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# FUZZY CONFORMABLE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-LOCAL CONDITIONS

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***Abstract.*** In this paper, we consider a conformable fractional Cauchy problem of integro-differential equations with non local conditions in a fuzzy metric space. We first establish the existence, uniqueness and continuous dependence on initial condition of the mild solution. We then examine a special case of non local conditions.

## 1. Introduction

In real-life many phenomena can be expressed by means of mathematical models based on differential, integral or integro-differential equations [8, 14, 18, 25, 26]. It is very remarkable that sometimes the triggering time of such phenomena is unknown. Thus in order to be able to study this type of phenomena, we have to suggest the so-called initial condition. For this raison, several types of initial conditions have been proposed in the literature, including nonlocal conditions [7,9,19], fuzzy initial conditions [11-13, 20] and fuzzy-nonlocal conditions [17, 21], which can be applied in many areas of science with better effects. In the other hand, in [15] the authors introduced a new fractional derivative named conformable fractional derivative in order to deal better with memory of dynamical systems. This novel fractional derivative attracts the attention of many researchers in various fields of applications [1-6, 10, 24]. For example, in the work [2] the authors gave, for the first time, the version of the conformable fractional derivative in a fuzzy metric space.

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In the present work, we are concerned with the following nonlocal fractional Cauchy problem

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + f(t, x(t)) + \int_0^t a(t-\sigma)h(\sigma, x(\sigma))d\sigma,\ x(0) = x_0 + g(x),\ t \in [0, \tau], \tag{1.1}$$

where $\tau$ is a positive real number and $\frac{d^{\alpha}x(t)}{dt^{\alpha}}$ presents the fuzzy conformable fractional derivative of order $\alpha \in (0, 1)$ ( [2]). $A$ is the infinitesimal generator of a fuzzy semigroup $(T(t))_{t\geq 0}$ on a fuzzy metric space $(E^n, d)$ ( [17]) and $x_0$ is an element of $E^n$. The functions $f, h : [0, \tau] \times E^n \longrightarrow E^n$, $a : [0, \tau] \longrightarrow [0, 1]$ and $g : \mathcal{C}_{\tau} \longrightarrow E^n$ satisfied some conditions, with $\mathcal{C}_{\tau}$ is the complete metric space of the continuous functions defined from $[0, \tau]$ into $E^n$ equipped with the metric $d_{\tau}(y, x) = \sup\limits_{t\in[0,\tau]} d(y(t), x(t))$. The expression $x(0) = x_0 + g(x)$ presents the fuzzy-nonlocal condition.
The rest of this paper is organized as follows. In section 2, we recall some needed definitions and results. Section 3 is devoted to prove the main results concerning the existence, uniqueness and continuous dependence on initial condition of the mild solution of Cauchy problem (1.1).

## 2. Preliminaries

The Hausdorff metric $d_H$ in the space $P_k(\mathbb{R}^n)$ of the family of all nonempty compact convex subsets of $\mathbb{R}^n$ is defined by

$$d_H(B, A) = \max[\sup_{b\in B}\inf_{a\in A} \mid b - a \mid, \sup_{a\in A}\inf_{b\in B} \mid b - a \mid],$$

where $\mid . \mid$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(P_k(\mathbb{R}^n), d_H)$ is a complete and separable space [23]. The fuzzy set $E^n$ is defined as follows

$$E^n = \{\mu_u : \mathbb{R}^n \longrightarrow [0, 1] \text{ satisfies } (1-4) \text{ below}\},$$

(1) $\mu_u$ is normal, i.e., there exists a $x_0 \in \mathbb{R}^n$ such that $\mu_u(x_0) = 1$.
(2) $\mu_u$ is convex, i.e., $\mu_u(\lambda x + (1-\lambda)y) \geq \min\{\mu_u(x), \mu_u(y)\}$ for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^n$.
(3) $\mu_u$ is upper semi-continuous.
(4) $\overline{\{x \in \mathbb{R}^n,\ \mu_u(x) > 0\}}$ is compact in $\mathbb{R}^n$.

