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PARTIAL STABILITY ANALYSIS OF SOME CLASSES OF TIME–VARYING SYSTEMS

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Abstract. In this work, we characterize the partial stability of nonlinear systems via comparison functions. Further, we develop the partial stability of linear time–varying systems within Lyapunov techniques. Moreover, the indirect method of Lyapunov is also investigated to derive the local partial exponential stability. We state sufficient conditions on partial uniform exponential stability and partial practical uniform exponential stability of solutions of linear time–varying perturbed systems. Finally, an illustrative numerical example is provided.

1. Introduction

Differential equations are a fundamental means for describing the time-varying process. The study of differential equations in dynamical systems and control is of the greatest significance, since differential equations arise in all disciplines of science, medicine, biology, engineering, and economics. The theory of stability of nonlinear differential equations has become a highly prevalent topic in a lot of areas of physics and engineering. Different authors tackle the problem of stability of differential equations, see [23, 24, 25].

Recently, special attention was given to the study of the stability properties of the solution of linear systems. Determine the quality characteristics of the solution of linear time–invariant systems is more easy than the perception of similar characteristics of linear time–varying systems. This is very tough and complicated, since it demands the evolution of the transition system matrix. The study of stability analysis of linear time–varying systems is of increasing interest ([2], [10], [12]-[15]). One reason is the

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growing importance of adaptative controllers. The underlying closed-loop adaptative system often is a linear time–varying system.

Some differential equations cannot be demonstrated to fill the stability properties against all unknown system variables. Though, it is very important to analyze if it is still possible to prove certain stability criteria for some of the variables in the problem which is called “Partial stability”.

The issue of the partial stability of the zero equilibrium point arises naturally in applications, the basic findings in this area belong to V. V. Rumyantsev, see [18, 19, 20], which has formed the theory of partial stability of differential equation. Moreover, he has shown the applicability of his results to more general model stability issues of distributed parameter systems. When the origin is not an equilibrium point, we can study the stability of dynamic systems in a small neighborhood of origin in terms of convergence of solution to a small ball. It is defined as “Practical stability.” The practical stability, in the sense introduced in [1, 9, 21, 22]. It is very important and very helpful to analyze stability or to design practical controllers for dynamic systems since controlling system to an idealized point are either expensive or impossible in the case of disturbances, the best we can hope for in such situations is to use practical stability. In practice, we may only need to stabilize a system in the phase space region where its implementation remains acceptable. It is well known that asymptotic stability is more important than stability. In addition, the desired system can oscillate towards the origin. Thus, the concept of practical stability is more appropriate in many cases than the stability of Lyapunov. In such a case, all state trajectories are bounded and approach a sufficiently small neighborhood of the origin.

Generally, we know certain information about the upper bound of the disturbance term whose size affects the size of the ball.

A number of authors deal with the problem of practical stability, see [3, 4, 8, 5, 6, 7, 11].

One of our major findings in this paper is to prove the partial uniform exponential stability as well as the partial practical uniform exponential stability of linear time–varying perturbed systems based on the method of Lyapunov.

The manuscript is organized as follows. In Section 2, we introduce some necessary notations and preliminaries about partial stability. In Section 3, we prove the partial uniform stability, the partial uniform asymptotic stability, the partial uniform exponential stability by using comparison functions. In Section 4, we investigate the problem of partial stability of linear time–varying systems using Lyapunov techniques, the indirect method of Lyapunov is also investigated to derive the local partial exponential stability. In Section 5, we set certain stability criteria for partial uniform exponential stability and the partial practical uniform exponential stability of linear time–varying systems based on Lyapunov techniques. In Section 6, we give an example demonstrating the applicability of our abstract theory. Finally, in Section 7, some conclusions are achieved.

2. Preliminaries

Let us introduce a few notations and preliminaries about partial stability analysis of nonlinear time–varying systems.

Notations:

$\mathcal{C}(\mathcal{I},\mathcal{S})$: Set of continuous functions $f:\mathcal{I}\to\mathcal{S}$ from an open set $\mathcal{I}\subseteq \mathrm{R}$ to a vector space $\mathcal{S}$.

$\mathcal{C}^k(\mathcal{I},\mathcal{S})$: The set of k–times continuously differentiable functions $f:\mathcal{I}\to\mathcal{S}$ from an open set $\mathcal{I}\subseteq \mathrm{R}$ to a vector space $\mathcal{S}$.

I : Identity matrix in $\mathrm{R}^{m\times m}$.

$||y|| := \sqrt{y^T y}$: Euclidean norm of $y\in\mathrm{R}^m$.

Let's consider the following nonlinear system varying in time:

$$\dot{y}(\mathrm{t}) = \mathcal{F}(\mathrm{t}, y(\mathrm{t})), \quad \mathrm{t}\geq \mathrm{t}_0 \geq 0, \tag{2.1}$$

where $\mathcal{F}:\mathrm{R}_+\times\mathrm{R}^m\to\mathrm{R}^m$ is continuous, locally Lipschitz in y, uniformly in t.

Denote $y=(y_1,y_2)$, where $y_1=(y_{1_1},\cdots,y_{1_k})\in\mathrm{R}^k,\ y_2=(y_{2_1},\cdots,y_{2_q})\in\mathrm{R}^q,\ k>0,\ q\geq 0,\ k+q=m$;

$$||y_1|| = \sqrt{y_{1_1}^2+\cdots+y_{1_k}^2},\ ||y_2|| = \sqrt{y_{2_1}^2+\cdots+y_{2_q}^2},\ ||y|| = \left(||y_1||^2+||y_2||^2\right)^{\frac{1}{2}}.$$

System (2.1) may be written as:

$$\begin{cases} \dot{y}_1(\mathrm{t}) = \mathcal{F}_1(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})) \\ \\ \dot{y}_2(\mathrm{t}) = \mathcal{F}_2(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})), \end{cases}$$

with initial condition $y(\mathrm{t}_0) := y_0 = (y_{1_0}, y_{2_0}),\ \mathcal{F} := (\mathcal{F}_1,\mathcal{F}_2)$, and

$$\mathcal{F}_1:\mathrm{R}_+\times\mathrm{R}^k\times\mathrm{R}^q\to\mathrm{R}^k, \quad \mathcal{F}_2:\mathrm{R}_+\times\mathrm{R}^k\times\mathrm{R}^q\to\mathrm{R}^q.$$

Under the previous assumptions, there exist a unique global solution:

$$y(\mathrm{t},\mathrm{t}_0,y_0) = (y_1(\mathrm{t},\mathrm{t}_0,y_0), y_2(\mathrm{t},\mathrm{t}_0,y_0))$$

correspondent to the initial condition $y_0\in\mathrm{R}^m$. In the ensuing, we use $y(\mathrm{t},\mathrm{t}_0,y_0) = (y_1(\mathrm{t},\mathrm{t}_0,y_0), y_2(\mathrm{t},\mathrm{t}_0,y_0))$, or merely $y(\mathrm{t}) = (y_1(\mathrm{t}), y_2(\mathrm{t}))$ to denote a solution to the Eq.(2.1).

In the sequel, we define the main concepts of partial stability in the case where the origin is an equilibrium point of the Eq.(2.1), i.e., $\mathcal{F}(\mathrm{t},0)=0$, for all $\mathrm{t}\geq\mathrm{t}_0\geq 0$.

Definition 2.1. The equilibrium point $y=0$ of the nonlinear time–varying equation (2.1) is:

i) Stable with respect to y_1, if for each $\eta>0$ and $\mathrm{t}_0\in\mathrm{R}_+$ there exists $\eta=\eta(\varepsilon,\mathrm{t}_0)>0$, such that for any $y_0\in\mathrm{R}^m$,

$$\|y_0\|<\eta \Rightarrow \|y_1(\mathrm{t},\mathrm{t}_0,y_0)\|<\varepsilon, \quad \forall \mathrm{t}\geq\mathrm{t}_0\geq 0.$$

ii) Uniformly stable with respect to y_1, if it is stable and η depends only on ε.

iii) Uniformly attractive with respect to y_1, if there exists $\mu > 0$ such that for all $\varepsilon > 0$ there exists $\mathrm{T} := \mathrm{T}(\varepsilon) > 0$ such that for any $\mathrm{t}_0 \in \mathrm{R}_+$, and $y_0 \in \mathrm{R}^m$ with $\|y_0\| < \mu$, the inequality $\|y_1(\mathrm{t}, \mathrm{t}_0, y_0)\| < \varepsilon$ holds for $\mathrm{t} \geq \mathrm{t}_0 + \mathrm{T}$.

iv) Globally uniformly attractive with respect to y_1, if (iii) is satisfied for any $\mu > 0$.

v) Uniformly asymptotically stable with respect to y_1, if it is uniformly stable with respect to y_1, and uniformly attractive with respect to y_1.

vi) Globally uniformly asymptotically stable with respect to y_1, if it is uniformly stable with respect to y_1, and globally uniformly attractive with respect to y_1.

vii) Uniformly exponentially stable with respect to y_1, if there exist nonnegative constants $\mu, \alpha_1, \alpha_2 > 0$, such that for each $\|y_0\| \leq \mu$ and any $\mathrm{t}_0 \in \mathrm{R}_+$,

$$\|y_1(\mathrm{t}, \mathrm{t}_0, y_0)\| \leq \alpha_1 \|y_0\| e^{-\alpha_2(\mathrm{t}-\mathrm{t}_0)}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

viii) Globally uniformly exponentially stable with respect to y_1, if (vii) is satisfied for any $\mu > 0$.

