Poincare Journal of Analysis & Applications
Vol. 9, No. 2 (2022), 183-199

GENERATION OF FRACTALS AND ANTI-FRACTALS USING THAKUR ITERATIVE SCHEME

AMIT KUMAR†, SUDESH KUMARI AND RENU CHUGH

Date of Receiving	:	24. 09. 2021
Date of Revision	:	13. 08. 2022
Date of Acceptance	:	20. 08. 2022

Abstract. In this paper, we exhibit utilization of fixed point iterative techniques towards generating fractals (Mandelbrot and Julia sets) and anti-fractals (tricorns, multicorns and anti-Julia sets) for the complex polynomials $P(z) = z^k + c$ and antipolynomials $A(z) = \bar{z}^d + c$ respectively, where c is complex number and $k, d \geq 2$. We find escape criteria to generate fractals and anti-fractals. Moreover, we graphically envisage and perceive the dynamics of these fractals and anti-fractals. Several attractive aesthetic shapes have been obtained which explore the geometry of fractals and anti-fractals.

1. Introduction

A fractal is a geometric graphics which follows the similar pattern at different scales [7]. French mathematician Gaston Julia [6] gave a new direction to the fractal theory. In 1975, Mandelbrot broadened the idea of Julia and used the term fractal for the first time. He introduced Mandelbrot set by taking "c" as a boundary of the complex quadratic function $z^2 + c$ [19]. The Julia and Mandelbrot sets can be perceived as images of fractal arts and they are generated with the assistance of fractal creating programming because it provides phenomena calculative abilities [2, 9]. The word fractal is almost half-century-old and the field of fractals is additionally named as fractal geometry. Due to the progression of innovation and the expansion of existing cultural necessities, some authors tried to improve the idea of fractals by adding more complex features to the systems involved. Also, the dynamics of antipolynomial $\bar{z} \mapsto \bar{z}^d + c$ leads to interesting tricorns, multicorns and anti-Julia sets [7, 21].

2010 *Mathematics Subject Classification.* 34H10, 37M10, 37B25, 37F45.
Key words and phrases. Thakur iterative scheme, Mandelbrot sets, Julia sets, tricorns, multicorns, anti-Julia sets.
The authors are thankful to the honorable editor and reviewer for their valuable and insightful comments that improved the quality of the paper.
Communicated by. M. C. Joshi
† *Corresponding author*

The exploration of the unpredictable behaviour of fractals using various mathematical methodologies and models is a prominent topic among researchers and technologists. The visual beauty, intricacy and self- likeness of the fractals are remarkable. Julia and Mandelbrot sets were presented via one-step feedback process [7] for quadratic [6, 7], cubic [7] and higher degree complex polynomials [7]. In 2002, Rani [23] introduced superior fractals which is a new class of fractals. In 2004, Rani and Kumar [24, 25] used Mann iteration, two-step fixed point iterative procedure, to generate superior Julia and Mandelbrot sets. These sets were introduced by Rani et al. [1] in Noor orbit, Kwun et al. [16, 17] in CR and Jungck-CR iteration schemes with s-convexity, Kumari et al. [14, 15] via Jungck SP iteration. For more information, one may refer to [3, 4, 8, 10, 13, 18, 20, 22]. Fractals have many applications, for instance, in biological science [11], telecom [5], living system [21] and hardware and electrical designing [12].

In 2014, Thakur et al. [26] introduced Thakur iteration which is a three step iterative procedure and proved that Thakur iteration is faster than all of the Picard, Mann, Agarwal et al., and the Abbas et al. iteration processes. Motivated by this fact we introduce a new class of fractals using this three-step iterative procedure. In this paper, we introduce and envision a new class of fractals and antifractals via Thakur iteration. Many graphical patterns of Mandelbrot, Julia, tricorns, multicorns and anti-Julia sets are presented by using the proposed three-step iterative procedure.

2. Preliminaries

Throughout the paper, we take $Q(z) = z^2 + c$ as a quadratic function of complex variable, where z is complex variable and $c \in \mathbb{C}$. And $Q(z) = z^n + c$, the function of n^{th} degree. In 2007, Thakur et al. [26] introduced a new iterative scheme "Thakur-iteration" which is a three-step procedure given as:

Definition 2.1. [26] Let $C \subset \mathbb{C}$ be a non-empty convex subset of $\mathbb{C}$ and $T : C \to C$ be an operator. Then, for arbitrary $z_1 \in C$, the Thakur-iteration process is defined by

$$\begin{aligned} z_{n+1} &= (1-\alpha_n)Tz_n + \alpha_n Tx_n, \\ x_n &= (1-\beta_n)y_n + \beta_n Ty_n, \\ y_n &= (1-\gamma_n)z_n + \gamma_n Tz_n,\ n \geq 0, \end{aligned} \tag{2.1}$$

where α_n, β_n and γ_n are real sequences in $(0, 1)$. In this paper, for simplicity, we take $\alpha_n = \alpha$, $\beta_n = \beta$, $\gamma_n = \gamma$ and $x_0 = x,\ y_0 = y,\ z_0 = z$.

Definition 2.2. [7] The filled in Julia set of the function Q is defined as

$$K(Q) = \left\{ z \in C : Q^k(z) \text{ does not tend to infinity } \right\},$$

where $\mathbb{C}$ is the complex space, $Q^k(z)$ is the k^{th} iterate of the function Q and $K(Q)$ denotes the filled in Julia set. The Julia set of the function Q is defined to be the boundary of $K(Q)$, i.e.,

$$J(Q) = (K(Q)),$$

where $J(Q)$ denotes the Julia set.