Where $\mu$ is a membership grade function of the fuzzy set $u$ ( [27]). For $\beta \in (0, 1]$, the $\beta-$cup of $u \in E^n$ is defined by $[u]^{\beta} = \{x \in \mathbb{R}^n,\ \mu_u(x) \geq \beta\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}^n,\ \mu_u(x) > 0\}}$. By using the Zadeh extension principle, we can show that $u + v \in E^n$, $\lambda u \in E^n$, $[u + v]^{\beta} = [u]^{\beta} + [v]^{\beta}$ and $[\lambda u]^{\beta} = \lambda[u]^{\beta}$ for all $u, v \in E^n$ and $\lambda \in \mathbb{R}$. According to [22], the mapping $d(u, v) := \sup\limits_{0\leq\beta\leq 1} d_H([u]^{\beta}, [v]^{\beta})$ becomes a metric on $E^n$. Moreover, we have $d(u + w, v + w) = d(u, v)$ for $u, v, w \in E^n$ and $(E^n, d)$ is a complete metric space.
Let $u, v \in E^n$, if there exists $w \in E^n$ such that $v = u + w$ then we call $w$ the Hukuhara difference of $v$ and $u$. This difference $w$ will be noted by $v \ominus u$ in the sequel of this paper.

**Definition 2.1.** ( [16]) A fuzzy function $x : [0, \tau] \longrightarrow E^n$ is Hukuhara differentiable at $t \in [0, \tau]$ if there exists an element $\frac{dx(t)}{dt} \in E^n$ such that the limits

$$\lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\varepsilon}(x(t+\varepsilon t) \ominus x(t)) \text{ and } \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\varepsilon}(x(t) \ominus x(t-\varepsilon t))$$

exist and equal to $\frac{dx(t)}{dt}$ in $(E^n, d)$.

**Definition 2.2.** ( [2]) Let $\alpha \in ]0, 1]$, $t \in ]0, \tau]$ and $x : [0, \tau] \longrightarrow E^n$ be a fuzzy function. If there exists an element $\frac{d^\alpha x(t)}{dt^\alpha} \in E^n$ such that the limits

$$\lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\varepsilon}(x(t+\varepsilon t^{1-\alpha}) \ominus x(t)) \text{ and } \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\varepsilon}(x(t) \ominus x(t-\varepsilon t^{1-\alpha}))$$

exist and equal to $\frac{d^\alpha x(t)}{dt^\alpha}$ in $(E^n, d)$, then we call $\frac{d^\alpha x(t)}{dt^\alpha}$ the fuzzy conformable fractional derivative of order $\alpha$ of the function $x$ at $t$. Moreover, if $\lim\limits_{t \longrightarrow 0^+} \dfrac{d^\alpha x(t)}{dt^\alpha}$ exists, then we define $\frac{d^\alpha x(0)}{dt^\alpha}$ as follows

$$\frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha}.$$

A function $x : [0, \tau] \longrightarrow E^n$ is said to be $(\alpha)-$differentiable over $[0, \tau]$ if $\frac{d^\alpha x(t)}{dt^\alpha}$ exists for all $t \in [0, \tau]$.

**Remark 2.3.** ( [2]) If $x$ is a Hukuhara differentiable function, then we have

$$\frac{d^\alpha x(t)}{dt^\alpha} = t^{1-\alpha} \frac{dx(t)}{dt}.$$

**Definition 2.4.** ( [2]) The fuzzy conformable fractional integral $I^\alpha(x)$ of a function $x$ is defined as follows

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s) ds,$$

where the integral symbol "$\displaystyle\int$" is the fuzzy integral defined in the works [11, 16].

A function $x : [0, \tau] \longrightarrow E^n$ is said to be $(\alpha)-$integrable over $[0, \tau]$ if $I^\alpha(x)(t) \in E^n$ for all $t \in [0, \tau]$.