Definition 2.2. [13] The solution of the subsystem with respect to the variable y_2 is said to be globally uniformly bounded, if for every $\nu > 0$ there exists $\lambda = \lambda(\nu)$, such that for all $\mathrm{t}_0 \geq 0$,

$$||y_{2_0}|| \leq \nu \Rightarrow ||y_2(\mathrm{t}, \mathrm{t}_0, y_0)|| \leq \lambda, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

with $y_{2_0} = y_2(\mathrm{t}_0, \mathrm{t}_0, y_0)$.

Below, we define the following comparison functions.

Definition 2.3. [16] Let $b \in \mathrm{R}_+ \backslash \{0\}$ and $\bar{\alpha}(\cdot) : [0, b) \to [0, +\infty)$ a continuous function is said to pertain to the class $\mathcal{K}$, if:

(1) $\bar{\alpha}$ is strictly increasing.
(2) $\bar{\alpha}(0) = 0$.

It is said to pertain to class $\mathcal{K}_\infty$, if $b = +\infty$ and $\bar{\alpha}(r) \to +\infty$ as $r \to +\infty$.

Definition 2.4. [16] A continuous function

$$\begin{aligned} \bar{\beta}(\cdot, \cdot) \ : \ \mathrm{R}_+ \times \mathrm{R}_+ \ &\to \ \mathrm{R}_+ \\ (s, \mathrm{t}) \ &\to \ \bar{\beta}(s, \mathrm{t}), \end{aligned}$$

is said to belong to class $\mathcal{KL}$, if:

(1) For all $\mathrm{t} \in \mathrm{R}_+$, $s \mapsto \bar{\beta}(s, \mathrm{t})$ pertains to class $\mathcal{K}$.
(2) For all $s \in [0, b]$, $\mathrm{t} \mapsto \bar{\beta}(s, \mathrm{t})$ is decreasing.
(3) For all $s \in [0, b]$, $\bar{\beta}(s, \mathrm{t}) \to 0$ as $\mathrm{t} \to +\infty$.

3. Partial stability of nonlinear time–varying systems via comparison functions

We aim to characterize the notion of partial stability through comparison functions.

Theorem 3.1. *The equilibrium point $y = 0$ of system* (2.1) *is:*

i) *Uniformly stable with respect to y_1, if and only if, there exist a class $\mathcal{K}$ function $\bar{\alpha}(\cdot)$ and a positive constant $\bar{c}$ irrespective of* t_0, *such that*

$$\|y_1(\mathrm{t}, \mathrm{t}_0, y_0)\| \leq \bar{\alpha}(\|y_0\|), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0, \quad \forall \|y_0\| < \bar{c}.$$

ii) *Globally uniformly stable with respect to* y_1*, if and only if, the previous inequality is fulfilled for all* y_0*.*

Proof. " $\Leftarrow$ " Fix $\mathrm{t}_0 \in \mathrm{R}_+$, suppose that there exists a class $\mathcal{K}$ function $\bar{\alpha}(\cdot)$, where

$$||y_1(\mathrm{t}, \mathrm{t}_0, y_0)|| \leq \bar{\alpha}(||y_0||), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0, \quad \forall ||y_0|| < \bar{c}.$$

Let $\varepsilon \geq 0$ and $\eta = \min\left(\bar{c}, \bar{\alpha}^{-1}(\varepsilon)\right)$, then for all $||y_0|| < \bar{c}$, one obtains

$$||y_1(\mathrm{t}, \mathrm{t}_0, y_0)|| \leq \bar{\alpha}(||y_0||) < \bar{\alpha}(\eta) \leq \bar{\alpha}(\bar{\alpha}^{-1}(\varepsilon)) = \varepsilon, \ \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

That is, $y = 0$ is uniformly stable with respect to y_1.

" $\Rightarrow$ " Fix $\mathrm{t}_0 \in \mathrm{R}_+$, suppose that for all $\varepsilon > 0$ there exists $\eta := \eta(\varepsilon) > 0$, such that for all $\mathrm{t} \geq \mathrm{t}_0$:

$$||y_0|| < \eta \Rightarrow ||y_1(\mathrm{t}, \mathrm{t}_0, y_0)|| < \varepsilon.$$

Now, we consider by $\bar{\eta}(\epsilon)$ the upper bound of all $\eta(\epsilon)$ given above. Hence, for all $\mathrm{t} \geq \mathrm{t}_0 \geq 0$, we obtain the following inequalities:

$$||y_0|| < \bar{\eta}(\epsilon) \Rightarrow ||y_1(\mathrm{t}, \mathrm{t}_0, y_0)|| < \epsilon.$$

Then, if $\eta_1 > \bar{\eta}(\epsilon)$, so there exists at least one initial condition $\widetilde{y}_0$, such that

$$\begin{aligned} &||\widetilde{y}_0|| < \eta_1, \\ &\sup_{\mathrm{t} \geq \mathrm{t}_0} ||y_1(\mathrm{t}, \mathrm{t}_0, \widetilde{y}_0)|| \geq \varepsilon. \end{aligned}$$

Clearly $\bar{\eta}(\epsilon)$ is a defined positive function, strictly increasing, and $\bar{\eta}(\epsilon) \to 0$ as $\epsilon \to 0$. Nevertheless, $\bar{\eta}(\epsilon)$ is not necessarily continuous. Therefore, we may pick $\xi(r)$ a class $\mathcal{K}$ function, such that

$$\xi(r) \leq k\bar{\eta}(r), \quad 0 < k < 1.$$

Now, we set the function $\bar{\alpha}(\cdot)$ as the following:

$$\bar{\alpha}(r) = \xi^{-1}(r), \quad \bar{\alpha}(\cdot) \in \mathcal{K}.$$

So, for $||y_0|| < \bar{c}$ and $\varepsilon = \bar{\alpha}(||y_0||)$, we obtain the following inequalities:

$$||y_0|| = \bar{\alpha}^{-1}(\varepsilon) = \xi(\varepsilon) \leq k\bar{\eta}(\varepsilon) < \bar{\eta}(\varepsilon) < \bar{c}.$$

As $y = 0$ is uniformly stable with respect to y_1, it yields that

$$||y_1(\mathrm{t}, \mathrm{t}_0, y_0)|| \leq \epsilon = \alpha(||y_0||), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

thus proving the theorem. □

Theorem 3.2. *The equilibrium point* $y = 0$ *of system* (2.1) *is:*

i) *Uniformly asymptotically stable with respect to* y_1*, if and only if, there exist a class* $\mathcal{KL}$ *function* $\bar{\beta}(\cdot, \cdot)$ *and a positive constant* $\bar{c}$ *independent of* t_0*, such that*

$$\|y_1(\mathrm{t}, \mathrm{t}_0, y_0)\| \leq \bar{\beta}(\|y_0\|, \mathrm{t} - \mathrm{t}_0), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0, \quad \forall \|y_0\| < \bar{c}.$$

ii) *Globally uniformly asymptotically stable with respect to* y_1*, if and only if, the previous inequality is fulfilled for all* y_0*.*

Proof. " $\Leftarrow$ " Fix $\mathrm{t}_0 \in \mathrm{R}_+$, suppose that there exists a class $\mathcal{KL}$ function $\bar{\beta}(\cdot,\cdot)$, such that for all $||y_0|| < \bar{c}$:

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| < \bar{\beta}(||y_0||,\mathrm{t}-\mathrm{t}_0), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

Hence, it yields that

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| \leq \bar{\beta}(||y_0||,0).$$

Using Theorem 3.1, we deduce $y = 0$ is uniformly stable with respect to y_1.
Furthermore, for all $||y_0|| < \bar{c}$, one obtains

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| \leq \bar{\beta}(\bar{c},\mathrm{t}-\mathrm{t}_0), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

which implies that $y_1(\mathrm{t},\mathrm{t}_0,y_0) \to 0$ as $\mathrm{t} \to \infty$ uniformly on t.
That is, $y = 0$ is uniformly asymptotically stable with respect to y_1.