Definition 2.3. [7] The Mandelbrot set M consists of all parameters $c \in \mathbb{C}$ for which the filled in Julia set of $Q(z)$ is connected, i.e.,

$$M = \{c \in C : K(Q(0)) \text{is connected}\}.$$

We choose the initial point 0, as 0 is the only critical point of Q.

3. Escape Criteria for Complex Polynomials using Thakur-Iterative Scheme

In this section, we find the escape criterion for generating and analyzing the Julia sets, Mandelbrot sets and their variants for general functions of k^{th} degree, i.e., $P(z) = z^k + c$ in respect of the Thakur-iteration. Moreover, some particular cases have also been considered.

Theorem 3.1. *Assume*

$$|z| \geq |c| > \max\left\{\left(\frac{2}{2\alpha-1}\right)^{1/k-1}, \left(\frac{2}{\beta}\right)^{1/k-1}, \left(\frac{2}{\gamma}\right)^{1/k-1}\right\},$$

$$\begin{aligned}
z_1 &= (1-\alpha)P(z) + \alpha P(x), \\
z_2 &= (1-\alpha)P(z_1) + \alpha P(x_1), \\
z_3 &= (1-\alpha)P(z_2) + \alpha P(x_2), \\
&\vdots \\
z_{n+1} &= (1-\alpha)P(z_n) + \alpha P(x_n),
\end{aligned}$$

where $x_n = (1-\beta)y_n + \beta P(y_n)$ *and* $y_n = (1-\gamma)z_n + \gamma P(z_n)$ *for* $n = 0, 1, 2, 3, \ldots$ *and* $P(z) = z^k + c$ *is* k^{th} *degree complex polynomial, then* $|z_n| \to \infty$ *as* $n \to \infty$.

Proof. From (2.1), consider

$$\begin{aligned}
|y| &= |(1-\gamma)z + \gamma P(z)|, \\
&= |(1-\gamma)z + \gamma(z^k + c)|, \\
&\geq |\gamma z^k + (1-\gamma)z| - \gamma|c|, \\
&\geq \gamma|z^k| - (1-\gamma)|z| - \gamma|c|.
\end{aligned}$$

As we assume that $|z| \geq |c|$, we have

$$\begin{aligned}
|y| &\geq \gamma|z^k| - |z| + \gamma|z| - \gamma|z|, \\
&= |z^k| - |z|, \\
&= |z|^k - |z|, \\
i.e.,\ |y| &\geq |z|(\gamma|z|^{k-1} - 1). \qquad (3.1)
\end{aligned}$$

Since, we assume that $|z| > \left(\frac{2}{\gamma}\right)^{1/k-1}$, i.e., $(\gamma|z|^{k-1} - 1) > 1$, using this in (3.1), we get

$$|y| > |z|. \qquad (3.2)$$

Again from Eq. (2.1),

$$\begin{aligned}
|x| &= |(1-\beta)y + \beta P(y)|, \\
&= |(1-\beta)y + \beta(y^k + c)|, \\
&\geq \beta|y|^k - \beta|c| - (1-\beta)|y|.
\end{aligned}$$

As $|z| \geq |c|$ and also from (3.2), $|y| > |z|$ gives $|y| > |c|$, using this in above inequality, we obtain

$$\begin{aligned} |x| &\geq \beta|y|^k - \beta|y| - (1-\beta)|y|, \\ &= \beta|y|^k - |y|. \end{aligned}$$

Therefore,

$$|x| \geq |y|(\beta|y|^{k-1} - 1). \tag{3.3}$$

Using (3.2) in (3.3), we get

$$|x| \geq |z|(\beta|z|^{k-1} - 1).$$

As we assume that $|z| > \left(\frac{2}{\beta}\right)^{1/k-1}$, i.e., $(\beta|z|^{k-1} - 1) > 1$, using this in the above inequality, we get

$$|x| > |z|. \tag{3.4}$$

Now, from Eq. (2.1) consider

$$\begin{aligned} |z_1| &= |\alpha P(x) + (1-\alpha)P(z)|, \\ &\geq \alpha|(x^k + c)| - (1-\alpha)|(z^k + c)|, \\ &\geq \alpha|x|^k - \alpha|c| - (1-\alpha)|z|^k - (1-\alpha)|c|. \end{aligned}$$

As $|x| > |z|$ and $|z| > |c|$, we get

$$\begin{aligned} |z_1| &\geq \alpha|z|^k - \alpha|z| - (1-\alpha)|z|^k - (1-\alpha)|z| \\ &= |z|^k(2\alpha - 1) + |z|(-\alpha - 1 + \alpha) \\ &= |z|^k(2\alpha - 1) - |z| \\ &= |z|((2\alpha - 1)|z|^{k-1} - 1). \end{aligned}$$

Since $|z| > \left(\frac{2}{2\alpha-1}\right)^{1/k-1}$, i.e., $(2\alpha - 1)|z|^{k-1} - 1 \geq 1 + \lambda > 1$ for some $\lambda > 0$. Consequently, we get

$$|z_1| > (1+\lambda)|z|.$$

Repeating this argument n time, we get

$$|z_n| > (1+\lambda)^n|z|. \tag{3.5}$$

Therefore, the orbit of z, under the Thakur-iteration of k^{th} degree polynomial $P(z) = z^k + c$, tends to infinity as n tends to infinity. □

From the above theorem, we obtain the following corollaries.