**Proposition 2.5.** ( [2]) Let $x : [0, \tau] \longrightarrow E^n$ be a Hukuhara differentiable function such that the derivative $\frac{d^\alpha x(.)}{dt^\alpha}$ is $(\alpha)-$integrable over $[0, \tau]$, then for each $t \in ]0, \tau]$ we have

$$x(t) = x(0) + I^\alpha(\frac{d^\alpha x(t)}{dt^\alpha}).$$

**Proposition 2.6.** ( [2]) Let $x : [0, \tau] \longrightarrow E^n$ be a continuous and $(\alpha)-$integrable function, then for every $t \in ]0, \tau]$, the function $I^\alpha(x)(.)$ is $(\alpha)-$differentiable and

$$\frac{d^\alpha (I^\alpha(x)(t))}{dt^\alpha} = x(t).$$

**Definition 2.7.** ( [17]) A one parameter family $(T(t))_{t \geq 0}$ of bounded linear operators on $E^n$ is called a semigroup if

(1) $T(0) = I$, ($I$ is the identity mapping on $E^n$).
(2) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.

(3) $\lim\limits_{t\longrightarrow 0^+} d(T(t)u,u)=0$ for all $u\in E^n$.
(4) There exist two constants $M>0$ and $\omega$ such that

$$d(T(t)v,T(t)u)\leq M\exp(\omega t)d(v,u) \text{ for all } u,v\in E^n.$$

**Definition 2.8.** ( [17]) The infinitesimal generator $A$ of a semigroup $(T(t))_{t\geq 0}$ is defined by

$$D(A)=\{x\in E^n,\ \lim_{h\longrightarrow 0^+}\frac{T(h)x\ominus x}{h} \text{ exists in } E^n\},$$

$$Ax=\lim_{h\longrightarrow 0^+}\frac{T(h)x\ominus x}{h}.$$

## 3. Main Results

To obtain the main results, we will need the following assumptions.

$(H_1)$**:** The function $f(.,z):[0,\tau]\longrightarrow E^n$ is continuous and there exists a constant $L_1>0$ such that
$d(f(t,y(t)),f(t,x(t)))\leq L_1 d(y(t),x(t))$ for all $z\in E^n$, $t\in[0,\tau]$ and $x,y\in\mathcal{C}_\tau$.

$(H_2)$**:** The function $h(.,z):[0,\tau]\longrightarrow E^n$ is continuous and there exists a constant $L_2>0$ such that
$d(h(t,y(t)),h(t,x(t)))\leq L_2 d(y(t),x(t))$ for all $z\in E^n$, $t\in[0,\tau]$ and $x,y\in\mathcal{C}_\tau$.

$(H_3)$**:** There exists a constant $L_3>0$ such that $d(g(y),g(x))\leq L_3 d_\tau(y,x)$ for all $x,y\in\mathcal{C}_\tau$.

As in the works [4–6], we first give the definition of mild solutions for Cauchy problem (1.1).

**Definition 3.1.** A function $x\in\mathcal{C}_\tau$ is called a mild solution of Cauchy problem (1.1), if the following assertion is true

$$x(t)=T(\frac{t^\alpha}{\alpha})(x_0+g(x))+\int_0^t s^{\alpha-1}T(\frac{t^\alpha-s^\alpha}{\alpha})\bigg(f(s,x(s))+\int_0^s a(s-\sigma)h(\sigma,x(\sigma))d\sigma\bigg)ds,$$

$t\in[0,\tau]$.

Now, we give the first result concerning the existence and uniqueness of a mild solution.

**Theorem 3.2.** *Assume that* $(H_1)-(H_3)$ *hold, then Cauchy problem* (1.1) *has an unique mild solution, provided that*

$$(L_1\frac{\tau^\alpha}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha+1}+L_3)M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))<1.$$

*Proof.* To prove this result, we will use the Banach contraction principle. To do so, we define the operator $\Gamma:\mathcal{C}_\tau\longrightarrow\mathcal{C}_\tau$ as follows

$$\Gamma(x)(t)=T(\frac{t^\alpha}{\alpha})(x_0+g(x))+\int_0^t s^{\alpha-1}T(\frac{t^\alpha-s^\alpha}{\alpha})\bigg(f(s,x(s))+\int_0^s a(s-\sigma)h(\sigma,x(\sigma))d\sigma\bigg)ds.$$

