$, from a given K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.19. *Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$, with bounds r_1, r_2. Further, let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be an invertible right linear operator such that T^{-1} is a bounded right linear operator and $KT = TK$. If $||I - T|| < \frac{r_1}{r_2}$. Then $\{u_i\}_{i\in\mathcal{I}}$ and $\{Tu_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let $\sigma \subset \mathcal{I}$ be any subset, consider

$$\begin{aligned}
\left(\sum_{i\in\sigma} |\langle u_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle Tu_i|u\rangle|^2\right)^{\frac{1}{2}} &= \left(\sum_{i\in\sigma} |\langle u_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle u_i|u\rangle - \langle u_i|(I-T^*)u|^2\right)^{\frac{1}{2}} \\
&\geq \left(\sum_{i\in\mathcal{I}} |\langle u_i|u\rangle|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in\sigma^c} |\langle u_i|(I-T^*)u\rangle|^2\right)^{\frac{1}{2}} \\
&\geq \sqrt{r_1}||u|| - \sqrt{r_2}||(I-T^*)u|| \\
&\geq \left(\sqrt{r_1} - \sqrt{r_2}||I-T^*||\right)||u||,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

In the next theorem, we have studied a necessary and sufficient condition under which K-frames of $\mathbb{H}^R(\mathfrak{Q})$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.20. *Let $K, T, U \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$, where T is an onto operator and $TK = KT$. Further, let $\{u_j\}_{j\in\mathbb{N}}$ and $\{p_j\}_{j\in\mathbb{N}}$ be K-frames of $\mathbb{H}^R(\mathfrak{Q})$. If for any subset $J \subset \mathbb{N}$,*

$$\sum_{j\in J} |\langle Tp_j|u\rangle|^2 \leq \sum_{j\in J} |\langle Up_j|u\rangle|^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Then $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ if and only if $\{Tu_j\}_{j\in\mathbb{N}}$, $\{Up_j\}_{j\in\mathbb{N}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1 and r_2, respectively. Then for any subset $\sigma \subset \mathbb{N}$,

$$\sum_{i\in\sigma} |\langle Tu_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle Up_i|u\rangle|^2 \leq r_2\left(||T||^2 + ||U||^2\right)||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

and for lower bound inequality, consider

$$\begin{aligned} r_1||K^*u||^2 \leq r_1||T^*K^*u||^2 &= r_1||K^*T^*u||^2 \\ &\leq \sum_{i\in\sigma} |\langle u_i|T^*u\rangle|^2 + \sum_{i\in\sigma^c} |\langle p_i|T^*u\rangle|^2 \\ &\leq \sum_{i\in\sigma} |\langle u_i|T^*u\rangle|^2 + \sum_{i\in\sigma^c} |\langle p_i|U^*u\rangle|^2,\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

For sufficient condition, take $T = U = I$. □

If $\{u\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ and $T_1, T_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be invertible right linear operators. The family $\{T_1u_i\}_{i\in\mathcal{I}}$, $\{T_2p_i\}_{i\in\mathcal{I}}$ may not form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Example 4.21. Let $\mathbb{H}^R(\mathfrak{Q})$ be a three dimensional right quaternionic Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^3$. Further, let $\{u_i\}_{i=1}^4 = \{e_1, 2e_1, e_2, e_3\}$, $\{p_i\}_{i=1}^4 = \{3e_1, e_1, e_2, e_3\}$ and $K, T_1, T_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be such that $K(e_1) = e_1, K(e_2) = e_1, K(e_3) = e_2$, $T_1 = I$, $T_2(e_1) = e_2, T_2(e_2) = e_3, T_2(e_3) = e_1$. Then $\{Ku_i\}_{i=1}^4$, $\{Kp_i\}_{i=1}^4$ form a K-frame of $\mathbb{H}^R(\mathfrak{Q})$. Further, any partition $\{\sigma_i\}_{i=1}^2$ of $\{1, 2, 3, 4\}$, we have

$$||K^*u||^2 \leq \sum_{i\in\sigma_1} |\langle u_i|u\rangle|^2 + \sum_{i\in\sigma_2} |\langle p_i|u\rangle|^2 \leq 14||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Thus, $\{Ku_i\}_{i=1}^4$ and $\{Kp_i\}_{i=1}^4$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$. But $\{T_1u_i\}_{i=1}^4 = \{e_1, 2e_1, e_2, e_3\}$, $\{T_2p_i\}_{i=1}^4 = \{3e_2, e_2, e_3, e_1\}$ does not form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$. As, for partition, $\sigma_1 = \{1, 2\}$ and $u = e_2$, the lower bound condition does not hold.

In the next result, we have studied a condition under which a bounded image of a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.22. *Let $\{u_i\}_{i\in\mathcal{I}}, \{p_i\}_{i\in\mathcal{I}}$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively. Further, let $T_1, T_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be surjective right linear operators such that $T_iK = KT_i$, $i = 1, 2$. If $R(K)$ is closed and $\sqrt{r_1} > \sqrt{r_2}||T_1 - T_2||\,||K^\dagger||\,||T_1^\dagger||$. Then $\{T_1u_i\}_{i\in\mathcal{I}}$, $\{T_2p_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $R(K)$ with universal bounds,*

$$\frac{\left[\sqrt{r_1} - r_2||T_1 - T_2||\,||K^\dagger||\,||T_1^\dagger||\right]^2}{}||T_1^\dagger||, \quad r_2\Big(||T_1||^2 + ||T_2||^2\Big).$$

Proof. First, we will show that $\{T_1u_i\}_{i\in\mathcal{I}}$ and $\{T_2p_i\}_{i\in\mathcal{I}}$ forms a K-frames of $R(K)$. For any $u \in R(K)$

$$\frac{r_1||K^*u||^2}{||T_1^\dagger||^2} \leq r_1||T_1^*K^*u||^2 = r_1||K^*T_1^*u||^2 \leq \sum_{i\in\mathcal{I}} |\langle T_1u_i|u\rangle|^2 \leq r_2||T_1||^2||u||^2.$$

Similarly, we can show that $\{T_2p_i\}_{i\in\mathcal{I}}$ is a K-frame of $R(K)$. Now, let $\{\sigma_i\}_{i=1}^2$ be any partition of $\mathcal{I}$ and $u \in R(K)$, consider

$$\sum_{i\in\sigma_1} |\langle T_1u_i|u\rangle|^2 + \sum_{i\in\sigma_2} |\langle T_2p_i|u\rangle|^2 \leq r_2(||T_1||^2 + ||T_2||^2)||u||^2.$$

$$\begin{aligned}\left(\sum_{i\in\sigma_1}|\langle T_1u_i|u\rangle|^2+\sum_{i\in\sigma_2}|\langle T_2p_i|u\rangle|^2\right)^{\frac{1}{2}} &\geq \left(\sum_{i\in\sigma_1}|\langle u_i|T_1^*u\rangle|^2+\sum_{i\in\sigma_2}|\langle p_i|T_1^*u\rangle|^2\right)^{\frac{1}{2}}\\ &- \left(\sum_{i\in\sigma_2}|\langle p_i|(T_1-T_2)^*u\rangle|^2\right)^{\frac{1}{2}}\\ &\geq \frac{\sqrt{r_1}-\sqrt{r_2}||T_1-T_2||\,||K^\dagger||\,||T_1^\dagger||}{||T_1^\dagger||}||K^*u||.\end{aligned}$$

Hence $\{T_1u_i\}_{i\in\mathcal{I}}$, $\{T_2p_i\}_{i\in\mathcal{I}}$ form a woven K-frame of $R(K)$. □

If we take $T_1 = T_2$ in Theorem 4.22, we have the following corollary.