Corollary 3.2. Let $|c| > \left(\frac{2}{2\alpha-1}\right)^{1/k-1}$, $|c| > \left(\frac{2}{\beta}\right)^{1/k-1}$ and $|c| > \left(\frac{2}{\gamma}\right)^{1/k-1}$, where $\frac{1}{2} < \alpha \leq 1$, $0 < \beta, \gamma \leq 1$, then the orbit of z under the Thakur-iteration of k^{th} degree polynomial $P(z) = z^k + c\,; k \geq 2$ escapes to infinity as n tends to infinity.

Corollary 3.3. Suppose

$$|z_n| \geq \max\left\{|c|, \left(\frac{2}{2\alpha - 1}\right)^{1/k-1}, \left(\frac{2}{\beta}\right)^{1/k-1}, \left(\frac{2}{\gamma}\right)^{1/k-1}\right\},$$

for some $k \geq 2$ then $|z_{n+1}| > (1+\lambda)|z_n|$ which implies that $|z_n| \to \infty$ as $n \to \infty$.

Corollary 3.4. For $k = 2$, if

$$|z| \geq |c| > \max\left\{|c|, \frac{2}{2\alpha - 1}, \frac{2}{\beta}, \frac{2}{\gamma}\right\},$$

where $\frac{1}{2} < \alpha < 1, 0 < \beta, \gamma \leq 1$ and $c \in \mathbb{C}$, then orbit of z under the Thakur-iteration of quadratic polynomial $P(z) = z^2 + c$ escapes to infinity as n tends to infinity.

Corollary 3.5. The orbit of z under the Thakur-iteration of cubic polynomial $P(z) = z^3 + c$ escapes to infinity as n tends to infinity if

$$|z| \geq |c| > \max\left\{|c|, \left(\frac{2}{2\alpha - 1}\right)^{1/2}, \left(\frac{2}{\beta}\right)^{1/2}, \left(\frac{2}{\gamma}\right)^{1/2}\right\},$$

where $\frac{1}{2} < \alpha < 1, 0 < \beta, \gamma \leq 1$ and $c \in \mathbb{C}$.

Remark 1. To generate tricorns, we replace $P(z) = z^2 + c$ by $A(\bar{z}) = \bar{z}^2 + c$ and find that for $k = 2$, if

$$|z| > \max\left\{|c|, \frac{2}{2\alpha - 1}, \frac{2}{\beta}, \frac{2}{\gamma}\right\},$$

where $\frac{1}{2} < \alpha < 1, 0 < \beta, \gamma \leq 1$ and $c \in \mathbb{C}$, the orbit of z under the Thakur-iteration escapes to infinity as n tends to infinity.

Remark 2. To generate multicorns and anti-Julia sets, we replace the polynomial $P(z) = z^n + c$ to the antipolynomial $A(\bar{z}) = \bar{z}^n + c$, for $n = 2, 3, 4, \cdots$, and find that for

$$|z| > \max\left\{\left(|c|, \frac{2}{2\alpha - 1}\right)^{1/n-1}, \left(\frac{2}{\beta}\right)^{1/n-1}, \left(\frac{2}{\gamma}\right)^{1/n-1}\right\},$$

where $\frac{1}{2} < \alpha < 1, 0 < \beta, \gamma \leq 1$ and $c \in \mathbb{C}$, the orbit of z tends to infinity.

4. Algorithm for generating Fractals

In this section, we present algorithm to generate Julia and Mandelbrot sets for quadratic, cubic and higher degree complex polynomials. With the help of derived results, we use the following algorithms to generate all fractals (Mandelbrot sets and Julia sets), tricorns, multicorns and anti-Julia sets:

(1) **Setup :**
Choose a complex number $c = \mu + i\,\nu$.
Initialize values of variables α, β, γ.
Take $z_0 = x + \iota y$ as an initial point.

(2) **Iterate :**

$$\begin{aligned} z_{n+1} &= (1 - \alpha_n)P(z_n) + \alpha_n P(x_n), \\ x_n &= (1 - \beta_n)P(y_n) + \beta_n P(y_n), \\ y_n &= (1 - \gamma_n)z_n + \gamma_n P(z_n), n \geq 1, \end{aligned}$$

where $P(z_n) = z^n + c, n = 2, 3, 4, ...,$ $\frac{1}{2} < \alpha < 1, 0 < \beta, \gamma \leq 1$ and $c \in \mathbb{C}$.

(3) **Stop :**
$|z_j| >$ escape radius $= \max\left\{|c|, \left(\frac{2}{2\alpha-1}\right)^{1/n-1}, \left(\frac{2}{\beta}\right)^{1/n-1}, \left(\frac{2}{\gamma}\right)^{1/n-1}\right\}$.

(4) **Count :**
Number of iterations to escape.

(5) **Color :**
Depends on the number of iterations required to escape.

With the help of this algorithm, we make a program in Mathematica 12.0 to generate fractals. To make source program, firstly, we use Block construct. Then, we input the values of parameters $\alpha, \beta, \gamma, \mu$ and ν ended with semicolons. Iterate the given polynomial $P(z)$ of degree n by implementing the general escape criterion derived in Theorem 1 to generate fractals of n^{th} degree polynomials ($n = 2, 3, 4, \cdots$) using while loop. In the last, to plot required fractals, we apply Density Plot which includes number of iterations, range of x-axis and y-axis, no. of points and color function.
Note: To generate tricorns, multicorns and anti-Julia sets, we replace polynomial $P(z)$ by antipolynomial $A(\bar{z})$.

4.1. Source Program to Generate Mandelbrot Set. Here, we provide a source program in Mathematica to generate Quadratic Mandelbrot set.