By using the properties of the metric $d$ one has

$$\begin{aligned}
&d(\Gamma(y)(t), \Gamma(x)(t)) \\
&\leq M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))d\Big(g(y), g(x)\Big) \\
&\quad + M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha})) \int_0^t s^{\alpha-1} d\Big(f(s, y(s)), f(s, x(s))\Big)ds \\
&\quad + M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha})) \int_0^t \int_0^s s^{\alpha-1} d\Big(a(s-\sigma)h(\sigma, y(\sigma)), a(s-\sigma)h(\sigma, x(\sigma))\Big)d\sigma ds.
\end{aligned}$$

According to assumptions $(H_1) - (H_3)$, we obtain

$$\begin{aligned}
d(\Gamma(y)(t), \Gamma(x)(t)) &\leq L_3 M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))d_{\tau}(y, x) \\
&\quad + L_1 M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))d_{\tau}(y, x) \int_0^t s^{\alpha-1} ds \\
&\quad + L_2 M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))d_{\tau}(y, x) \int_0^t \int_0^s s^{\alpha-1} d\sigma ds \\
&\leq (L_1 \frac{\tau^{\alpha}}{\alpha} + L_2 \frac{\tau^{\alpha+1}}{\alpha+1} + L_3) M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))d_{\tau}(y, x).
\end{aligned}$$

Taking the supremum for $t$, we get

$$d_{\tau}(\Gamma(y), \Gamma(x)) \leq (L_1 \frac{\tau^{\alpha}}{\alpha} + L_2 \frac{\tau^{\alpha+1}}{\alpha+1} + L_3) M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))d_{\tau}(y, x).$$

Since $(L_1 \frac{\tau^{\alpha}}{\alpha} + L_2 \frac{\tau^{\alpha+1}}{\alpha+1} + L_3) M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha})) < 1$, then the operator $\Gamma$ has an unique fixed point in $\mathcal{C}_{\tau}$, which is the mild solution of Cauchy problem (1.1). $\square$

Here, we will give some results concerning the continuous dependence on initial condition of mild solutions.

**Theorem 3.3.** *Assume that the conditions of Theorem 3.2 are satisfied. Let $x_0$, $y_0 \in E^n$ and denote by $x$, $y$ the solutions associated with $x_0$ and $y_0$, respectively. Then we have*

$$d_{\tau}(y, x) \leq \frac{M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))}{1 - (L_1 \frac{\tau^{\alpha}}{\alpha} + L_2 \frac{\tau^{\alpha+1}}{\alpha+1} + L_3) M \sup_{t\in[0,\tau]} (\exp(\frac{\omega t^{\alpha}}{\alpha}))} d(y_0, x_0).$$

*Proof.* We have the following inequality

$$
\begin{aligned}
&d(y(t),x(t))\\
&\leq M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))d\Big(y_0+g(y),x_0+g(x)\Big)\\
&+M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\int_0^t s^{\alpha-1}d\Big(f(s,y(s)),f(s,x(s))\Big)ds\\
&+M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\int_0^t\int_0^s s^{\alpha-1}d\Big(a(s-\sigma)h(\sigma,y(\sigma)),a(s-\sigma)h(\sigma,x(\sigma))\Big)d\sigma ds.
\end{aligned}
$$

Then, we get

$$
d(y(t),x(t))\leq M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\Big[d(y_0,x_0)+(L_1\frac{\tau^{\alpha}}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha+1}+L_3)d_\tau(y,x)\Big].
$$

Taking the supremum for $t$, we obtain

$$
d_\tau(y,x)\leq M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\Big[d(y_0,x_0)+(L_1\frac{\tau^{\alpha}}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha+1}+L_3)d_\tau(y,x)\Big].
$$

Finally, we obtain the following desired estimate

$$
d_\tau(y,x)\leq\frac{M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))}{1-(L_1\frac{\tau^{\alpha}}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha+1}+L_3)M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))}d(y_0,x_0). \qquad \square
$$

The next theorem gives us another estimate of the continuous dependence on initial condition of mild solution.