Corollary 4.23. Let $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively. If $T\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a surjective right linear operator such that $TK = KT$. If $R(K)$ is closed. Then $\{Tu_i\}_{i\in\mathcal{I}}$, $\{Tp_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $R(K)$ with universal bounds $\frac{r_1}{||T^\dagger||^2}$ and $r_2||T||^2$, respectively.

If we take $K = I$ in Theorem 4.22, we have the following corollary.

Corollary 4.24. Let $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ be a woven frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively. Further, let $T_1, T_2\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be invertible right linear operators on $\mathbb{H}^R(\mathfrak{Q})$ and $\sqrt{r_1} > \sqrt{r_2}||T_1-T_2||\,||T_1^{-1}||$. Then $\{T_1u_i\}_{i\in\mathcal{I}}$, $\{T_2p_i\}_{i\in\mathcal{I}}$ forms a woven frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $\frac{\left[\sqrt{r_1}-\sqrt{r_2}||T_1-T_2||||T_1^{-1}||\right]^2}{||T_1^\dagger||^2}$, $r_2\Big(||T_1||^2+||T_2||^2\Big)$.

In the next result, we have obtained a new family of woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ from a given family of a woven K-frame and a Bessel sequence of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.25. *Let $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively. Further, let $\{h_{1i}\}_{i\in\mathcal{I}}$ and $\{h_{2i}\}_{i\in\mathcal{I}}$ be Bessel sequences of $\mathbb{H}^R(\mathfrak{Q})$ with bounds R_2^1, R_2^2, respectively. If for any subset $\sigma\subset\mathcal{I}$ and any $u\in\mathbb{H}^R(\mathfrak{Q})$,*

$$\sum_{i\in\mathcal{I}}|\langle h_{1i}|u\rangle|^2+2Re\left(\sum_{i\in\sigma}\langle u_i|u\rangle\overline{\langle h_{1i}|u\rangle}\right)\geq 0,\sum_{i\in\mathcal{I}}|\langle h_{2i}|u\rangle|^2+2Re\left(\sum_{i\in\sigma}\langle p_i|u\rangle\overline{\langle h_{2i}|u\rangle}\right)\geq 0.$$

Then $\{u_i+h_{1i}\}_{i\in\mathcal{I}}$, $\{p_i+h_{2i}\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proof. For any partition $\{\sigma\}_{i=1}^2\subset\mathcal{I}$, consider

$$\begin{aligned}\sum_{i\in\sigma_1}|\langle u_i+h_{1i}|u\rangle|^2 &+ \sum_{i\in\sigma_2}|\langle p_i+h_{2i}|u\rangle|^2=\sum_{i\in\sigma_1}|\langle u_i|u\rangle|^2\\ &+ \sum_{i\in\sigma_1}|\langle h_{1i}|u\rangle|^2+2Re\left(\sum_{i\in\sigma_1}\langle u_i|u\rangle\overline{\langle h_{1i}|u\rangle}\right)\\ &+ \sum_{i\in\sigma_2}|\langle p_i|u\rangle|^2+\sum_{i\in\sigma_2}|\langle h_{2i}|u\rangle|^2+2Re\left(\sum_{i\in\sigma_2}\langle u_i|u\rangle\overline{\langle h_{2i}|u\rangle}\right)\\ &\geq \sum_{i\in\sigma_1}|\langle u_i|u\rangle|^2+\sum_{i\in\sigma_2}|\langle p_i|u\rangle|^2\geq r_1||K^*u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

For upper bound inequality, consider

$$\begin{aligned}\sum_{i\in\sigma_1}|\langle u_i+h_{1i}|u\rangle|^2+\sum_{i\in\sigma_2}|\langle p_i+h_{2i}|u\rangle|^2 &\leq 2\sum_{i\in\sigma_1}|\langle u_i|u\rangle|^2\\ &+ 2\sum_{i\in\sigma_2}|\langle p_i|u\rangle|^2+2\sum_{i\in\sigma_1}|\langle h_{1i}|u\rangle|^2\\ &+ 2\sum_{i\in\sigma_2}|\langle h_{2i}|u\rangle|^2\\ &\leq 2(r_2+R_2^1+R_2^2)||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

□

In the next result, we have studied a condition under which a bounded image of a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.26. *Let $\{u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ be a family of woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively. If $U\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a right linear operator with a closed range such that $UK=KU$. Further, if $R(K^*)\subset R(U)$. Then $\{Uu_{ij}\}_{j\in\mathcal{J},i\in[m]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $r_1||U^\dagger||^2$ and $r_2||U||^2$, respectively.*

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$, we have

$$r_1||U^\dagger||^{-2}||K^*u||^2\leq r_1||K^*U^*u||^2\leq\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle Uu_{ij}|u\rangle|^2\leq r_2||U||^2||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

□

Corollary 4.27. Let $\{u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$. Further, let $U\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a surjective right linear operator such that $UK=KU$. Then $\{Uu_{ij}\}_{j\in\mathcal{J},i\in[m]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

In the next result, we have proved that if bounded image of a tight woven K-frame is a woven K-frame. Then the operator is a surjective right linear operator.

Theorem 4.28. *Let $\{u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ be a r-tight woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ and $U\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be such that $UK=KU$. If $R(K^*)=\mathbb{H}^R(\mathfrak{Q})$ and $\{Uu_{ij}\}_{j\in\mathcal{J},i\in[m]}$ is a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$. Then U is a surjective right linear operator.*

Proof. Let R_1,R_2 be the universal frame bounds corresponding to the woven K-frame $\{Uu_{ij}\}_{j\in\mathcal{J},i\in[m]}$. For any partition $\{\sigma_i\}_{i\in[m]}\subset\mathcal{J}$, we have

$$R_1||K^*u||^2\leq\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle Uu_{ij}|u\rangle|^2\leq R_2||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

Since $r||K^*u||^2=\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|u\rangle|^2$, hence

$$||U^*K^*u||^2=r^{-1}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle Uu_{ij}|u\rangle|^2\geq r^{-1}R_1||K^*u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

As K^* is an onto right linear operator. Thus the above inequality hold for all $u\in\mathbb{H}^R(\mathfrak{Q})$. Hence U is a surjective right linear operator. □

In the next result, we have studied the necessary and sufficient conditions under which a bounded image of a Parseval woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ forms a Parseval woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proposition 4.29. Let $\{u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ be a Parseval woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ and $U\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be such that $UK=KU$ and $R(K^*)=\mathbb{H}^R(\mathfrak{Q})$. Then $\{Uu_{ij}\}_{j\in\mathcal{J},i\in[m]}$ is a Parseval woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ if and only if U is a co-isometry.