$iter[x, y, lim] := Block[c, z, ct, \alpha, \beta, \gamma$

$c = \mu + \iota\nu;$

$z = z;$

$\alpha = 0.99;$

$\beta = 0.02;$

$\gamma = 0.01;$

$ct = 0;$

$While[(Abs[z] < Max[Abs[c], 2/\beta, 2/\gamma, 2/(2\alpha - 1)])\&\&(ct <= \lim), + + ct;$

$b = z * z + c;$

$q = (1 - \gamma) * z + \gamma * b;$

$a = q * q + c;$

$p = (1 - \beta) * q + \beta * a;$

$d = p * p + c;$

$z = (1 - \alpha) * b + \alpha * d;];$

$Return[ct];$

$]$

Density Plot $[-iter[x, y, 25], \{x, x\min, x\max\}, \{y, y\min, y\max\}, PlotPoints \to 200, Mesh \to False, ColorFunction \to (Hue[1 - \#]\&)]$

4.2. Source Program to Generate Julia Set. Here, we provide a source program in Mathematica to generate Quadratic Mandelbrot set.

$iter[x, y, lim] := Block[c, z, ct, \alpha, \beta, \gamma$

$c = \mu + \iota\nu;$

$z = x + \iota y;$

$\alpha = 0.99;$
$\beta = 0.02;$
$\gamma = 0.01;$
$ct = 0;$
$\mu = 0.5;$
$\nu = 0.3;$
$While[(Abs[z] < Max[Abs[c], 2/\beta, 2/\gamma, 2/(2\alpha - 1)])\&\&(ct <= \lim), ++ct;$
$b = z * z + c;$
$q = (1 - \gamma) * z + \gamma * b;$
$a = q * q + c;$
$p = (1 - \beta) * q + \beta * a;$
$d = p * p + c;$
$z = (1 - \alpha) * b + \alpha * d;];$
$Return[ct];$
]
Density Plot $[-iter[x, y, 25], \{x, x\min, x\max\}, \{y, y\min, y\max\}, PlotPoints \to 200, Mesh \to False, ColorFunction \to (Hue[1 - \#]\&)]$

Remark 3. To generate tricorns and multicorns, we replace the variable z by its conjugate $\bar{z}$ that is $z \to \bar{z}$ and the rest pseudocode remains same.

Note: We construct all fractals and antifractals using the same source programs only by changing the values of parameters as described in the caption of each figure.

5. Generation of Mandelbrot sets in Thakur-iteration

Using source program given in Subsection 4.1, we generate following Mandelbrot sets via Thakur-iteration by running the program in Mathematica 12.0. We found several captivating new fractals having various geometric shapes. However, we have chosen some figures of Mandelbrot sets for $P(z) = z^k + c, k = 2, 3, 4, \cdots$ at:

(1) $(\alpha, \beta, \gamma) = (0.6, 0.2, 0.3)$ (see Fig. 1), for $k = 2$.
(2) $(\alpha, \beta, \gamma) = (0.68, 0.1, 0.22)$ (see Fig. 2), for $k = 2$.
(3) $(\alpha, \beta, \gamma) = (0.51, 0.2, 0.05)$ (see Fig. 3), for $k = 2$.
(4) $(\alpha, \beta, \gamma) = (0.98, 0.2, 0.2)$ (see Fig. 4), for $k = 2$.
(5) $(\alpha, \beta, \gamma) = (0.82, 0.1, 0.99)$ (see Fig. 5), for $k = 3$.
(6) $(\alpha, \beta, \gamma) = (0.82, 0.02, 0.08)$ (see Fig. 6), for $k = 3$.
(7) $(\alpha, \beta, \gamma) = (0.98, 0.05, 0.05)$ (see Fig. 7), for $k = 3$.
(8) $(\alpha, \beta, \gamma) = (0.8, 0.9, 0.05)$ (see Fig. 8), for $k = 5$.
(9) $(\alpha, \beta, \gamma) = (0.69, 0.1, 0.15)$ (see Fig. 9), for $k = 5$.
(10) $(\alpha, \beta, \gamma) = (0.99, 0.1, 0.1)$ (see Fig. 10), for $k = 5$.
(11) $(\alpha, \beta, \gamma) = (0.99, 0.02, 0.02)$ (see Fig. 11), for $k = 8$.
(12) $(\alpha, \beta, \gamma) = (0.85, 0.6, 0.07)$ (see Fig. 12), for $k = 10$.
(13) $(\alpha, \beta, \gamma) = (0.99, 0.12, 0.01)$ (see Fig. 13), for $k = 15$.

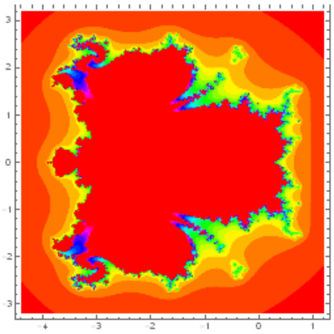

FIGURE 1. Quadratic Mandelbrot set for (α, β, γ)=(0.6, 0.2, 0.3)

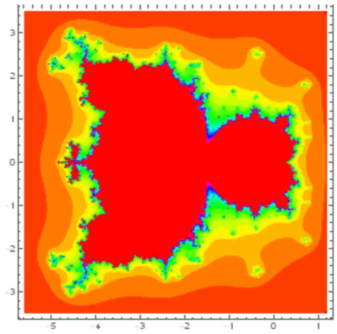

FIGURE 2. Quadratic Mandelbrot set for (α, β, γ)=(0.68, 0.1, 0.22)