" $\Leftarrow$ " Now, we suppose that $y = 0$ is uniformly asymptotically stable with respect to y_1.
That means that $y = 0$ is uniformly stable with respect to y. Then, using Theorem 3.1, we deduce that there exists a class $\mathcal{K}$ function $\bar{\alpha}(\cdot)$ and a positive constant $\bar{c}$ independently of t_0, such that

$$\forall r \in [0,\bar{c}],\ \forall ||y_0|| < r,\ ||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| \leq \bar{\alpha}(||y_0||) < \bar{\alpha}(r), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0. \tag{3.1}$$

Furthermore, since $y = 0$ is uniformly attractive with respect to y_1, then for each $\eta > 0$ there is $\mathrm{T} := \mathrm{T}(\eta,r) > 0$, such that

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| < \eta, \quad \forall \mathrm{t} \geq \mathrm{t}_0 + \mathrm{T}(\eta,r).$$

Now, let consider $\bar{\mathrm{T}}(\eta,r)$ as the lower bound of all $\mathrm{T}(\eta,r)$ given above.
That is, we obtain

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| < \eta, \quad \forall \mathrm{t} \geq \mathrm{t}_0 + \bar{\mathrm{T}}(\eta,r).$$

Further, we retain

$$\sup_{\mathrm{t}_0 \leq \mathrm{t} \leq \bar{\mathrm{T}}(\eta,r)} ||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| \geq \eta.$$

It is evident that, the function $\bar{\mathrm{T}}(\eta,r)$ is nonnegative, non–increasing in η, non–decreasing in r, and

$$\bar{\mathrm{T}}(\eta,r) = 0, \quad \forall \eta \geq \bar{\alpha}(r).$$

We define the function $W_r(\cdot)$ as the following:

$$W_r(\eta) = \frac{2}{\eta}\int_{\frac{\eta}{2}}^{\eta} \bar{\mathrm{T}}(s,r)ds + \frac{r}{\eta} \geq \bar{\mathrm{T}}(\eta,r) + \frac{r}{\eta}.$$

Note that $W_r(\eta)$ is a positive function, and fulfils the following properties:

- For all fixed r, $\eta \mapsto W_r(\eta)$ is a continuous function, strictly increasing and $\lim\limits_{\eta \to +\infty} W_r(\eta) = 0$.
- $W_r(\eta)$ is strictly increasing.

Let U_r expressed as follows: $U_r = W_r^{-1}$. One has,

$$\bar{\mathrm{T}}(U_r(s),r) < W_r(U_r(s)) = s.$$

Consequently, one obtains

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| \le U_r(\mathrm{t}-\mathrm{t}_0), \quad \forall ||y_0|| < r, \quad \forall \mathrm{t} \ge \mathrm{t}_0 \ge 0. \tag{3.2}$$

Combining this result with (3.1), we deduce

$$||y_1(\mathrm{t},\mathrm{t}_0,y_0)|| \le \sqrt{\bar{\alpha}(||y_0||)U_{\bar{c}}(\mathrm{t}-\mathrm{t}_0)}, \quad \forall ||y_0|| < \bar{c}, \quad \forall \mathrm{t} \ge \mathrm{t}_0 \ge 0.$$

Finally, we close that $\bar{\beta}(r,s)$ is defined as below:

$$\bar{\beta}(r,s) = \sqrt{\alpha(r)U_{\bar{c}}(s)}. \qquad \square$$

Remark 3.3. If the class $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ is defined as following:

$$\bar{\beta}(r,s) = kr\exp(-\gamma s), \quad k > 0, \quad \gamma > 0.$$

Then, $y = 0$ is exponentially stable with respect to y_1.

4. Linear time–varying systems and linearization

Our objective now is to investigate the partial uniform exponential stability of linear time–varying systems within the method of Lyapunov.

Theorem 4.1. *Let μ_1, μ_2, μ_3, d, be a nonnegative constants, and assume that there exists a continuously differentiable function $\mathcal{V} : \mathrm{R}_+ \times \mathrm{R}^m \to \mathrm{R}$, such that for all* $\mathrm{t} \ge \mathrm{t}_0 \ge 0$, *and all $y = (y_1, y_2) \in \mathrm{R}^m$, we have*

$$\mu_1\|y_1\|^d \le \mathcal{V}(\mathrm{t},y) \le \mu_2\|\bar{y}\|^d, \tag{4.1}$$

$$\dot{\mathcal{V}}(\mathrm{t},y) \le -\mu_3||\bar{y}||^d, \tag{4.2}$$

where $d \ge 2$ and $||\bar{y}|| = (y_{1_1}^2 + \cdots + y_{1_p}^2)^{\frac{1}{2}}, \quad k \le p \le m$.
Then, system (2.1) is globally uniformly exponentially stable with respect to y_1.

Proof. Based on conditions (4.1) and (4.2), it follows that

$$\dot{\mathcal{V}}(\mathrm{t},x) \le -\mu_3\|\bar{y}\|^d \le -\frac{\mu_3}{\mu_2}\mathcal{V}(\mathrm{t},y).$$

Based on the Comparison lemma [16], it yields that for all $\mathrm{t} \ge \mathrm{t}_0 \ge 0$,

$$\mathcal{V}(\mathrm{t},y) \le \mathcal{V}(\mathrm{t}_0,y_0)e^{-\dfrac{\mu_3}{\mu_2}(\mathrm{t}-\mathrm{t}_0)}.$$

Using condition (4.1) once more, one obtains

$$\begin{aligned}
\|y_1(\mathrm{t},\mathrm{t}_0,y_0)\| &\le \left(\frac{\mathcal{V}(\mathrm{t},y(\mathrm{t}))}{\mu_1}\right)^{\dfrac{1}{d}} \\
&\le \left(\frac{\mathcal{V}(\mathrm{t}_0,y(\mathrm{t}_0))e^{-\dfrac{\mu_3}{\mu_2}(\mathrm{t}-\mathrm{t}_0)}}{\mu_1}\right)^{\frac{1}{d}} \\
&\le \left(\frac{\mu_2||\bar{y}(\mathrm{t}_0)||^d e^{-\frac{\mu_3}{\mu_2}(\mathrm{t}-\mathrm{t}_0)}}{\mu_1}\right)^{\frac{1}{d}} \le \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{d}} ||y(\mathrm{t}_0)||e^{\frac{-\mu_3}{\mu_2 d}(\mathrm{t}-\mathrm{t}_0)}.
\end{aligned}$$

Thus, for all $\mathrm{t} \geq \mathrm{t}_0 \geq 0$, we have

$$\|y(\mathrm{t}, \mathrm{t}_0, y_0)\| \leq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{d}} ||y(\mathrm{t}_0)|| \exp\left(\frac{-\mu_3}{\mu_2 d}(\mathrm{t} - \mathrm{t}_0)\right).$$

Finally, we deduce that $y = 0$ is globally uniformly exponentially stable with respect to y_1. □

The following Corollary presents a particular specialization of Theorem 4.1 that is more in the style of the classical Lyapunov stability. Although, this result is more restrictive for deducing the partial uniform exponential stability.

Corollary 4.2. *Let μ_1, μ_2, μ_3 and d be a nonnegative constants and assume that there exist a continuously differentiable function $\mathcal{V} : \mathrm{R}_+ \times \mathrm{R}^k \to \mathrm{R}$, such that for all $\mathrm{t} \geq \mathrm{t}_0 \geq 0$, and all $y_1 \in \mathrm{R}^k$, we have*

$$\mu_1 ||y_1||^d \leq \mathcal{V}(\mathrm{t}, y_1) \leq \mu_2 ||y_1||^d,$$

$$\frac{\partial \mathcal{V}}{\partial t}(\mathrm{t}, y_1) + \frac{\partial \mathcal{V}}{\partial y_1}(\mathrm{t}, y_1)\mathcal{F}_1(\mathrm{t}, y_1, y_2) \leq -\mu_3 ||y_1||^d.$$

Then, system (2.1) *is globally uniformly exponentially stable with respect to y_1.*

Proof. The proof is a direct result of Theorem 4.1 with $\mathcal{V}(\mathrm{t}, y)$ is replaced by $\mathcal{V}(\mathrm{t}, y_1)$. □

Consider the following linear time–varying system:

$$\dot{y}(\mathrm{t}) = \mathcal{A}(\mathrm{t})y(\mathrm{t}), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0, \tag{4.3}$$

where

$$\mathcal{A}(\mathrm{t}) = \begin{pmatrix} \mathcal{A}_1(\mathrm{t}) & 0 \\ 0 & \mathcal{A}_2(\mathrm{t}) \end{pmatrix}, \quad y := (y_1, y_2) \in \mathrm{R}^k \times \mathrm{R}^q.$$

- $\mathcal{A}_1(\mathrm{t})$ is an $k \times k$ matrix whose entries are all real valued continuous function of $\mathrm{t} \in \mathrm{R}_+$.
- $\mathcal{A}_2$ is an $q \times q$ matrix whose entries are all real valued continuous function of $\mathrm{t} \in \mathrm{R}_+$.