In the next theorem, we have obtained a woven K^*-frame of $\mathbb{H}^R(\mathfrak{Q})$ from a given woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.30. *Let $\{u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ be a family of woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively and $R(K)$ is closed. Then $\{K^*u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ form a woven K^*- frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $r_1||K^\dagger||^{-2}$ and $r_2||K||^2$, respectively.*

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$, we have

$$r_1||K^\dagger||^2||Ku||^2\le r_1||K^*Ku||^2\le\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle K^*u_{ij}|u\rangle|^2\le r_2||K||^2||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

□

In the next result, we have studied a condition under which a bounded image of a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$, under two different operators forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.31. *Let $\{u_{1j}\}_{j\in\mathcal{J}}$ be a K-frame with bounds r_1^1, r_2^1 and $\{u_{2j}\}_{j\in\mathcal{J}}$ be another K-frame with bounds r_1^2, r_2^2. Let both forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1, r_2. Further, let $U_1, U_2\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be right linear operators such that $U_iK=KU_i$, $N(K^*)\subset N(U_i^*), i=1,2$. If*

$$||U_1^\dagger||\,||U_2-U_1||\,||\widetilde{K^*}^{-1}||^2<\sqrt{\frac{r_1}{r_2^2}}.$$

Then $\{U_1u_{1j}\}_{j\in\mathcal{J}}$, $\{U_2u_{2j}\}_{j\in\mathcal{J}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$, with universal bounds $\left(\sqrt{r_1}||U_1^\dagger||^{-1}-\sqrt{r_2^2}||U_2-U_1||\,||\widetilde{K^*}^{-1}||\right)^2$ and $r_1^1||U_1||^2+r_2^2||U_2||^2$, where $\widetilde{K^*}: N^\perp(K^*)\to R(K^*)$ is restriction of K^* on $N^\perp(K^*)$.

Proof. For any $u\in Ker^\perp(K^*)$ and for any partition, $\sigma\subset\mathcal{J}$. Consider

$$\begin{aligned}\left(\sum_{j\in\sigma}|\langle U_1u_{1j}|u\rangle|^2+\sum_{j\in\sigma^c}|\langle U_2u_{2j}|u\rangle|^2\right)^{1/2} &\ge \left(\sum_{j\in\sigma}|\langle u_{1j}|U_1^*u\rangle|^2+\sum_{j\in\sigma^c}|\langle u_{2j}|U_1^*u\rangle|^2\right)^{1/2}\\ &- \left(\sum_{j\in\sigma^c}|\langle u_{2j}|(U_2^*-U_1^*)u\rangle|^2\right)^{1/2}\\ &\ge \left(\sqrt{r_1}||U_1^\dagger||^{-1}-\sqrt{r_2^2}||U_2-U_1||\,||\widetilde{K^*}^{-1}||\right)||K^*u||.\end{aligned}$$

For $u \in \mathbb{H}^R(\mathfrak{Q})$, $u = u_1 + u_2$ where $u_1 \in Ker(K^*)$, $u_2 \in Ker^{\perp}(K^*)$ also $Ker(K^*) \subset Ker(U_i^*)$, $i = 1, 2$. Therefore above lower inequality hold for every $u \in \mathbb{H}^R(\mathfrak{Q})$, for upper bound inequality consider

$$\begin{aligned}\sum_{j\in\sigma}|\langle U_1u_{1j}|u\rangle|^2+\sum_{j\in\sigma^c}|\langle U_2u_{2j}|u\rangle|^2 &\leq \sum_{j\in\mathcal{J}}|\langle u_{1j}|U_1^*u\rangle|^2+\sum_{j\in\mathcal{J}}|\langle u_{2j}|U_2^*u\rangle|^2\\ &\leq (r_2^1||U_1||^2+r_2^2||U_2||^2)||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

□

In the next theorem, we have obtained a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ from a given woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.32. *Let $\{u_{1j}\}_{j\in\mathcal{J}}$, $\{u_{2j}\}_{j\in\mathcal{J}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1, r_2. Further, let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a positive right linear operator. For any $\sigma \subset \mathcal{I}$, let S_σ be the frame operator of $\{u_{1j}\}_{j\in\sigma} \cup \{u_{2j}\}_{j\in\sigma}$ and $TS_\sigma = S_\sigma T$. Then $\{u_{ij} + Tu_{ij}\}_{j\in\mathcal{J},i\in[2]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and $r_2||I + T||^2$, respectively.*

Proof. For any subset $\sigma \subset \mathcal{J}$, we have

$$\begin{aligned}\sum_{j\in\sigma}|\langle u_{1j}+Tu_{1j}|u\rangle|^2 &+ \sum_{j\in\sigma^c}|\langle u_{2j}+Tu_{2j}|u\rangle|^2\\ &\leq r_2||(I+T)^*u||^2\\ &\leq r_2||I+T||^2||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

Let S_σ^1 be the frame operator corresponding to $\{u_{1j} + Tu_{1j}\}_{j\in\sigma} \cup \{u_{2j} + Tu_{2j}\}_{j\in\sigma^c}$. Then

$$S_\sigma^1u = S_\sigma u + TS_\sigma u + S_\sigma T^*u + TS_\sigma T^*u \geq S_\sigma u,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Thus, for any $u \in \mathbb{H}^R(\mathfrak{Q})$ we have

$$\sum_{j\in\sigma}|\langle u_{1j}+Tu_{1j}|u\rangle|^2+\sum_{j\in\sigma^c}|\langle u_{2j}+Tu_{2j}|u\rangle|^2=\langle S_\sigma^1u|u\rangle\geq\langle S_\sigma u|u\rangle\geq r_1||K^*u||^2.$$

□

In the next result, we have obtained a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ from given families of woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.33. *Let $\{u_{ij}\}_{j\in\mathcal{J},i\in\mathbb{N}_2}$ and $\{p_{ij}\}_{j\in\mathcal{J},i\in\mathbb{N}_2}$ be the woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$, with universal bounds r_1, r_2 and R_1, R_2, respectively. Further, let $T_1, T_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be co-isometric right linear operators with $T_iK = KT_i$, $i = 1, 2$. For any $\sigma \subset \mathcal{I}$, let U_σ, P_σ be the analysis operators corresponding to $\{u_{1j}\}_{j\in\sigma} \cup \{u_{2j}\}_{j\in\sigma}$ and $\{p_{1j}\}_{j\in\sigma} \cup \{p_{2j}\}_{j\in\sigma}$. If $U_\sigma^*P_\sigma = 0$. Then $\{T_1u_{ij} + T_2p_{ij}\}_{j\in\mathcal{J},i\in\mathbb{N}_2}$ is a woven K-frame with universal bounds $r_1 + R_1$ and $2(r_2||T_1||^2 + R_2||T_2||^2)$, respectively.*