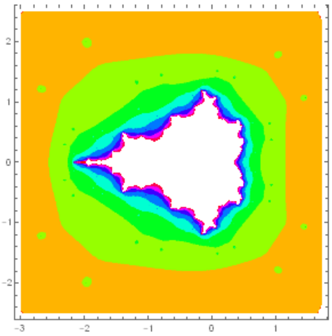

FIGURE 3. Quadratic Mandelbrot set for (α, β, γ)=(0.51, 0.2, 0.05)

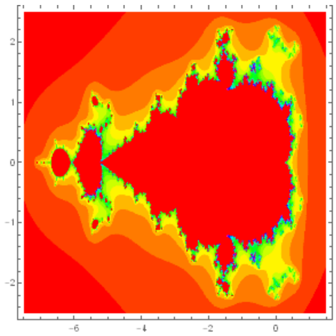

FIGURE 4. Cubic Mandelbrot set for (α, β, γ)=(0.98, 0.2, 0.2)

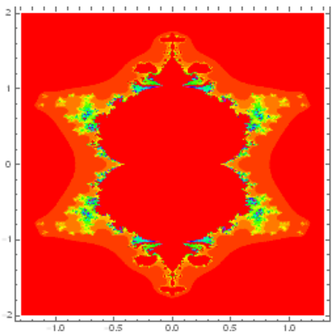

FIGURE 5. Cubic Mandelbrot set for (α, β, γ)=(0.82, 0.1, 0.99)

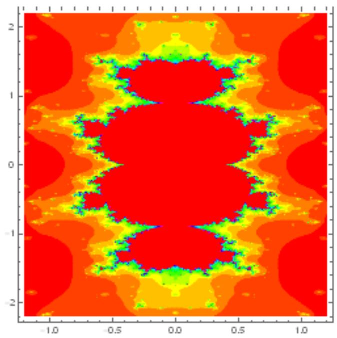

FIGURE 6. Cubic Mandelbrot set for (α, β, γ)=(0.82, 0.02, 0.08)

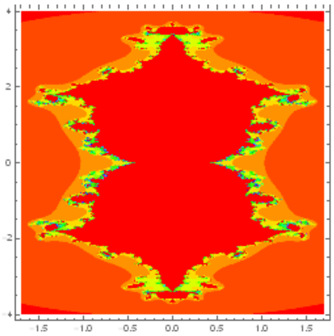

FIGURE 7. Cubic Mandelbrot set for (α, β, γ)=(0.98, 0.05, 0.05)

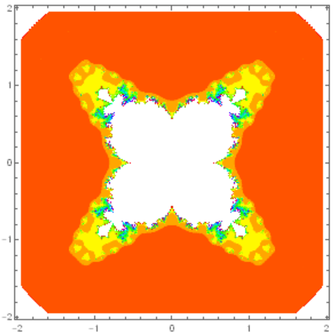

FIGURE 8. Fifth order Mandelbrot set for (α, β, γ)=(0.8, 0.9, 0.05)

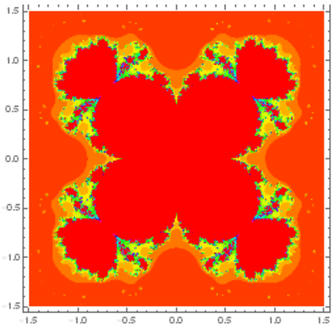

FIGURE 9. Fifth order Mandelbrot set for (α, β, γ)=(0.69, 0.1, 0.15)

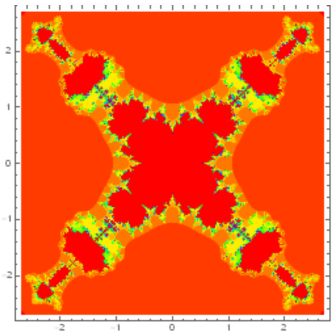

FIGURE 10. Fifth order Mandelbrot set for (α, β, γ)=(0.99, 0.1, 0.1)

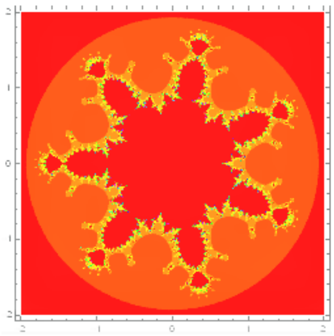

FIGURE 11. Eight order Mandelbrot set for (α, β, γ)=(0.99, 0.02, 0.02)

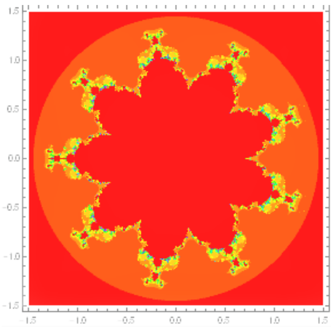

FIGURE 12. Tenth order Quadratic Mandelbrot set for (α, β, γ)=(0.85, 0.6, 0.07)

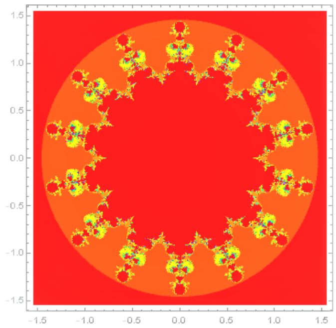

FIGURE 13. Fifteenth order Mandelbrot set for (α, β, γ)=(0.99, 0.12, 0.01)