The above-mentioned system can be considered as following:

$$\begin{cases} \dot{y}_1(\mathrm{t}) = \mathcal{A}_1(\mathrm{t})y_1(\mathrm{t}) \\ \\ \dot{y}_2(\mathrm{t}) = \mathcal{A}_2(\mathrm{t})y_2(\mathrm{t}), \end{cases} \tag{4.4}$$

with initial condition $y(\mathrm{t}_0) := y_0 := (y_{1_0}, y_{2_0}) \in \mathrm{R}^k \times \mathrm{R}^q$.

The following theorem characterizes the uniform exponential stability with respect to y_1 of linear time–varying systems.

Theorem 4.3. *System* (4.3) *is globally uniformly exponentially stable with respect to y_1, if and only if, there exist positive constants k and γ, such that*

$$\|\Phi_1(\mathrm{t}, \mathrm{t}_0)\| \leq k e^{-\gamma(\mathrm{t} - \mathrm{t}_0)}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

where $\Phi_1(\mathrm{t}, \mathrm{t}_0)$ *is the state transition matrix of the subsystem with respect to the variable* y_1*.*

Proof. " $\Rightarrow$ " The subsystem solution relative to the variable y_1 with initial condition $y_1(\mathrm{t}_0, \mathrm{t}_0, y_0) = y_{1_0}$ is delivered by

$$y_1(\mathrm{t}) = \Phi_1(\mathrm{t}, \mathrm{t}_0)y_{1_0}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

where $\Phi_1(\mathrm{t}, \mathrm{t}_0)$ is the state transition matrix of $\mathcal{A}_1(\mathrm{t})$.

Assume that the origin is globally uniformly exponentially stable with respect to y_1. Then, based on Theorem 3.2 there exists a class $\mathcal{KL}$ function $\bar{\beta}(\cdot, \cdot)$, such that

$$\|y_1(\mathrm{t})\| \leq \bar{\beta}(\|y_0\|, \mathrm{t} - \mathrm{t}_0), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0, \quad \forall y_0 \in \mathrm{R}^m.$$

Based on the definition of the matrix norm, one obtains

$$\|\Phi_1(\mathrm{t}, \mathrm{t}_0)\| = \max_{\|y_1\|=1} \|\Phi_1(\mathrm{t}, \mathrm{t}_0)y_1\| \leq \max_{\|y\|=1} \|\Phi_1(\mathrm{t}, \mathrm{t}_0)y_1\| \leq \max_{\|y\|=1} \bar{\beta}(\|y\|, \mathrm{t} - \mathrm{t}_0) = \bar{\beta}(1, \mathrm{t} - \mathrm{t}_0).$$

As $\lim_{s \to +\infty} \bar{\beta}(1, s) = 0$, then there exists $\mathrm{T} > 0$ such that $\bar{\beta}(1, \mathrm{T}) \leq \dfrac{1}{e}$.

Let $\mathcal{N}$ be the smallest whole positive in this way: $\mathrm{t} \leq \mathrm{t}_0 + \mathcal{N}\mathrm{T}$. We split $[\mathrm{t}_0, \mathrm{t}_0 + (\mathcal{N} - 1)\mathrm{T}]$ into $(\mathcal{N} - 1)$ equal sub–intervals of width T each. That is, we obtain

$$\Phi_1(\mathrm{t}, \mathrm{t}_0) = \Phi_1(\mathrm{t}, \mathrm{t}_0 + (\mathcal{N} - 1)\mathrm{T})\Phi_1(\mathrm{t}_0 + (\mathcal{N} - 1)\mathrm{T}, \mathrm{t}_0 + (\mathcal{N} - 2)\mathrm{T}) \cdots \Phi_1(\mathrm{t}_0 + \mathrm{T}, \mathrm{t}_0).$$

Thus, it follows that

$$\begin{aligned} \|\Phi_1(\mathrm{t}, \mathrm{t}_0)\| &\leq \|\Phi_1(\mathrm{t}, \mathrm{t}_0 + (\mathcal{N} - 1)\mathrm{T})\| \prod_{k=1}^{\mathcal{N}-1} \|\Phi_1(\mathrm{t}_0 + k\mathrm{T}, \mathrm{t}_0 + (k-1)\mathrm{T})\| \\ &\leq \bar{\beta}(1, 0) \prod_{k=1}^{\mathcal{N}-1} \bar{\beta}(1, \mathrm{T}) \\ &\leq \bar{\beta}(1, 0) \prod_{k=1}^{\mathcal{N}-1} \frac{1}{e} \\ &\leq e\bar{\beta}(1, 0)e^{-\mathcal{N}}. \end{aligned}$$

Since $\mathrm{t} \leq \mathrm{t}_0 + \mathcal{N}\mathrm{T}$, then we have $\mathcal{N} \geq \frac{\mathrm{t} - \mathrm{t}_0}{\mathrm{T}}$, which implies that

$$\Phi_1(\mathrm{t}, \mathrm{t}_0) \quad \leq \quad e\bar{\beta}(1, 0)e^{-\frac{1}{\mathrm{T}}(\mathrm{t} - \mathrm{t}_0)}.$$

Consequently, there exists $k = e\bar{\beta}(1, 0)$ and $\gamma = \frac{1}{\mathrm{T}}$, such that

$$\|\Phi_1(\mathrm{t}, \mathrm{t}_0)\| \leq ke^{-\gamma(\mathrm{t} - \mathrm{t}_0)}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

" $\Leftarrow$ " Since, we have

$$\|\Phi_1(\mathrm{t}, \mathrm{t}_0)\| \leq ke^{-\gamma(\mathrm{t} - \mathrm{t}_0)}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

and also

$$y_1(\mathrm{t}) = \Phi_1(\mathrm{t}, \mathrm{t}_0)y_{1_0}, \quad \forall y_{1_0} \in \mathrm{R}^k.$$

Then, it yields that

$$\begin{aligned}\|y_1(\mathrm{t},\mathrm{t}_0,y_0)\| &= ||\Phi_1(\mathrm{t},\mathrm{t}_0)y_{1_0}|| \\ &\le ||\Phi_1(\mathrm{t},\mathrm{t}_0)||\,||y_{1_0}|| \\ &\le ||\Phi_1(\mathrm{t},\mathrm{t}_0)||\,||y_0|| \\ &\le k\|y_0\|e^{-\gamma(\mathrm{t}-\mathrm{t}_0)}, \quad \forall \mathrm{t}\ge \mathrm{t}_0\ge 0, \quad \forall y_0\in \mathrm{R}^m.\end{aligned}$$

Thus, $y=0$ is globally uniformly exponentially stable with respect to y_1. □

Within the Lyapunov method, we present this result. The demonstration of the next theorem is the same as the classical stability result

Theorem 4.4. *Suppose that the origin is an equilibrium point uniformly exponentially stable with respect to* y_1 *of system* (4.3)*, also we suppose that* $\mathcal{A}_1(\mathrm{t})$ *is continuous and bounded. Then,*
for any matrix $\mathrm{Q}(\mathrm{t})$ *continuous, positive definite, symmetric and bounded, there exists a continuously, differentiable, bounded, positive definite, symmetric matrix* $\mathrm{P}(\mathrm{t})$ *satisfies:*

$$\dot{\mathrm{P}}(\mathrm{t})+\mathrm{P}(\mathrm{t})\mathcal{A}_1(\mathrm{t})+\mathcal{A}_1^T(\mathrm{t})\mathrm{P}(\mathrm{t})=-\mathrm{Q}(\mathrm{t}). \tag{4.5}$$

So, it yields that

$$\mathcal{V}(\mathrm{t},y_1)=y_1^T\mathrm{P}(\mathrm{t})y_1 \tag{4.6}$$

is a Lyapunov candidate of system (4.3) *which satisfies all assumptions of Corollary 4.2.*

4.1. Lyapunov's indirect method. Lyapunov's indirect method uses linearisation of a system to determine the local partial stability of the origin system.