Proof. Clearly, $U_\sigma T_1^*$ and $P_\sigma T_2^*$ be the analysis operators corresponding to $\{T_1 u_{1j}\}_{j\in\sigma} \cup \{T_1 u_{2j}\}_{j\in\sigma^c}$ and $\{T_2 p_{1j}\}_{j\in\sigma} \cup \{T_2 p_{2j}\}_{j\in\sigma^c}$, respectively. Thus, any $u \in \mathbb{H}^R(\mathfrak{Q})$

$$\begin{aligned}\sum_{j\in\sigma} |\langle T_1 u_{1j} + T_2 p_{1j} | u\rangle|^2 \quad &+ \quad \sum_{j\in\sigma^c} |\langle T_1 u_{2j} + T_2 p_{2j} | u\rangle|^2 \\ &= \quad ||U_\sigma T_1^* u + P_{\sigma^c} T_2^* u||^2 \\ &\leq \quad 2(r_2||T_1||^2 + R_2||T_2||^2)||u||^2.\end{aligned}$$

Let S_σ be the frame operator corresponding to $\{T_1 u_{1j} + T_2 p_{1j}\}_{j\in\sigma} \cup \{T_1 u_{2j} + T_2 p_{2j}\}_{j\in\sigma^c}$,

$$\begin{aligned}\sum_{j\in\sigma} |\langle T_1 u_{1j} + T_2 p_{1j} | u\rangle|^2 \quad &+ \quad \sum_{j\in\sigma^c} |\langle T_1 u_{2j} + T_2 p_{2j} | u\rangle|^2 \\ &= \quad \langle U_\sigma^* U_\sigma T_1^* u | T_1^* u\rangle + \langle P_{\sigma^c}^* P_{\sigma^c} T_2^* u | T_2^* u\rangle \\ &\geq \quad r_1||K^* T_1^* u||^2 + R_1||K^* T_2^* u||^2 \\ &= \quad (r_1 + R_1)||K^* u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

□

Corollary 4.34. Let $\{u_{ij}\}_{j\in\mathcal{J}, i\in\mathbb{N}_2}$ and $\{p_{ij}\}_{j\in\mathcal{J}, i\in\mathbb{N}_2}$ be the woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1, r_2 and R_1, R_2, respectively. For any $\sigma \subset \mathcal{I}$, let U_σ, P_σ be the analysis operators corresponding to $\{u_{1j}\}_{j\in\sigma} \cup \{u_{2j}\}_{j\in\sigma}$ and $\{p_{1j}\}_{j\in\sigma} \cup \{p_{2j}\}_{j\in\sigma}$. If $U_\sigma^* P_\sigma = 0$. Then $\{u_{ij} + p_{ij}\}_{j\in\mathcal{J}, i\in\mathbb{N}_2}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $r_1 + R_1$ and $2(r_2 + R_2)$, respectively.

Theorem 4.35. *Let $\{u_{ij}\}_{j\in\mathcal{J}, i\in[m]}$ be a woven K-frame of $R(K)$ with universal bounds r_1 and r_2, respectively, and $R(K)$ is closed. Then $\{KK^\dagger u_{ij}\}_{j\in\mathcal{J}, i\in[m]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and $r_2||KK^\dagger||^2$, respectively.*

Proof. Let $\{\sigma\}_{i\in[m]}$ be any partition of $\mathcal{J}$, we have

$$\begin{aligned}r_1||K^* u||^2 \quad &= \quad r_1||K^*(KK^\dagger)^* u||^2 \\ &\leq \quad \sum_{i\in[m]} \sum_{j\in\sigma_i} |\langle KK^\dagger u_{ij} | u\rangle|^2 \\ &\leq \quad r_2||KK^\dagger u||^2 \\ &\leq \quad r_2||KK^\dagger||^2||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

□

In the next result, we have studied conditions under which a woven K-frame of a closed subspace of $\mathbb{H}^R(\mathfrak{Q})$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.36. *Let $\{u_{ij}\}_{j\in\mathcal{J}, i\in[m]}$ be a family of woven K-frame with universal bounds r_1 and r_2, for a closed right subspace M of $\mathbb{H}^R(\mathfrak{Q})$. Then*

(1) If P is the orthogonal projection onto M and $R(K) \subset M$. Then $\{Pu_{ij}\}_{j\in\mathcal{J}, i\in[m]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with same bounds.

(2) If M is dense in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{u_{ij}\}_{j\in\mathcal{J}, i\in[m]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with the same bounds.

Proof. (1) For any $u \in \mathbb{H}^R(\mathfrak{Q})$, $Pu \in M$ and any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$, we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle Pu_{ij}|u\rangle|^2 \leq r_2||P(u)||^2 \leq r_2||u||^2.$$

Since $u \in \mathbb{H}^R(\mathfrak{Q})$, $u = u_1 + u_2$ where $u_1 \in M$ and $u_2 \in M^\perp$, we have

$$\begin{aligned}\sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle Pu_{ij}|u\rangle|^2 &\geq r_1||K^*P(u_1+u_2)||^2 \\ &= r_1||K^*u_1||^2 \\ &= r_1||K^*u||^2.\end{aligned}$$

(2) Let there is a partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$ and an element $u \in \mathbb{H}^R(\mathfrak{Q})$ such that $\sum\limits_{i\in[m]}\sum\limits_{j\in\sigma_i} |\langle u_{ij}|u\rangle|^2 > r_2||u||^2$. Thus, there exists a finite set $\mathcal{I} \subset \mathcal{J}$ such that $\sum\limits_{i\in[m]}\sum\limits_{j\in\sigma_i\cap\mathcal{I}} |\langle u_{ij}|u\rangle|^2 > r_2||u||^2$. Since M is a dense subset of $\mathbb{H}^R(\mathfrak{Q})$, there exists $u \in M$ satisfies later inequality which leads to a contradiction. Hence, upper inequalities hold. Any $u \in \mathbb{H}^R(\mathfrak{Q})$ and every $\epsilon > 0$ there exists $w \in M$ such that $||u - w|| < \epsilon$. Consider

$$\begin{aligned}\left(\sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle u_{ij}|u\rangle|^2\right)^{1/2} &= ||\{\langle u_{ij}|u\}_{j\in\sigma_i,i\in[m]}|| \\ &\geq ||\{\langle u_{ij}|w\}_{j\in\sigma_i,i\in[m]}|| - ||\{\langle u_{ij}|u-w\}_{j\in\sigma_i,i\in[m]}|| \\ &\geq \sqrt{r_1}||K^*u|| - 2\sqrt{r_2}\epsilon,\end{aligned}$$

since $\epsilon > 0$ is arbitrary. Therefore

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle u_{ij}|u\rangle|^2 \geq r_1||K^*u||^2.$$

□

If $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ be two Bessel sequences in $\mathbb{H}^R(\mathfrak{Q})$. Then for any $\sigma \subset \mathcal{I}$, $\{u_i\}_{i\in\sigma} \cup \{p_i\}_{i\in\sigma^c}$ forms a Bessel sequence. Then the corresponding frame operator $S^\sigma_{u,p}$ is given by

$$S^\sigma_{u,p}(u) = \sum_{i\in\sigma} u_i\langle u_i|u\rangle + \sum_{i\in\sigma^c} p_i\langle p_i|u\rangle,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

In the next result, we have obtained a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ from a given K-frames of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.37. *Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame with bounds r_1^1, r_2^1 and $\{p_i\}_{i\in\mathcal{I}}$ be another K-frame with bounds r_1^2, r_2^2. If there exists $\alpha, \beta \in [0,1)$ such that $||S^\sigma_{u,p}u - K^*u|| \leq \alpha||S^\sigma u|| + \beta||K^*u||$,*
$u \in \mathbb{H}^R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ form a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $\frac{(1-\beta)^2}{(1+\alpha)^2||T||^2}$ and $r_2^1 + r_2^2$, where T is the synthesis operator corresponding to $\{u_i\}_{i\in\sigma} \cup \{p_i\}_{i\in\sigma^c}$.