**Theorem 3.4.** *Assume that the conditions of Theorem* 3.2 *are satisfied. Let* $x_0$*,* $y_0\in E^n$ *and denote by* $x$*,* $y$ *the solutions associated with* $x_0$ *and* $y_0$*, respectively. Then we have*

$$
\begin{aligned}
&d_\tau(y,x)\\
&\leq\frac{M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\exp\left(M(L_1\frac{\tau^{\alpha}}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\right)}{1-L_3M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\exp\left(M(L_1\frac{\tau^{\alpha}}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\right)}d(y_0,x_0),
\end{aligned}
$$

*provided that*

$$
L_3M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\exp\left(M(L_1\frac{\tau^{\alpha}}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^{\alpha}}{\alpha}))\right)<1.
$$

*Proof.* According to assumptions $(H_1)-(H_3)$, we obtain

$$\begin{aligned}d(y(t),x(t)) \leq M \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))[d(x_0,y_0)+L_3 d_\tau(y,x)]\\ + L_1 M \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\int_0^t s^{\alpha-1}\sup_{\xi\in[0,s]} d(y(\xi),x(\xi))ds\\ + L_2 M \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\int_0^t\int_0^s s^{\alpha-1}\sup_{\xi\in[0,s]} d(y(\xi),x(\xi))d\sigma ds.\end{aligned}$$

Then one has

$$\begin{aligned}\sup_{\xi\in[0,t]} d(y(\xi),x(\xi)) \leq M \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))[d(x_0,y_0)+L_3 d_\tau(y,x)]\\ + (L_1+L_2\tau) M \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\int_0^t s^{\alpha-1}\sup_{\xi\in[0,s]} d(y(\xi),x(\xi))ds.\end{aligned}$$

By using the Gronwall's inequality, we obtain

$$\begin{aligned}\sup_{\xi\in[0,t]} d(y(\xi),x(\xi)) \leq M[d(x_0,y_0)+L_3 d_\tau(y,x)]\\ \times \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\exp\left(M(L_1\frac{\tau^\alpha}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\right).\end{aligned}$$

Taking the supremum for $t$, we get

$$\begin{aligned}d_\tau(y,x) \leq M[d(x_0,y_0)+L_3 d_\tau(y,x)]\\ \times \sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\exp\left(M(L_1\frac{\tau^\alpha}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\right).\end{aligned}$$

Finally, we deduce the following estimate

$$\begin{aligned}&d_\tau(y,x)\\ &\leq \frac{M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\exp\left(M(L_1\frac{\tau^\alpha}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\right)}{1-L_3 M\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\exp\left(M(L_1\frac{\tau^\alpha}{\alpha}+L_2\frac{\tau^{\alpha+1}}{\alpha})\sup_{t\in[0,\tau]}(\exp(\frac{\omega t^\alpha}{\alpha}))\right)} d(y_0,x_0). \square\end{aligned}$$

To finish we propose to study a special case of nonlocal conditions as a concrete application of our abstract results. Precisely, we will examine the non-local condition of the form

$$g(x)=\sum_{i=1}^{n-1} | c_i | \int_{t_i}^{t_{i+1}} s^{\alpha-1}x(s)ds,$$

where $c_i$, $i=1,2,...,(n-1)$, are given constants and $0<t_1<t_2<...<t_n<\tau$.