Let us go back to the system (2.1), where $\mathcal{F}_1 : \mathrm{R}_+\times U(r_1)\times U(r_2)\to \mathrm{R}^k$ and $\mathcal{F}_2 : \mathrm{R}_+\times U(r_1)\times U(r_2)\to \mathrm{R}^q$ are continuously differentiable functions, where

$$U(r_1)=\{y_1\in \mathrm{R}^k \mid ||y_1||<r_1\},\ \text{ and } U(r_2)=\{y_2\in \mathrm{R}^q \mid ||y_2||<r_2\},\ \ r_1,r_2>0.$$

Suppose that $y=0$ is an equilibrium point of system (2.1).

Notice that, is always possible to decompose $\mathcal{F}_1(\mathrm{t},y_1,y_2)$ in the following form:

$$\mathcal{F}_1(\mathrm{t},y_1,y_2)=\mathcal{F}_1(\mathrm{t},y_1,0)+\mathcal{G}_1(\mathrm{t},y_1,y_2)y_2,$$

where

$$\mathcal{G}_1(\mathrm{t},y_1,y_2)=\int_0^1\frac{\partial \mathcal{F}_1}{\partial y_2}(\mathrm{t},y_1,\lambda y_2)d\lambda,$$

and $\mathcal{F}_2(\mathrm{t},y_1,y_2)$ in the following form:

$$\mathcal{F}_2(\mathrm{t},y_1,y_2)=\mathcal{F}_2(\mathrm{t},0,y_2)+\mathcal{G}_2(\mathrm{t},y_1,y_2)y_1,$$

where

$$\mathcal{G}_2(\mathrm{t},y_1,y_2)=\int_0^1\frac{\partial \mathcal{F}_2}{\partial y_1}(\mathrm{t},\lambda y_1,y_2)d\lambda.$$

As a result, one gets the following system:

$$\begin{cases}\dot{y}_1(\mathrm{t})=\mathcal{F}_1(\mathrm{t},y_1,0)+\mathcal{G}_1(\mathrm{t},y_1,y_2)y_2\\[2ex] \dot{y}_2(\mathrm{t})=\mathcal{F}_2(\mathrm{t},0,y_2)+\mathcal{G}_2(\mathrm{t},y_1,y_2)y_1.\end{cases}$$

By the mean value theorem, one obtains

$$\mathcal{F}_1(\mathrm{t}, y_1, 0) = \mathcal{F}_1(\mathrm{t}, 0, 0) + \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, \chi_1, 0) y_1,$$

where χ_1 is a point in the line segment linking y_1 to the origin.

Since, $\mathcal{F}_1(\mathrm{t}, 0, 0) = 0$ then we may write $\mathcal{F}_1(\mathrm{t}, y_1, 0)$ as the following:

$$\begin{aligned}\mathcal{F}_1(\mathrm{t}, y_1, 0) &= \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, \chi_1, 0) y_1 \\ &= \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, 0, 0) y_1 + \left[\frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, \chi_1, 0) - \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, 0, 0)\right] y_1.\end{aligned}$$

Hence, it yields that

$$\mathcal{F}_1(\mathrm{t}, y_1, 0) = \mathcal{A}_1(\mathrm{t}) y_1 + \mathcal{H}_1(\mathrm{t}, y_1, 0),$$

where $\mathcal{A}_1(\mathrm{t}) = \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, 0, 0)$ and $\mathcal{H}_1(\mathrm{t}, y_1, 0) = \left[\frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, \chi_1, 0) - \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, 0, 0)\right] y_1$.

Furthermore, we assume that the Jacobian matrix $\left[\frac{\partial \mathcal{F}_1}{\partial y_1}\right]$ is Lipschitz on $U(r_1)$, uniformly in t, thus

$$||\frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, y_1, y_2) - \frac{\partial \mathcal{F}_1}{\partial y_1}(\mathrm{t}, u_1, y_2)|| \leq l_1 ||y_1 - u_1||, \text{for all } y_1, u_1 \in U(r_1),\ y_2 \in U(r_2),\ \forall \mathrm{t} \geq 0.$$

As the above process, we obtain that

$$\mathcal{F}_2(\mathrm{t}, 0, y_2) = \mathcal{F}_2(\mathrm{t}, 0, 0) + \frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, \chi_2) y_2,$$

where χ_2 is a point in the line segment linking y_2 to the origin.

As $\mathcal{F}_2(\mathrm{t}, 0, 0) = 0$ so we may write $\mathcal{F}_2(\mathrm{t}, 0, y_2)$ as the follows:

$$\begin{aligned}\mathcal{F}_2(\mathrm{t}, 0, y_2) &= \frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, \chi_2) y_2 \\ &= \frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, 0) y_2 + \left[\frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, \chi_2) - \frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, 0)\right] y_2.\end{aligned}$$

Hence, it follows

$$\mathcal{F}_2(\mathrm{t}, 0, y_2) = \mathcal{A}_2(\mathrm{t}) y_2 + \mathcal{H}_2(\mathrm{t}, 0, y_2),$$

where $\mathcal{A}_2(\mathrm{t}) = \frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, 0)$ and $\mathcal{H}_2(\mathrm{t}, 0, y_2) = \left[\frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, \chi_2) - \frac{\partial \mathcal{F}_2}{\partial y_2}(\mathrm{t}, 0, 0)\right] y_2$.

Finally, in a small neighborhood of origin, we can approximate the nonlinear system (2.1) as follows:

$$\begin{cases} \dot{y}_1(\mathrm{t}) = \mathcal{A}_1(\mathrm{t}) y_1(\mathrm{t}) + \mathcal{H}_1(\mathrm{t}, y_1, 0) + \mathcal{G}_1(\mathrm{t}, y_1, y_2) y_2 \\ \\ \dot{y}_2(\mathrm{t}) = \mathcal{A}_2(\mathrm{t}) y_2(\mathrm{t}) + \mathcal{H}_2(\mathrm{t}, 0, y_2) + \mathcal{G}_2(\mathrm{t}, y_1, y_2) y_1. \end{cases}$$

For small deviations from the equilibrium point, system performance is roughly controlled by linear terms. The next theorem states Lyapunov's indirect method to show the local partial exponential stability of the origin.

Theorem 4.5. *Considering the system (2.1), we suppose that $y = 0$ is an equilibrium point,*
where $\mathcal{F}_1 : \mathrm{R}_+ \times U(r_1) \times U(r_2) \to \mathrm{R}^k$ and $\mathcal{F}_2 : \mathrm{R}_+ \times U(r_1) \times U(r_2) \to \mathrm{R}^q$ are continuously differentiable. We suppose that:

- $\left[\dfrac{\partial \mathcal{F}_1}{\partial y}\right]$ *is bounded.*
- $\dfrac{\partial \mathcal{F}_1}{\partial y_1}$ *is Lipshitz on $U(r_1)$, uniformly in* t.

Furthermore, we suppose that the origin is an equilibrium point exponentially stable with respect to y_1 of the following linear system:

$$\begin{cases} \dot{y}_1(\mathrm{t}) = \mathcal{A}_1(\mathrm{t})y_1(\mathrm{t}) \\ \dot{y}_2(\mathrm{t}) = \mathcal{A}_2(\mathrm{t})y_2(\mathrm{t}). \end{cases} \tag{4.7}$$

Then, it is an equilibrium point locally exponentially stable with respect to y_1 of the system (2.1).