Proof. Let $\sigma \subset \mathcal{I}$ be any subset of $\mathcal{I}$,
$$\sum_{i\in\sigma} |\langle u_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle p_i|u\rangle|^2 \le (r_2^1 + r_2^2)||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$
Consider,
$$||S^\sigma_{u,p}(u)|| \ge ||K^*u|| - ||S^\sigma_{u,p}(u) - K^*(u)|| \ge (1-\beta)||K^*u|| - \alpha||S^\sigma_{u,p}u||.$$
Thus, $\frac{1-\beta}{1+\alpha}||K^*u|| \le ||S^\sigma_{u,p}u||$. Hence
$$\frac{(1-\beta)^2}{(1+\alpha)^2||T||^2}||K^*u||^2 \le \sum_{i\in\sigma} |\langle u_i|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle p_i|u\rangle|^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$
□

In the next result, we have given conditions under which a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proposition 4.38. Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^1, r_2^1 and $\{p_i\}_{i\in\mathcal{I}}$ be another K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^2, r_2^2.

(1) If $S^\sigma_{u,p}$ is bounded below. Then $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.
(2) If there exits $\alpha, \beta \in [0, 1)$ such that $||S^\sigma_{u,p}u - K^*u|| \le \alpha||S^\sigma u|| + \beta||K^*u||$, $u \in \mathbb{H}^R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.
(3) If $R(K) \subset R(S^\sigma_{u,p})$. Then $\{u_i\}_{i\in\mathcal{I}}$ and $\{p_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

In the next theorem, we have obtained a woven K-frame from a given woven K-frames.

Theorem 4.39. *Let $\{u_{ij}\}_{j\in\mathcal{J}, i\in\mathbb{N}_3}$ be K-frames of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^i, r_2^i and T_i be corresponding synthesis operators. Let $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a positive right linear operator with closed range. Further, let $\{u_{1j}\}_{j\in\mathcal{J}}$, $\{u_{2j}\}_{j\in\mathcal{J}}$ and $\{u_{1j}\}_{j\in\mathcal{J}}$, $\{u_{3j}\}_{j\in\mathcal{J}}$ both are woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$ with lower bounds r_1^{12} and r_1^{13}, respectively. If $K(u) = \sum_{j\in\mathcal{J}} u_{3j}\langle u_{2j}|u\rangle$, $u \in \mathbb{H}^R(\mathfrak{Q})$ and $r_1^{12} + r_1^{13} > \Big(r_2^1 + 2\sqrt{r_2^2 r_2^3}\Big)||K^\dagger||^2$. Then $\{u_{1j}\}_{j\in\mathcal{J}}$, $\{u_{2j} + u_{3j}\}_{j\in\mathcal{J}}$ forms a woven K-frame of $R(K)$ with universal bounds $\Big(r_1^{12} + r_1^{13} - (r_2^1 + 2\sqrt{r_2^2 r_2^3})||K^\dagger||^2\Big)$ and $\Big(r_2^1 + 2(r_2^2 + r_2^3)\Big)$, respectively.*

Proof. For any $\sigma \subset \mathcal{I}$, let $T_2^\sigma(\{q_i\}_{i\in\mathcal{I}}) = \sum_{i\in\sigma} u_{2i}q_i$ and $T_3^\sigma(\{q_i\}_{i\in\mathcal{I}}) = \sum_{i\in\sigma} u_{3i}q_i, \{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q})$. As, $||T_2^\sigma|| \le ||T_2||$ and $||T_3^\sigma|| \le ||T_3||$. Further, for any $u \in R(K)$,
$$\begin{aligned}
\sum_{i\in\sigma} |\langle u_{1i}|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle u_{2i} + u_{3i}|u\rangle|^2 \ &\ge\ r_1^{12}||K^*u||^2 + r_1^{13}||K^*u||^2 - r_2^1||u||^2 \\
&-\ \Big|\langle \sum_{i\in\sigma} u_{3i}\langle u_{2i}|u\rangle + \sum_{i\in\sigma} u_{2i}\langle u_{3i}|u\rangle|u\rangle\Big| \\
&\ge\ (r_1^{12} + r_1^{13})||K^*u||^2 - r_2^1||u||^2 \\
&-\ ||u||\,||T_3^\sigma T_2^* u + T_2^\sigma T_3^* u|| \\
&\ge\ (r_1^{12} + r_1^{13} - (r_2^1 + 2\sqrt{r_2^2 r_2^3})||K^\dagger||^2)||K^*u||^2.
\end{aligned}$$

$\square$

Theorem 4.40. *Let $\{u_{ij}\}_{j\in\mathcal{J},i\in[m]}$ be a family of K-frames of $\mathbb{H}^R(\mathfrak{Q})$. Further, let $T, U \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be right linear operators with closed ranges such that, $KUT^\dagger = UT^\dagger K$ and $R(K^*) \subset R(U)$. If $\{Tu_{ij}\}_{j\in\mathcal{J},i\in[m]}$ is a woven K-frames of $R(T)$. Then $\{Uu_{ij}\}_{j\in\mathcal{J},i\in[m]}$ is a woven K-frame of $R(U)$.*

Proof. Define $L : R(T) \to R(U)$ as $L(u) = UT^\dagger(u)$. Since, $N(T) = N(U)$ we have

$$R(T^*) = (N(T))^\perp = (N(U))^\perp = R(U^*). \text{Thus,}$$
$$N(UT^\dagger) = N(TT^\dagger) = (R(T))^\perp,$$
$$T^\dagger T = P_{R(T^*)} = P_{R(U^*)} = U^\dagger U,$$
$$LT = UT^\dagger T = UU^\dagger U = U.$$

Hence,

$$N(L) = N(UT^\dagger) \cup R(T) = (R(T))^\perp \cup R(T) = \{0\}.$$

Therefore, L is an invertible right linear operator on $R(T)$. Suppose R_1, R_2 be universal bounds of $\{Tu_{ij}\}_{j\in\mathcal{I},i\in[m]}$. Then for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$ and any $u \in R(U)$,

$$\begin{aligned}
R_1||L^{-1}||^{-2}||K^*u||^2 &\leq R_1||K^*L^*u||^2 \\
&\leq \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle Uu_{ij}|u\rangle|^2 \\
&= \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle Tu_{ij}|L^*u\rangle|^2 \\
&\leq R_2||L||^2||u||^2,\ u \in R(U).
\end{aligned}$$

$\square$

In the next result, we have studied a condition under which given K-frames of $\mathbb{H}^R(\mathfrak{Q})$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.41. *Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^1, r_2^1 and $\{p_i\}_{i\in\mathcal{I}}$ be another K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^2, r_2^2. If there exists $\alpha, \beta \in [0, 1)$ such that*

$$(\sum_{i\in J} |\langle p_i - u_i|u\rangle|^2)^{1/2} \leq \alpha(\sum_{i\in J} |\langle u_i|u\rangle|^2)^{1/2} + \beta(\sum_{i\in J} |\langle p_i|u\rangle|^2)^{1/2},\ u \in \mathbb{H}^R(\mathfrak{Q})$$

for any $J \subset \mathcal{I}$. Then $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $\left(r_1^1\, min\{1, (\frac{1-\alpha}{1-\beta})^2\}\right)$ and $r_2^1 + r_2^2$, respectively.