6. Generation of Julia sets in Thakur-iteration

Using source program given in Subsection 4.1, we generate following Julia sets via Thakur-iteration by running the program in Mathematica 12.0. We found several fascinating new fractals having various geometric shapes of Julia sets for $P(z) = z^k + c, k = 2, 3, 4 \cdots$, at:

(1) $(c, \alpha, \beta, \gamma) = (-0.9 + 0.1i, 0.91, 0.6, 0.9)$ (see Fig. 14), for $k = 2$.
(2) $(c, \alpha, \beta, \gamma) = (0.45 + 0.25i, 0.96, 0.1, 0.35)$ (see Fig. 15), for $k = 2$.
(3) $(c, \alpha, \beta, \gamma) = (0.45 + 0.25i, 0.96, 0.10, 0.2)$ (see Fig. 16), for $k = 2$.
(4) $(c, \alpha, \beta, \gamma) = (0.02 + 0.01i, 0.75, 0.1, 0.99)$ (see Fig. 17), for $k = 3$.
(5) $(c, \alpha, \beta, \gamma) = (0.01 + 0.01i, 0.99, 0.09, 0.9)$ (see Fig. 18), for $k = 3$.
(6) $(c, \alpha, \beta, \gamma) = (0.1 + 0.1i, 0.51, 0.07, 0.07)$ (see Fig. 19), for $k = 6$.
(7) $(c, \alpha, \beta, \gamma) = (0.9 + 0.1i, 0.91, 0.6, 0.9)$ (see Fig. 20), for $k = 10$.
(8) $(c, \alpha, \beta, \gamma) = (-0.02 + 0.02i, 0.99, 0.99, 0.02)$ (see Fig. 21), for $k = 13$.
(9) $(c, \alpha, \beta, \gamma) = (-0.1 - 0.2i, 0.99, 0.01, 0.01)$ (see Fig. 22), for $k = 16$.
(10) $(c, \alpha, \beta, \gamma) = ((0.48 + 0.72i, 0.501, 0.1, 0.1))$ (see Fig. 23), for $k = 20$.

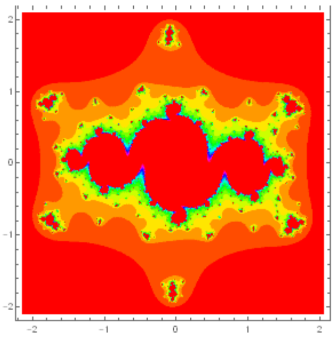

FIGURE 14. Quadratic Julia set for $(c, \alpha, \beta, \gamma)$=(-0.9+0.1i, 0.91, 0.6, 0.9)

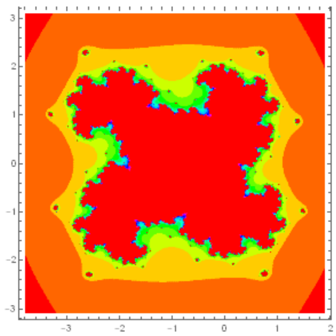

FIGURE 15. Quadratic Julia set for $(c, \alpha, \beta, \gamma)$=(0.45+0.25i, 0.96,0.1, 0.35)

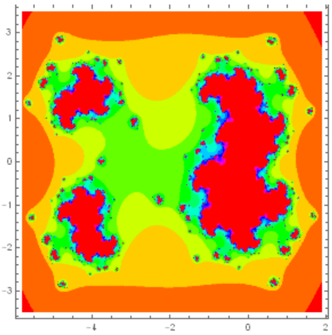

FIGURE 16. Quadratic Julia set for $(c, \alpha, \beta, \gamma)$=(0.45+0.25i ,0.96, 0.10, 0.2)

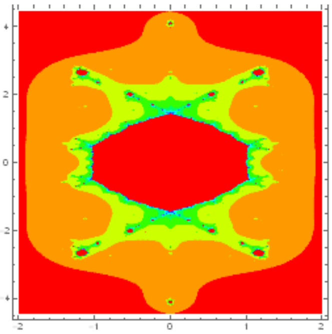

FIGURE 17. Cubic Julia set for $(c, \alpha, \beta, \gamma)$=(0.02+0.01i, 0.75, 0.1, 0.99)

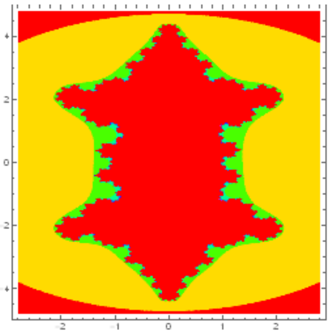

FIGURE 18. Cubic Julia set for $(c, \alpha, \beta, \gamma)$=(0.01+0.01i, 0.99, 0.09, 0.9)

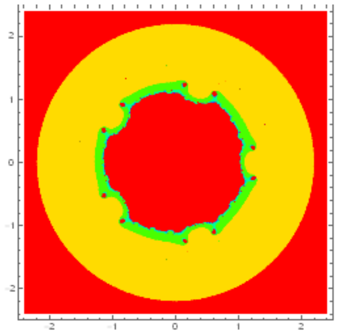

FIGURE 19. Sixth order Julia set for $(c, \alpha, \beta, \gamma)$=(0.1+0.1i, 0.51, 0.07, 0.07)

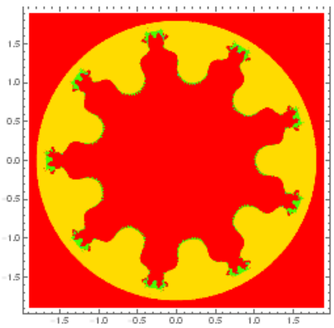

FIGURE 20. Tenth order Julia set for $(c, \alpha, \beta, \gamma)$=(0.9+0.1i, 0.91, 0.6, 0.9)