Proof. The linear system (4.7) verifies all assumptions of Theorem 4.4. Therefore, we may use $\mathcal{V}(\mathrm{t}, y_1) = y_1^T \mathrm{P}(\mathrm{t}) y_1$ as a function Lyapunov candidate to the system (2.1). The derivative of $\mathcal{V}(\mathrm{t}, y_1)$ along the pathways of system (2.1) is given by:

$$\begin{aligned} \dot{\mathcal{V}}(\mathrm{t}, y_1) &= y_1^T \mathrm{P}(\mathrm{t})\mathcal{F}_1(\mathrm{t}, y_1, y_2) + \mathcal{F}_1^T(\mathrm{t}, y_1, y_2)\mathrm{P}(\mathrm{t})y_1 + y_1^T \dot{\mathrm{P}}(\mathrm{t})y_1 \\ &= y_1^T(\mathrm{P}(\mathrm{t})\mathcal{A}_1(\mathrm{t}) + \mathcal{A}_1^T(\mathrm{t}) + \dot{\mathrm{P}}(\mathrm{t}))y_1 + 2y_1^T \mathrm{P}(\mathrm{t})\mathcal{H}_1(\mathrm{t}, y_1, 0) + 2y_1^T \mathrm{P}(\mathrm{t})\mathcal{G}_1(\mathrm{t}, y_1, y_2)y_2 \\ &= -y_1^T \mathrm{Q}(\mathrm{t})y_1 + 2y_1^T \mathrm{P}(\mathrm{t})\mathcal{H}_1(\mathrm{t}, y_1, 0) + 2y_1^T \mathrm{P}(\mathrm{t})\mathcal{G}_1(\mathrm{t}, y_1, y_2)y_2 \\ &\leq -c_3||y_1||^2 + 2c_2 l_1 ||y_1||^3 + 2c_2 l_2 ||y_1|| ||y_2|| \\ &\leq -\left(c_3 - 2c_2 l_1 \rho_1 - 2c_2 l_2 \frac{\rho_2}{\rho_1}\right) ||y_1||^2, \quad \text{for all } ||y_1|| < \rho_1 \text{ and } ||y_2|| < \rho_2. \end{aligned}$$

Choosing, $\rho_1 < r_1$, and $\rho_2 < \min\left\{ r_2, \dfrac{\rho_1}{2c_2 l_2}(c_3 - 2c_2 l_1 \rho_1) \right\}$.

That is, all the requirements of the theorem 4.4 are satisfied for $||y_1|| < \rho_1$ and $||y_2|| < \rho_2$, that is the origin is exponentially stable. □

Remark 4.6. The subsystem for the variable y_2 may be taken in the manner that all trajectories are uniformly bounded according to Definition 2.2. In such a situation, it is impossible to establish exponential stability for all variables although it is crucial to analyze whether it is still possible to prove certain stability properties with respect to some of the variables.

5. Stability analysis for a class of linear time–varying perturbed systems via Lyapunov functions

Suppose some parameters of linear time–varying system (4.4) are excited or perturbed, and the perturbed system has the following form:

$$\begin{cases} \dot{y}_1(\mathrm{t}) = \mathcal{A}_1(\mathrm{t})y_1(\mathrm{t}) + g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})) \\ \dot{y}_2(\mathrm{t}) = \mathcal{A}_2 y_2(\mathrm{t}), \end{cases} \tag{5.1}$$

with the same initial conditions and $g : \mathrm{R}_+ \times \mathrm{R}^k \times \mathrm{R}^q \longrightarrow \mathrm{R}^k$ is continuous, locally Lipschitz in y, uniformly in t. We denote by $y(\mathrm{t}, \mathrm{t}_0, y_0) = (y_1(\mathrm{t}, \mathrm{t}_0, y_0), y_2(\mathrm{t}, \mathrm{t}_0, y_0))$ the solutions of the perturbed system (5.1) with initial condition $y(\mathrm{t}_0) = y_0 := (y_{1_0}, y_{2_0})$.

Suppose that the origin $y = (0, 0)$ is an equilibrium point of the perturbed system (5.1), that is $g(\mathrm{t}, 0, 0) = 0$ for all $\mathrm{t} \geq \mathrm{t}_0 \geq 0$.

Our structured approach in this section is to use the Lyapunov function (4.6) for the linear time–varying system (4.4) as a Lyapunov function candidate for the perturbed system (5.1) under some constraints in the perturbation term.

Theorem 5.1. *Consider the perturbed system (4.4). Assume that there exists* $\mathrm{P} \in \mathcal{C}^1(\mathrm{R}_+, \mathrm{R}^{m\times m})$*, such that*

$$\exists p_1,\ p_2 > 0 : \quad p_1\mathrm{I} \leq \mathrm{P}(\cdot) \leq p_2\mathrm{I}, \quad \forall \mathrm{t} \geq 0, \tag{5.2}$$

being the solution to the Lyapunov generalized equation (4.5), with $\mathrm{Q} = \mathrm{Q}^T \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{m\times m})$*, such that*

$$\exists q_1, q_2 > 0 : \quad q_1\mathrm{I} \leq \mathrm{Q}(\cdot) \leq q_2\mathrm{I}, \quad \forall \mathrm{t} \geq 0. \tag{5.3}$$

Additionally, we assume that

$$||g(\mathrm{t}, y_1, y_2)|| \leq \varphi(\mathrm{t})||y_1||, \quad \forall \mathrm{t} \geq 0, \tag{5.4}$$

where $\varphi(\cdot)$ is a continuous positive function, with

$$\varphi(\mathrm{t}) \to 0 \ \textit{as}\ \mathrm{t} \to +\infty. \tag{5.5}$$

We suppose that the Q *matrix is selected as follows: $q_1 > 2\widetilde{\varphi}p_2$, and we assume that $y_2(\mathrm{t}, \mathrm{t}_0, y_0)$ is globally uniformly bounded. Then, the perturbed system (5.1) is uniformly exponentially stable with respect to y_1.*

Proof. We consider the following Lyapunov–like function:

$$\mathcal{V}(\mathrm{t}, y) = y_1^T \mathrm{P}(\mathrm{t}) y_1,$$

as a Lyapunov function candidate for the linear time–varying perturbed system (5.1). The derivation of $\mathcal{V}(\mathrm{t}, y)$ along the trajectories of the perturbed system is given by the following:

$$\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) = y_1^T(\mathrm{t}) \left(\mathrm{P}(\mathrm{t})\mathcal{A}_1(\mathrm{t}) + \mathcal{A}_1^T\mathrm{P}(\mathrm{t}) + \dot{\mathrm{P}}(\mathrm{t})\right) y_1(\mathrm{t}) + 2y_1^T(\mathrm{t})\mathrm{P}(\mathrm{t})g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})).$$

Based upon (4.5), (5.2) and (5.3), we obtain

$$\begin{aligned}
\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) &= -y_1^T(\mathrm{t})\mathrm{Q}(\mathrm{t})y_1(\mathrm{t}) + 2y_1^T(\mathrm{t})\mathrm{P}(\mathrm{t})g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})) \\
&\leq -q_1||y_1(\mathrm{t})||^2 + 2||y_1(\mathrm{t})||\ ||\mathrm{P}(\mathrm{t})||\ ||g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t}))|| \\
&\leq -q_1||y_1(\mathrm{t})||^2 + 2p_2||y_1(\mathrm{t})||\ ||g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t}))||.
\end{aligned}$$

From the assumption (5.4), we deduce:

$$\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) \leq -q_1||y_1(\mathrm{t})||^2 + 2p_2\varphi(\mathrm{t})||y_1(\mathrm{t})||^2.$$

Since $\varphi(\mathrm{t}) \to 0$ as $\mathrm{t} \to +\infty$, then there exists a positive constant $\bar{\varphi}$, such that

$$||\varphi(\mathrm{t})|| \leq \bar{\varphi}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0. \tag{5.6}$$

That is, we obtain

$$\begin{aligned}\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) &\leq -q_1 y_1^T(\mathrm{t}) y_1(\mathrm{t}) + 2\bar{\varphi} p_2 y_1^T(\mathrm{t}) y_1(\mathrm{t}) \\ &= -\left(q_1 - 2\bar{\varphi} p_2\right) y_1^T(\mathrm{t}) y_1(\mathrm{t}).\end{aligned}$$

As $q_1 > 2\bar{\varphi} p_2$, hence one obtains

$$\begin{aligned}\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) &\leq -\frac{q_1 - 2\bar{\varphi} p_2}{p_2} y_1^T(\mathrm{t}) \mathrm{P}(\mathrm{t}) y_1(\mathrm{t}) \\ &= -\left(\frac{q_1}{p_2} - 2\bar{\varphi}\right) \mathcal{V}(\mathrm{t}, y(\mathrm{t})).\end{aligned}$$

Integration over s from t_0 to t, it follows that

$$\mathcal{V}(\mathrm{t}, y(\mathrm{t})) \leq \mathcal{V}(\mathrm{t}_0, y(\mathrm{t}_0)) \exp\left(-\left(\frac{q_1}{p_2} - 2\widetilde{\varphi}\right)(\mathrm{t} - \mathrm{t}_0)\right).$$

Now we want to infer an estimation for the norm of $y_1(\cdot)$.