Proof. For any subset $\sigma \subset \mathcal{I}$, consider

$$\begin{aligned}
\Big(\sum_{i\in\sigma^c} |\langle u_{2i}|u\rangle|^2\Big)^{1/2} \leq \Big(\sum_{i\in\sigma^c} |\langle u_{1i}|u\rangle|^2\Big)^{1/2} &- \Big(\sum_{i\in\sigma^c} |\langle u_{2i} - u_{1i}|u\rangle|^2\Big)^{1/2} \\
&\geq (1-\alpha)\Big(\sum_{i\in\sigma^c} |\langle u_{1i}|u\rangle|^2\Big)^{1/2} \\
&- \beta\Big(\sum_{i\in\sigma^c} |\langle u_{2i}|u\rangle|^2\Big)^{1/2},\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

Hence,

$$\sum_{i\in\sigma^c} |\langle u_{2i}|u\rangle|^2 \geq \Big(\frac{1-\alpha}{1+\beta}\Big)^2 \sum_{i\in\sigma^c} |\langle u_{1i}|u\rangle|^2.$$

Therefore,

$$\begin{aligned}
(r_2^1 + r_2^2)||u||^2 &\geq \sum_{i\in\sigma} |\langle u_{1i}|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle u_{2i}|u\rangle|^2 \\
&\geq min\{1, \Big(\frac{1-\alpha}{1+\beta}\Big)^2\} \left(\sum_{i\in\sigma} |\langle u_{1i}|u\rangle|^2 + \sum_{i\in\sigma^c} |\langle u_{2i}|u\rangle|^2\right) \\
&= min\{1, \Big(\frac{1-\alpha}{1+\beta}\Big)^2\} \sum_{i\in\mathcal{I}} |\langle u_{1i}|u\rangle|^2 \\
&\geq \left(r_1^1\, min\{1, \Big(\frac{1-\alpha}{1+\beta}\Big)^2\}\right) ||K^*u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

In the next result, we have obtained a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ from given woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.42. *Let $\{u_{ij}\}_{j\in\mathcal{J},i\in\mathbb{N}_2}$ and $\{p_{ij}\}_{j\in\mathcal{J},i\in\mathbb{N}_2}$ both are woven K-frames of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1, r_2 and R_1, R_2, respectively. For any $\sigma \subset \mathcal{J}$, let U_u^σ and U_p^σ be the analysis operators corresponding to $\{u_{1j}\}_{j\in\sigma} \cup \{u_{2j}\}_{j\in\sigma^c}$ and $\{p_{1j}\}_{j\in\sigma} \cup \{p_{2j}\}_{j\in\sigma^c}$, respectively. If there exists $0 \leq \alpha, \beta < 2$, such that for any $\sigma \subset \mathcal{J}$, we have*

$$\begin{aligned}
\sum_{j\in\sigma} |\langle u_{ij} - p_{1j}|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle u_{2j} - p_{2j}|u\rangle|^2 &\leq \alpha(\sum_{j\in\sigma} |\langle u_{1j}|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle u_{2j}|u\rangle|^2) \\
&+ \beta(\sum_{j\in\sigma} |\langle p_{1j}|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle p_{2j}|u\rangle|^2),\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

Then $\{u_{ij} + p_{ij}\}_{j\in\mathcal{J},i\in\mathbb{N}_2}$ is a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $((2-\alpha)r_1^1 + (2-\beta)r_1^2)$ and $2(r_2^1 + r_2^2)$, respectively.

Proof. For any subset $\sigma \subset \mathcal{J}$,

$$||U_u^\sigma u - U_p^\sigma u|| \leq \alpha||U_u^\sigma u||^2 + \beta||U_p^\sigma u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Therefore,

$$\begin{aligned}
\sum_{j\in\sigma} |\langle u_{1j} + p_{1j}|u\rangle|^2 &+ \sum_{j\in\sigma^c} |\langle u_{2j} + p_{2j}|u\rangle|^2 \\
&= ||U_u^\sigma u + U_p^\sigma u||^2 \\
&= 2||U_u^\sigma u||^2 + 2||U_p^\sigma u||^2 - ||U_u^\sigma u - U_p^\sigma u||^2 \\
&\geq (2-\alpha)r_1||K^*u||^2 + (2-\beta)R_1||K^*u||^2 \\
&= \Big((2-\alpha)r_1 + (2-\beta)R_1\Big)||K^*u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

In the next result, we have obtained a woven T-frame from a woven K-frame under certain condition.

Theorem 4.43. *Let $T, K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be right linear operators such that K has closed range. If there exists, $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$ such that*

$$||(T^* - K^*)u|| \leq \alpha_1||T^*u|| + \alpha_2||K^*u|| + \alpha_3||u||,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Then $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ forms a woven T-frame, if it is a woven K-frame, in $R(K)$.

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ be a woven K-frame with bounds r_1 and r_2, respectively. Then for any subset $\sigma \subset \mathbb{N}$,

$$r_1||K^*u||^2 \leq \sum_{j\in\sigma} |\langle u_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle p_j|u\rangle|^2 \leq r_2||u||^2, u \in\ R(K).$$

Further by Lemma 3.3, $\frac{u}{||K^\dagger||} \leq ||K^*u||,\ u \in R(K)$. Using the fact $||K^*u|| \geq ||T^*u|| - ||(T^* - K^*)u||$. We have $(1-\alpha_1)||T^*u|| \leq (1 + \alpha_2 + \alpha_3||K^\dagger||)||K^*u||$, hence

$$r_1\left(\frac{1-\alpha_1}{1+\alpha_2+\alpha_3||K^\dagger||}\right)^2 ||T^*u|| \leq \sum_{j\in\sigma} |\langle u_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle p_j|u\rangle|^2 \leq r_2||u||^2,\ u \in R(K).$$

□

Corollary 4.44. Let $T, K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be right linear operators. If there exists $\alpha_1, \alpha_2 \in (0,1)$ such that

$$||T^*u - K^*u|| \leq \alpha_1||T^*u|| + \alpha_2||K^*u||,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Then $\{u_j\}_{j\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ forms a woven T-frame, if it is a woven K-frame.