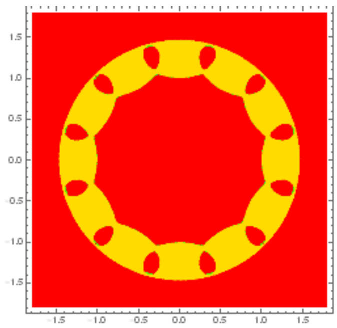

FIGURE 21. Thirteenth order Julia set for $(c, \alpha, \beta, \gamma)$= (-0.02+0.02i, 0.99, 0.99, 0.02)

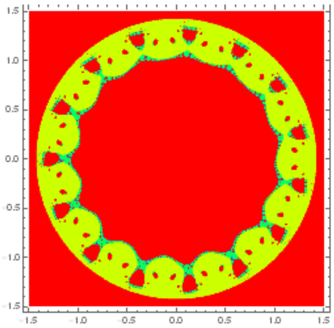

FIGURE 22. Sixteenth order Julia set for $(c, \alpha, \beta, \gamma)$=(-0.1-0.2i, 0.99, 0.01, 0.01)

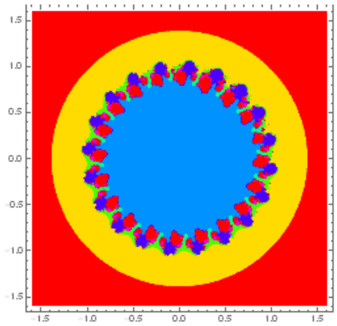

FIGURE 23. Twentieth order Julia set for $(c, \alpha, \beta, \gamma) = (0.48 + 0.72i, 0.501, 0.1, 0.1)$

7. Generation of tricorns and multicorns

In this section, some tricorns and multicorns are presented for the antipolynomial $\bar{z} \mapsto \bar{z}^d + c, d \geq 2$. Here, we present some graphics of these sets at:

(1) $(\alpha, \beta, \gamma) = (0.99, 0.09, 0.99)$ (see Fig. 24), for $k = 2$.
(2) $(\alpha, \beta, \gamma) = (0.8, 0.0.3, 0.01)$ (see Fig. 25), for $k = 2$.
(3) $(\alpha, \beta, \gamma) = (0.88, 0.07, 0.95)$ (see Fig. 26), for $k = 3$.
(4) $(\alpha, \beta, \gamma) = (0.68, 0.89, 0.99)$ (see Fig. 27), for $k = 5$.
(5) $(\alpha, \beta, \gamma) = (0.76, 0.55, 0.99)$ (see Fig. 28), for $k = 5$.
(6) $(\alpha, \beta, \gamma) = (0.8, 0.06, 0.25)$ (see Fig. 29), for $k = 8$.
(7) $(\alpha, \beta, \gamma) = (0.75, 0.9, 0.21)$ (see Fig. 30), for $k = 15$.
(8) $(\alpha, \beta, \gamma) = (0.8, 0.52, 0.21)$ (see Fig. 31), for $k = 20$.

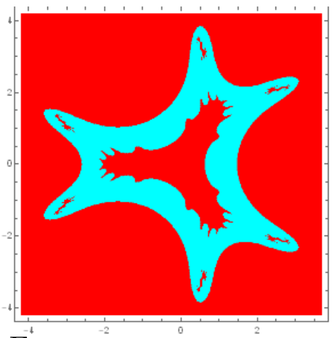

FIGURE 24. Tricorn set for (α, β, γ)=(0.99,0.09,0.99)

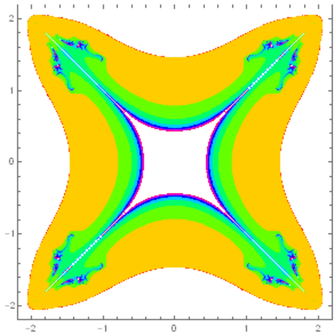

FIGURE 25. Tricorn set for (α, β, γ) = (0.8,0.0.3,0.01)

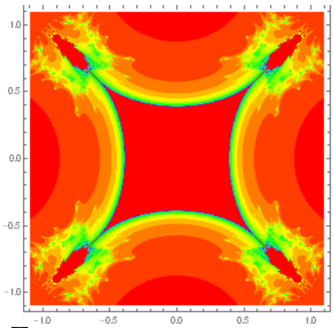

FIGURE 26. Multicorn set for (α, β, γ)=(0.88,0.07,0.95)

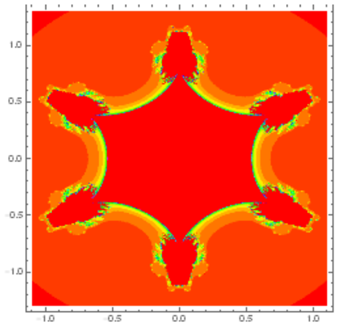

FIGURE 27. Multicorn set for (α, β, γ) =(0.68,0.89,0.99)

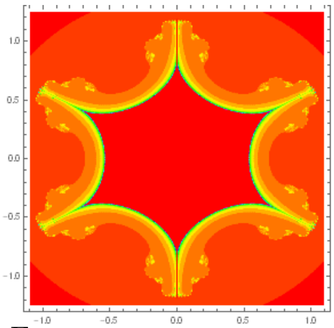

FIGURE 28. Multicorn set for (α, β, γ)=(0.76,0.55,0.99)