$$\begin{aligned}p_1 ||y_1(\mathrm{t})||^2 \leq y_1^T(\mathrm{t}) \mathrm{P}(\mathrm{t}) y_1(\mathrm{t}) &\leq \mathcal{V}(\mathrm{t}_0, y(\mathrm{t}_0)) \exp\left(-\left(\frac{q_1}{p_2} - 2\bar{\varphi}\right)(\mathrm{t} - \mathrm{t}_0)\right) \\ &\leq y_1^T(\mathrm{t}_0) \mathrm{P}(\mathrm{t}_0) y_1(\mathrm{t}_0) \exp\left(-\left(\frac{q_1}{p_2} - 2\bar{\varphi}\right)(\mathrm{t} - \mathrm{t}_0)\right) \\ &\leq p_2 ||y_1(\mathrm{t}_0)||^2 \exp\left(-\left(\frac{q_1}{p_2} - 2\bar{\varphi}\right)(\mathrm{t} - \mathrm{t}_0)\right) \\ &\leq p_2 ||y(\mathrm{t}_0)||^2 \exp\left(-\left(\frac{q_1}{p_2} - 2\widetilde{\varphi}\right)(\mathrm{t} - \mathrm{t}_0)\right).\end{aligned}$$

Finally, for all $\mathrm{t} \geq \mathrm{t}_0$, we conclude

$$||y_1(\mathrm{t})|| \leq \sqrt{\frac{p_2}{p_1}} ||y(\mathrm{t}_0)|| \exp\left(-\left(\frac{q_1}{2p_2} - \widetilde{\varphi}\right)(\mathrm{t} - \mathrm{t}_0)\right).$$

That is, the perturbed system (5.1) is uniformly exponentially stable with respect to y_1. □

5.1. Partial practical uniform exponential stability. Now, we develop the uniform exponential stability of a nontrivial solution of the perturbed system (5.1). Let us assume that $g(\mathrm{t}, 0, 0)$ does not necessarily mean zero.

We define the partial uniform exponential stability for the perturbed system (5.1), when the origin is no longer an equilibrium point. In that case, we dissect the stability of solutions with respect to a neighborhood of the origin.

The analysis of the partial uniform exponential stability of the perturbed system led back to the study of uniform exponential stability with respect to a part of the variables of a ball centered at the origin:

$$\mathrm{B}_r := \{y \in \mathrm{R}^m \mid \|y\| \leq r\}, \ r \geq 0.$$

Definition 5.1.

- It is said that the ball B_r is uniformly exponentially stable with respect to y_1, if there exist $\sigma_1 > 0$ and $\sigma_2 > 0$, such that for all $\mathrm{t}_0 \in \mathrm{R}_+$ and $y_0 \in \mathrm{R}^m$,

$$\| y_1(\mathrm{t}, \mathrm{t}_0, y_0) \| \leq \sigma_1 \| y_0 \| \exp(-\sigma_2(\mathrm{t} - \mathrm{t}_0)) + r, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

- The perturbed system (5.1) is said to be practically uniformly exponentially stable with respect to y_1, if there exists $r > 0$ such that the ball B_r is uniformly exponentially stable with respect to y_1.

Our purpose in the following is to investigate the partial practical uniform exponential stability of the perturbed system (5.1) using the method of Lyapunov.

Theorem 5.2. *Consider the perturbed system (5.1), we assume that there exists* $\mathrm{P} \in \mathcal{C}^1(\mathrm{R}_+, \mathrm{R}^{m\times m})$ *satisfies condition* (5.2) *being the solution of the generalized time–varying Lyapunov equation (4.5), with* $\mathrm{Q} = \mathrm{Q}^T \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{m\times m})$ *fulfilling (5.3). Furthermore, we suppose that* $g(\mathrm{t}, y_1, y_2)$ *fulfils the following requirement:*

$$||g(\mathrm{t}, y_1, y_2)|| \leq \varphi(\mathrm{t})||y_1|| + \Psi(\mathrm{t}), \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0,$$

where $\varphi(\cdot)$ *is a nonnegative continuous function satisfies condition (5.5), and* $\Psi(\cdot)$ *is a nonnegative continuous bounded function. Suppose the matrix* Q *is selected in such a way that* $q_1 > p_2(2\widetilde{\varphi} + 1)$, *and we presume that* $y_2(\mathrm{t}, \mathrm{t}_0, y_0)$ *is globally uniformly bounded. Hence, the perturbed system (5.1) is practically uniformly exponentially stable with respect to* y_1.

Let us remember the following generalized Gronwall lemma.

Lemma 5.3. [17] *Let* $\mathrm{H} : \mathrm{R}_+ \longrightarrow \mathrm{R}_+$ *be a continuous function,* α *is a positive real number and* λ *is a strictly positive real number. Assume that for all* $\mathrm{t} \in \mathrm{R}_+$, *and* $0 \leq h \leq \mathrm{t}$, *we have*

$$\mathrm{H}(\mathrm{t}) - \mathrm{H}(h) \leq \int_h^{\mathrm{t}} (-\lambda \mathrm{H}(s) + \alpha) ds.$$

Then,

$$\mathrm{H}(\mathrm{t}) \leq \frac{\alpha}{\lambda} + \mathrm{H}(0)\exp(-\lambda \mathrm{t}).$$

Remark 5.4. The previous lemma stays true if one replaces $\mathrm{H}(0)$ with $\mathrm{H}(\mathrm{t}_0)$, $0 \leq \mathrm{t}_0 \leq h \leq \mathrm{t}$.

Proof of Theorem 5.2. Let us consider the following Lyapunov function:

$$\mathcal{V}(\mathrm{t}, y) = y_1^T \mathrm{P}(\mathrm{t}) y_1,$$

as a Lyapunov function for the linear time–varying perturbed system (5.1).

Then, one obtains

$$\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) = y_1^T(\mathrm{t}) \left(\mathrm{P}(\mathrm{t})\mathcal{A}_1(\mathrm{t}) + \mathcal{A}_1^T \mathrm{P}(\mathrm{t}) + \dot{\mathrm{P}}(\mathrm{t})\right) y_1(\mathrm{t}) + 2y_1^T(\mathrm{t})\mathrm{P}(\mathrm{t})g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})).$$

Taking into account the standing hypothesis, it follows that

$$\begin{aligned}
\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) &= -y_1^T(\mathrm{t})\mathrm{Q}(\mathrm{t})y_1(\mathrm{t}) + 2y_1^T(\mathrm{t})\mathrm{P}(\mathrm{t})g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t})) \\
&\leq -q_1||y_1(\mathrm{t})||^2 + 2||y_1(\mathrm{t})||\ ||\mathrm{P}(\mathrm{t})||\ ||g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t}))|| \\
&\leq -q_1||y_1(\mathrm{t})||^2 + 2p_2||y_1(\mathrm{t})||\ ||g(\mathrm{t}, y_1(\mathrm{t}), y_2(\mathrm{t}))|| \\
&= -q_1 y_1^T(\mathrm{t})y_1(\mathrm{t}) + 2p_2\varphi(\mathrm{t})||y_1(\mathrm{t})||^2 + 2p_2||y_1(\mathrm{t})||\Psi(\mathrm{t}) \\
&\leq -q_1 y_1^T(\mathrm{t})y_1(\mathrm{t}) + 2p_2\varphi(\mathrm{t})||y_1(\mathrm{t})||^2 + p_2||y_1(\mathrm{t})||^2 + p_2\Psi^2(\mathrm{t}).
\end{aligned}$$

Since $\Psi(\mathrm{t})$ is a continuous nonnegative bounded function, then there exists $M > 0$, such that

$$\Psi(\mathrm{t}) \leq M, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

Moreover, from (5.6) we acquire that

$$\dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) \leq -\left(q_1 - p_2\left(2\widetilde{\varphi} + 1\right)\right) y_1^T(\mathrm{t}) y_1(\mathrm{t}) + p_2 M^2.$$

Since $p_2\left(2\widetilde{\varphi} + 1\right) \leq q_1$, then

$$\begin{aligned} \dot{\mathcal{V}}(\mathrm{t}, y(\mathrm{t})) &\leq -\frac{q_1 - p_2(2\widetilde{\varphi} + 1)}{p_2} y_1^T(\mathrm{t}) \mathrm{P}(\mathrm{t}) y_1(\mathrm{t}) + p_2 M^2 \\ &= -\mu \mathcal{V}(\mathrm{t}, y(\mathrm{t})) + p_2 M^2, \end{aligned} \tag{5.7}$$

where $\mu = \dfrac{q_1 - p_2(2c+1)}{p_2} > 0$.