Theorem 4.45. *Let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$ be a right linear operator and $\mathfrak{F} = \{u_j\}_{j\in\mathbb{N}}$, $\mathfrak{G} = \{p_j\}_{j\in\mathbb{N}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal lower and upper bounds r_1 and r_2, respectively. If there exists a subset $\mathbb{J} \subset \mathbb{N}$ and a constant C, $0 < C < \frac{r_1}{||T||^2}$ such that $\sum\limits_{j\in\mathbb{J}} |\langle Tu_j|u\rangle|^2 \leq C||TK^*T^*u||^2$, $u \in \mathbb{H}_1^R(\mathfrak{Q})$. Then $\{Tu_j\}_{j\in\mathbb{N}\setminus\mathbb{J}}$, $\{Tp_j\}_{j\in\mathbb{N}\setminus\mathbb{J}}$ forms a woven TKT^*-frame of $\mathbb{H}_1^R(\mathfrak{Q})$.*

Proof. By Theorem 4.17, $T\mathfrak{F}$, $T\mathfrak{G}$ forms a woven TKT^*-frame of $\mathbb{H}_1^R(\mathfrak{Q})$ with universal lower bound $\frac{r_1}{||T||^2}$. For any subset $\sigma \subset \mathbb{N} \setminus \mathbb{J}$, we have

$$\begin{aligned}\sum_{j\in\sigma} |\langle Tu_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle Tp_j|u\rangle|^2 &= \sum_{j\in\sigma\cup\mathbb{J}} |\langle Tu_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle Tp_j|u\rangle|^2 - \sum_{j\in\mathbb{J}} |\langle Tu_j|u\rangle|^2 \\ &\geq \left(\frac{r_1}{||T||^2} - C\right) ||(TKT^*)^*u||^2,\ u \in \mathbb{H}_1^R(\mathfrak{Q}).\end{aligned}$$

For upper bound inequality, consider

$$\sum_{j\in\sigma} |\langle Tu_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle Tp_j|u\rangle|^2 \leq \sum_{j\in\sigma\cup\mathbb{J}} |\langle Tu_j|u\rangle|^2 + \sum_{j\in\sigma^c} |\langle Tp_j|u\rangle|^2 \leq r_2||u||^2,\ u \in \mathbb{H}_1^R(\mathfrak{Q}).$$

□

If we take $\mathbb{H}^R(\mathfrak{Q}) = \mathbb{H}_1^R(\mathfrak{Q})$, $T = I$ in Theorem 4.45. We obtained the following result.

Corollary 4.46. Let $\mathfrak{F} = \{u_j\}_{j\in\mathbb{N}}$, $\mathfrak{G} = \{p_j\}_{j\in\mathbb{N}}$ be a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds r_1 and r_2, respectively. If there exists a subset $\mathbb{J} \subset \mathbb{N}$ and a constant C, $0 < C < r_1$ such that $\sum\limits_{j\in\mathbb{J}} |\langle u_j|u\rangle|^2 \leq C||K^*u||^2$, $u \in \mathbb{H}^R(\mathfrak{Q})$. Then $\{u_j\}_{j\in\mathbb{N}\setminus\mathbb{J}}$, $\{p_j\}_{j\in\mathbb{N}\setminus\mathbb{J}}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.47. *Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^1, r_2^1 and $\{p_i\}_{i\in\mathcal{I}}$ be a Q-frame with bounds r_1^2, r_2^2. Further, let $T_1, T_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be surjective right linear operators such that $T_iK = KT_i, i = 1, 2$. If $R(K)$ is closed and there exists $\alpha, \beta, \gamma \in [0, \infty)$ such that $r_1^1 > M\Big(\sqrt{r_2^1}||T_1|| + \sqrt{r_2^2}||T_2||\Big)||T_1^\dagger||^2||K^\dagger||^2$, where $M = \sqrt{r_2^1}\Big(\alpha\sqrt{r_2^1} + \beta\sqrt{r_2^2} + \gamma\Big) + \sqrt{r_2^2}||T_1 - T_2||$, satisfying*

$$||\sum_{i\in\mathcal{I}}(u_i - p_i)q_i|| \leq \alpha||\sum_{i\in\mathcal{I}} u_iq_i|| + \beta||\sum_{i\in\mathcal{I}} p_iq_i|| + \gamma\Big(\sum_{i\in\mathcal{I}} |q_i|^2\Big)^{1/2}, \ \forall\, \{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q}).$$

Then $\{T_1u_i\}_{i\in\mathcal{I}}$, $\{T_2p_i\}_{i\in\mathcal{I}}$ forms a woven frame of $R(K)$ with universal bounds $\frac{r_1^1 - M\Big(\sqrt{r_2^1}||T_1|| + \sqrt{r_2^2}||T_2||\Big)||T_1^\dagger||^2||K^\dagger||^2}{||T_1^\dagger||^2||K^\dagger||^2}$, $r_2^1||T_1||^2 + r_2^2||T_2||^2$.

Proof. Let $\{Q_i\}_{i=1}^4$, be the synthesis operators corresponding to Bessel sequences $\{u_i\}_{i\in\mathcal{I}}, \{p_i\}_{i\in\mathcal{I}}, \{T_1u_i\}_{i\in\mathcal{I}}$ and $\{T_2p_i\}_{i\in I}$, respectively. Then, $||Q_1 - Q_2|| \leq \alpha\sqrt{r_2^1} + \beta\sqrt{r_2^2} + \gamma$, and by direct calculation, $Q_3 = T_1Q_1, Q_4 = T_2Q_2$. Therefore,

$$\begin{aligned} ||Q_3 - Q_4|| &= ||T_1Q_1 - T_2Q_2|| \\ &\leq \sqrt{r_2^1}(\alpha\sqrt{r_2^1} + \beta\sqrt{r_2^2} + \gamma) + \sqrt{r_2^2}||T_1 - T_2|| \\ &= M. \end{aligned}$$

For any subset $\sigma \subset \mathcal{I}$, consider

$$||\sum_{i\in\sigma} T_1u_i\langle T_1u_i|u\rangle - \sum_{i\in\sigma} T_2p_i\langle T_2p_i|u\rangle|| \leq M\Big(\sqrt{r_2^1}||T_1|| + \sqrt{r_2^2}||T_2||\Big)||u||, \ u \in \mathbb{H}^R(\mathfrak{Q}).$$

For any partition $\{\sigma\}_{i=1}^2$ of $\mathcal{I}$ and any $u \in R(K)$ we have

$$\begin{aligned} \sum_{i\in\sigma_1} |\langle T_1u_i|u\rangle|^2 + \sum_{i\in\sigma_2} |\langle T_2p_i|u\rangle|^2 &= \sum_{i\in\mathcal{I}} |\langle u_i|T_1^*u\rangle|^2 + \langle\sum_{i\in\sigma_2} T_2p_i\langle T_2p_i|u\rangle \\ &- \sum_{i\in\sigma_2} T_1u_i\langle T_1u_i|u\rangle, u\rangle \\ &\geq r_1^1||K^*T_1^*u||^2 - ||\langle\sum_{i\in\sigma_2} T_2p_i\langle T_2p_i|u\rangle \\ &- \sum_{i\in\sigma_2} T_1u_i\langle T_1u_i|u\rangle||\,||u|| \\ &\geq \frac{r_1^1 - M\Big(\sqrt{r_2^1}||T_1|| + \sqrt{r_2^2}||T_2||\Big)||T^\dagger||^2||K^\dagger||^2}{||T_1^\dagger||^2||K^\dagger||^2}||u||^2. \end{aligned}$$

The upper bound inequality follow immediately,

$$\sum_{i\in\sigma_1} |\langle T_1u_i|u\rangle|^2 + \sum_{i\in\sigma_2} |\langle T_2p_i|u\rangle|^2 \leq (r_2^1||T_1||^2 + r_2^2||T_2||^2)||u||^2, \ u \in R(K).$$

□

If we take $T_1 = T_2 = I$ in Theorem 4.47. We obtained the following result.