**Proposition 3.5.** Assume that $(H_1)$ and $(H_2)$ hold, then Cauchy problem (1.1) has an unique mild solution, provided that there exists a constant $\varepsilon>1$ such that

$$\varepsilon M\exp(\frac{|\omega|\tau^\alpha}{\alpha})\left[\sum_{i=1}^{n-1} | c_i | + L_1 + L_2\tau\right] \leq | \omega | .$$

*Proof.* We define the operator $\Gamma : \mathcal{C}_\tau \longrightarrow \mathcal{C}_\tau$ by

$$\Gamma(x)(t) = T(\frac{t^\alpha}{\alpha})(x_0+g(x))+\int_0^t s^{\alpha-1}T(\frac{t^\alpha - s^\alpha}{\alpha})\bigg(f(s,x(s))+\int_0^s a(s-\sigma)h(\sigma,x(\sigma))d\sigma\bigg)ds.$$

We also define a new metric $d_\alpha$ in $\mathcal{C}_\tau$ as follows

$$d_\alpha(y,x) = d_\tau\bigg(\exp(\frac{-\theta(.)^\alpha}{\alpha})y, \exp(\frac{-\theta(.)^\alpha}{\alpha})x\bigg),$$

where

$$\theta = \varepsilon M \exp(\frac{\mid \omega \mid \tau^\alpha}{\alpha})\bigg[\sum_{i=1}^{n-1} \mid c_i \mid + L_1 + L_2\tau\bigg].$$

By using the properties of the metric $d$ one has

$$\begin{aligned}
&d(\Gamma(y)(t),\Gamma(x)(t))\\
&\leq M\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})d\bigg(g(y),g(x)\bigg)\\
&\quad + M\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})\int_0^t s^{\alpha-1}d\bigg(f(s,y(s)),f(s,x(s))\bigg)ds\\
&\quad + M\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})\int_0^t\int_0^s s^{\alpha-1}d\bigg(a(s-\sigma)h(\sigma,y(\sigma)),a(s-\sigma)h(\sigma,x(\sigma))\bigg)d\sigma ds.
\end{aligned}$$

According to assumptions $(H_1)$ and $(H_2)$, we obtain

$$\begin{aligned}
d(\Gamma(y)(t),\Gamma(x)(t)) \leq{}& M\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})d_\alpha(y,x)\int_0^\tau s^{\alpha-1}\sum_{i=1}^{n-1}\mid c_i \mid \exp(\frac{\theta s^\alpha}{\alpha})ds\\
&+ L_1M\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})d_\alpha(y,x)\int_0^\tau s^{\alpha-1}\exp(\frac{\theta s^\alpha}{\alpha})ds\\
&+ L_2M\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})d_\alpha(y,x)\int_0^\tau\int_0^\tau s^{\alpha-1}\exp(\frac{\theta s^\alpha}{\alpha})d\sigma ds.
\end{aligned}$$

Then, we obtain the following inequality

$$\begin{aligned}
&d(\Gamma(y)(t),\Gamma(x)(t))\\
&\leq \bigg[M\bigg(\sum_{i=1}^{n-1}\mid c_i \mid + L_1 + L_2\tau\bigg)\exp(\frac{\mid \omega \mid t^\alpha}{\alpha})\int_0^\tau s^{\alpha-1}\exp(\frac{\theta s^\alpha}{\alpha})ds\bigg]d_\alpha(y,x).
\end{aligned}$$

Consequently, we have

$$d(\Gamma(y)(t),\Gamma(x)(t)) \leq \frac{M\exp(\frac{|\omega|t^\alpha}{\alpha})\bigg[\sum_{i=1}^{n-1}\mid c_i \mid + L_1 + L_2\tau\bigg]\exp(\frac{\theta\tau^\alpha}{\alpha})}{\theta}d_\alpha(y,x).$$

Hence, we get

$$d_\alpha(\Gamma(y),\Gamma(x)) \leq \frac{M\exp(\frac{(|\omega|-\theta)\tau^\alpha}{\alpha})\bigg[\sum_{i=1}^{n-1}\mid c_i \mid + L_1 + L_2\tau\bigg]\exp(\frac{\theta\tau^\alpha}{\alpha})}{\theta}d_\alpha(y,x).$$

Therefore, we deduce the following inequality

$$d_\alpha(\Gamma(y), \Gamma(x)) \leq \frac{1}{\varepsilon} d_\alpha(y, x). \qquad \square$$

The above inequality shows that $\Gamma$ is a contraction operator. Thus the operator $\Gamma$ has an unique fixed point, which is the mild solution of Cauchy problem (1.1).

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