Integrating (5.7) from $h \in [\mathrm{t}_0, \mathrm{t}]$ to $\mathrm{t} \geq \mathrm{t}_0$, we obtain

$$\mathcal{V}(\mathrm{t}, y(\mathrm{t})) - \mathcal{V}(h, y(h)) \leq \int_h^{\mathrm{t}} \left(-\mu \mathcal{V}(s, y(s)) + p_2 M^2\right) ds.$$

Using the Lemma 5.3, then one obtains

$$\mathcal{V}(\mathrm{t}, y(\mathrm{t})) \leq \frac{p_2 M^2}{\mu} + \mathcal{V}(\mathrm{t}_0, y(\mathrm{t}_0)) \exp\left(-\mu(\mathrm{t} - \mathrm{t}_0)\right).$$

Hence, it yields that

$$\begin{aligned} p_1 ||y_1(\mathrm{t})||^2 \leq y_1^T(\mathrm{t}) \mathrm{P}(\mathrm{t}) y_1(\mathrm{t}) &\leq \mathcal{V}(\mathrm{t}_0, y(\mathrm{t}_0)) \exp(-\mu(\mathrm{t} - \mathrm{t}_0)) + \frac{p_2 M^2}{\mu} \\ &= y_1^T(\mathrm{t}_0) \mathrm{P}(\mathrm{t}_0) y_1(\mathrm{t}_0) \exp(-\mu(\mathrm{t} - \mathrm{t}_0)) + \frac{p_2 M^2}{\mu} \\ &\leq p_2 ||y_1(\mathrm{t}_0)||^2 \exp(-\mu(\mathrm{t} - \mathrm{t}_0)) + \frac{p_2 M^2}{\mu} \\ &\leq p_2 ||y(\mathrm{t}_0)||^2 \exp(-\mu(\mathrm{t} - \mathrm{t}_0)) + \frac{p_2 M^2}{\mu}. \end{aligned}$$

Consequently, we close that

$$||y_1(\mathrm{t})||^2 \leq \frac{p_2}{p_1} ||y(\mathrm{t}_0)||^2 \exp\left(-\mu(\mathrm{t} - \mathrm{t}_0)\right) + \frac{p_2 M^2}{\mu p_1}.$$

Since $(\alpha_1 + \alpha_2)^\epsilon \leq \alpha_1^\epsilon + \alpha_2^\epsilon$, for all $\alpha_1, \alpha_2 \geq 0$ and $\varepsilon \in]0, 1[$, then we conclude that for all $\mathrm{t} \geq \mathrm{t}_0 \geq 0$,

$$||y_1(\mathrm{t})|| \leq \left(\frac{p_2}{p_1}\right)^{\frac{1}{2}} ||y(\mathrm{t}_0)|| \exp\left(-\frac{\mu}{2}(\mathrm{t} - \mathrm{t}_0)\right) + M \left(\frac{p_2}{\mu p_1}\right)^{\frac{1}{2}}.$$

That is, the perturbed system (5.1) is practically uniformly exponentially stable with respect to y_1. $\square$

6. Example

We consider the ensuing time–varying system:

$$\begin{cases} \dot{z}(\mathrm{t}) = \mathcal{A}_1(\mathrm{t})z(\mathrm{t}) + g(\mathrm{t}, z, y) \\ \\ \dot{y}(\mathrm{t}) = \mathcal{A}_2(\mathrm{t})y(\mathrm{t}), \quad \mathrm{t} \geq 0, \end{cases} \tag{6.1}$$

where $z = (z_1, z_2) \in \mathrm{R}^2$, $y = (y_1, y_2) \in \mathrm{R}^2$.

$$\mathcal{A}_1(\mathrm{t}) = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{A}_2(\mathrm{t}) = \begin{pmatrix} \cos\mathrm{t} & \sin\mathrm{t} \\ -\sin\mathrm{t} & \cos\mathrm{t} \end{pmatrix}, \quad g(\mathrm{t}, z, y) = \begin{pmatrix} g_1(\mathrm{t}, z, y) \\ g_2(\mathrm{t}, z, y) \end{pmatrix},$$

where

$$\begin{cases} g_1(\mathrm{t}, z, y) = \dfrac{e^{-y_1}}{8} \dfrac{z_1^2}{1 + \sqrt{z_1^2 + z_2^2}} \\ \\ g_2(\mathrm{t}, z, y) = \dfrac{e^{-y_2}}{8} \dfrac{z_2^2}{1 + \sqrt{z_1^2 + z_2^2}}, \end{cases}$$

with initial value $z_0 = (z_{1_0}, z_{2_0})$, $y_0 = (y_{1_0}, y_{2_0})$.

The solution $y_2(\mathrm{t})$ is globally uniformly bounded. Indeed, we have

$$\Phi_2(\mathrm{t}, \mathrm{t}_0) = e^{\lambda_1} \begin{pmatrix} \cos\lambda_2 & \sin\lambda_2 \\ -\sin\lambda_2 & \cos\lambda_2 \end{pmatrix},$$

where $\Phi_2(\mathrm{t}, \mathrm{t}_0)$ is the transition matrix for the following subsystem:

$$\dot{y}(\mathrm{t}) = \mathcal{A}_2(\mathrm{t})y(\mathrm{t}),$$

and $\lambda_1 = \sin\mathrm{t} - \cos\mathrm{t}_0$, $\lambda_2 = \cos\mathrm{t}_0 - \cos\mathrm{t}$.

That is, one obtains

$$\|y(\mathrm{t})\| \leq e^2 \|y_0\|, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

Thus, the solution $y(\mathrm{t})$ is globally uniformly bounded, as we can see in Figure 1.

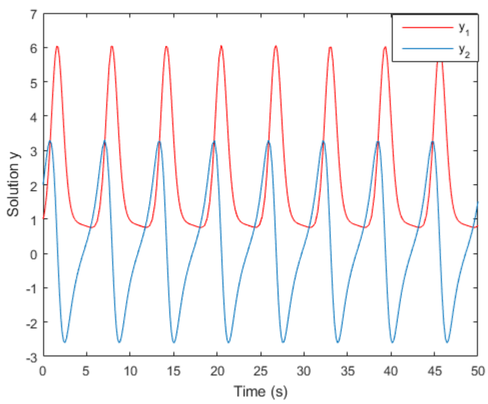

Figure 1: Time evolution of the state $y(\mathrm{t}) = (y_1(\mathrm{t}), y_2(\mathrm{t}))$ of system 6.1.

We select

$$\mathrm{P(t)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathrm{Q(t)} = \begin{pmatrix} 8 & -4 \\ -4 & 4 \end{pmatrix}.$$

We can prove that $\mathrm{P}(\cdot)$ and $\mathrm{Q}(\cdot)$ solve Eq.(4.5). Using Matlab, we deduce $q_1 = 1.5,\ q_2 = 10.5$ and $p_1 = p_2 = 2$.

In the other side, $||g(\mathrm{t}, z, y)||^2 = g_1^2(\mathrm{t}, z, y) + g_2^2(\mathrm{t}, z, y)$.
That is, one obtains

$$||g(\mathrm{t}, z, y)||^2 \leq \frac{1}{32}(z_1^2 + z_2^2).$$

Consequently, we find for all $\mathrm{t} \geq 0$,

$$||g(\mathrm{t}, z, y)|| \leq \frac{1}{\sqrt{32}}||z||.$$

Next, we may choose constant in Theorem 5.2 as follows: $\widetilde{\varphi} = \dfrac{1}{\sqrt{32}}$, it is clear that $q_1 > 2\widetilde{\varphi}p_2$.
Using Theorem 5.1 we conclude that the linear time–varying perturbed system (6.1) is uniformly exponentially stable with respect to the variable z, as shown in Figure 2.

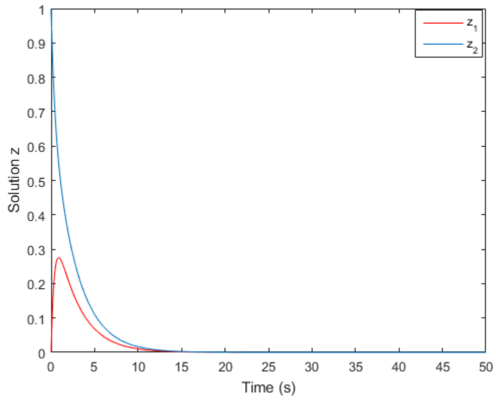

Figure 2: Time evolution of the state $z(\mathrm{t}) = (z_1(\mathrm{t}), z_2(\mathrm{t}))$ of system 6.1.

7. Conclusion

In that manuscript, we dealt with the issue of partial stability analysis. We characterize the partial uniform stability, the partial uniform asymptotic stability, the partial uniform exponential stability by using comparison functions. Further, we investigate the problem of partial stability of linear time–varying systems using Lyapunov techniques, the indirect method of Lyapunov is also investigated to derive the local partial exponential stability. Moreover, some stability criteria for partial uniform exponential stability as well as the partial practical uniform exponential stability of linear time–varying perturbed systems have been established based on Lypunov functions. We analyze an illustrative example to validate the methods devised.

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