Corollary 4.48. Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^1, r_2^1 and $\{p_i\}_{i\in\mathcal{I}}$ be a Q-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1^2, r_2^2. If $R(K)$ is closed and there exists $\alpha, \beta, \gamma \in [0, \infty)$ such that $r_1^1 > \sqrt{r_2^1}\Big(\alpha\sqrt{r_2^1} + \beta\sqrt{r_2^2} + \gamma\Big)\Big(\sqrt{r_2^1} + \sqrt{r_2^2}\Big)||K^\dagger||^2$ and

$$||\sum_{i\in\mathcal{I}}(u_i - p_i)q_i|| \leq \alpha||\sum_{i\in\mathcal{I}} u_i q_i|| + \beta||\sum_{i\in\mathcal{I}} p_i q_i|| + \gamma(\sum_{i\in\mathcal{I}} |q_i|^2)^{1/2}, \forall\, \{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q}).$$

Then $\{u_i\}_{i\in\mathcal{I}}$, $\{p_i\}_{i\in\mathcal{I}}$ form a woven frame of $R(K)$ with universal bounds $\frac{r_1^1 - \sqrt{r_2^1}\Big(\alpha\sqrt{r_2^1}+\beta\sqrt{r_2^2}+\gamma\Big)(\sqrt{r_2^1}+\sqrt{r_2^2})||K^\dagger||^2}{||K^\dagger||^2}$, $r_2^1 + r_2^2$.

If we take $\alpha = \beta = \gamma = 0$ in Theorem 4.47. We obtained the following results.

Corollary 4.49. Let $\{u_i\}_{i\in\mathcal{I}}$ be a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1, r_2. Further, let $T_1, T_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be surjective right linear operators such that $T_i K = K T_i$, $i = 1, 2$. If $R(K)$ is closed and $r_1 > M\sqrt{r_2}(||T_1|| + ||T_2||)||T_1^\dagger||^2||K^\dagger||^2$, where $M = \sqrt{r_2}||T_1 - T_2||$. Then $\{T_1 u_i\}_{i\in\mathcal{I}}$, $\{T_2 u_i\}_{i\in\mathcal{I}}$ forms a woven frame of $R(K)$ with universal frame bounds $\frac{r_1 - M\sqrt{r_2}\Big(||T_1||+||T_2||\Big)||T_1^\dagger||^2||K^\dagger||^2}{||T_1^\dagger||^2||K^\dagger||^2}$, $r_2(||T_1||^2 + ||T_2||^2)$.

Theorem 4.50. *Let $\{u_{ij}\}_{j\in\mathcal{I}, i\in[m]}$ be a family of K-frames of $\mathbb{H}^R(\mathfrak{Q})$ with bounds $r_1^i, r_2^i, i \in [m]$. further, Let $T_i's, i \in [m]$ be the corresponding synthesis operators. If for any $\sigma \subset \mathcal{I}$, $R(T_i^\sigma) \subset R(K), i \in [m]$ and there exists constants $\alpha_i, \beta_i, \gamma_i \geq 0$, $i \in [m]$ such that*

$$\sum_{i\in[m]/\{n\}} \Big(\sqrt{r_2^n} + \sqrt{r_2^i}\Big)\Big(\alpha_i\sqrt{r_2^n} + \beta_i\sqrt{r_2^i} + \gamma_i\Big)||\widetilde{K^*}^{-1}||^2 < r_1^n$$

and

$$||\sum_{j\in\mathcal{I}}(u_{uj} - u_{ij})q_j|| \leq \alpha_i||\sum_{j\in\mathcal{I}} u_{nj}q_j|| + \beta_i||\sum_{j\in\mathcal{I}} u_{ij}q_j|| + \gamma_i(\sum_{j\in\mathcal{I}} |q_j|^2)^{1/2}, \forall\, i \in [m]/\{n\}$$

for any $\{q_j\}_{j\in\mathcal{I}} \in l^2(\mathfrak{Q})$. Then $\{u_{ij}\}_{j\in\mathcal{I}, i\in[m]}$ forms a woven K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal bounds $r_1^n - \sum_{i\in[m]/\{n\}}\Big(\sqrt{r_2^n}+\sqrt{r_2^i}\Big)\Big(\alpha_i\sqrt{r_2^n}+\beta_i\sqrt{r_2^i}+\gamma_i\Big)||\widetilde{K^*}^{-1}||^2$ and $\sum_{i\in[m]} r_2^i$, where $\widetilde{K^*} : N(K^*)^\perp \to R(K^*)$ is the restriction of K^* on $N(K^*)^\perp$.

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{I}$ and any $u \in \mathbb{H}^R(\mathfrak{Q})$, we have $\sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle u_{ij}|u\rangle|^2 \leq \sum_{i\in[m]} r_2^i||u||^2$. Also for fixed, $n \in [m]$ and $i \in [m]/\{n\}$ we have $||T_n - T_i|| \leq \alpha_i\sqrt{r_2^n} + \beta_i\sqrt{r_2^i} + \gamma_i$. Hence $||T_n^{\sigma_i}T_n^* - T_i^{\sigma_i}T_i^*|| \leq$

$\left(\sqrt{r_2^n}+\sqrt{r_2^i}\right)\left(\alpha_i\sqrt{r_2^n}+\beta_i\sqrt{r_2^i}+\gamma_i\right)$. Therefore for any $u \in N(K^*)^\perp$,

$$\begin{aligned}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|u\rangle|^2 \quad &\geq \quad \sum_{j\in\mathcal{I}}|\langle u_{nj}|u\rangle|^2 \\ &- \quad ||u||\sum_{i\in[m]/\{n\}}||\sum_{j\in\sigma_i}\left(u_{nj}\langle u_{nj}|u\rangle - u_{ij}\langle u_{ij}\langle u_{ij}|u\rangle\right)|| \\ &\geq \quad \left(r_1^n - \sum_{i\in[m]/\{n\}}(\sqrt{r_2^n}+\sqrt{r_2^i})(\alpha_i\sqrt{r_2^n}+\beta_i\sqrt{r_2^i}+\gamma_i)||\widetilde{K^*}^{-1}||^2\right) \\ &||K^*u||^2 \quad .\end{aligned}$$

Every $u \in \mathbb{H}^R(\mathfrak{Q})$ can be written as $u = u_1 + u_2$, where $u_1 \in N(K^*), u_2 \in N(K^*)^\perp$. Since $R(T_i^\sigma) \subset R(K)$, for every subset $\sigma \subset \mathcal{I}$, $i \in [m]$. Hence,

$$\begin{aligned}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|u\rangle|^2 \quad &= \quad \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|u_1+u_2\rangle|^2 \\ &= \quad \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle u_{ij}|u_2\rangle|^2 \\ &\geq \quad \left(r_1^n - \sum_{i\in[m]/\{n\}}\left(\sqrt{r_2^n}+\sqrt{r_2^i}\right)\left(\alpha_i\sqrt{r_2^n}+\beta_i\sqrt{r_2^i}+\gamma_i\right)||\widetilde{K^*}^{-1}||^2\right) \\ &||K^*u||^2.\end{aligned}$$

□

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