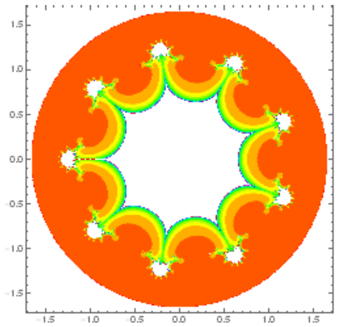

FIGURE 29. Multicorn set for (α, β, γ) =(0.8,0.06,0.25)

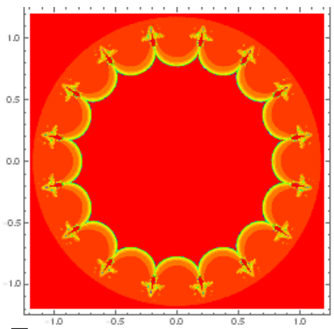

FIGURE 30. Multicorn set for (α, β, γ)=((0.75,0.9,0.21)

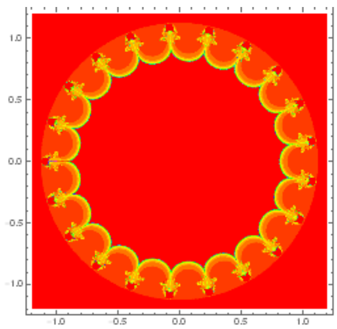

FIGURE 31. Multicorn set for (α, β, γ) =(0.8,0.52,0.21)

8. Generation of anti-Julia sets

In this section, some anti-Julia sets are presented for the antipolynomial $\bar{z} \mapsto \bar{z}^d + c, d \geq 2$. Here, we present some graphics of these sets at:

(1) $(c, \alpha, \beta, \gamma) = (0.42 + 0.4i, 0.99, 0.91, 0.92)$ (see Fig. 32), for $k = 2$.

(2) $(c, \alpha, \beta, \gamma) = (0.4 + 0.28i, 0.99, 0.41, 0.73)$ (see Fig. 33), for $k = 2$.
(3) $(c, \alpha, \beta, \gamma) = (0.13 + 0.2i, 0.93, 0.02, 0.03)$ (see Fig. 34), for $k = 2$.
(4) $(c, \alpha, \beta, \gamma) = (0.14 + 0.15i, 0.51, 0.02, 0.66)$ (see Fig. 35), for $k = 3$.
(5) $(c, \alpha, \beta, \gamma) = (0.74 - 0.7i, 0.99, 0.99, 0.99)$ (see Fig. 36), for $k = 3$.
(6) $(c, \alpha, \beta, \gamma) = (0.04 + 0.01i, 0.524, 0.01, 0.1)$ (see Fig. 37), for $k = 6$.
(7) $(c, \alpha, \beta, \gamma) = (0.01 + 0.01i, 0.99, 0.99, 0.03)$ (see Fig. 38), for $k = 10$.
(8) $(c, \alpha, \beta, \gamma) = (0.3 + 0.91i, 0.51, 0.83, 0.08)$ (see Fig. 39), for $k = 13$.

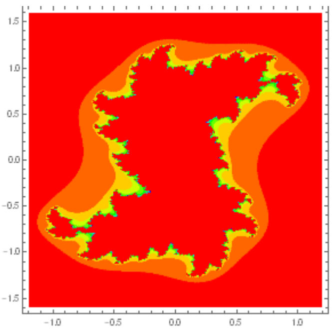

FIGURE 32. Quadratic Julia set for $(c, \alpha, \beta, \gamma)$=(0.42+0.4i, 0.99, 0.91, 0.92)

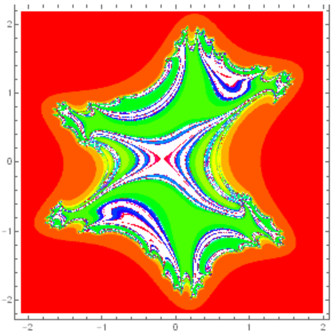

FIGURE 33. Quadratic Julia set for $(c, \alpha, \beta, \gamma)$ =(0.4+0.28i, 0.99, 0.41, 0.73)

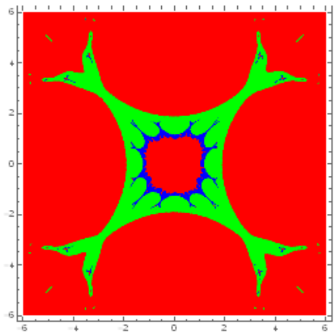

FIGURE 34. Quadratic Julia set for $(c, \alpha, \beta, \gamma)$=(0.13+0.2i, 0.93, 0.02, 0.03)

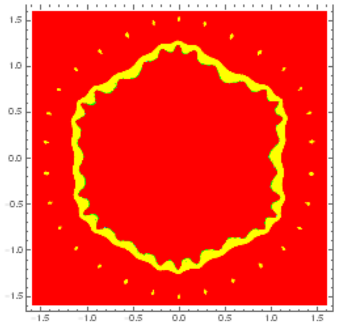

FIGURE 35. Cubic Julia set for $(c, \alpha, \beta, \gamma)$ =(0.14+0.15i, 0.51, 0.02, 0.66)

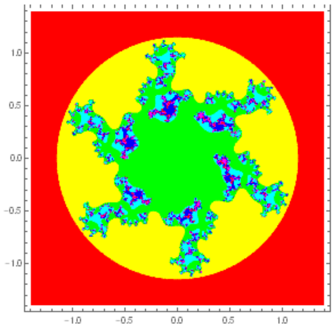

FIGURE 36. Cubic Julia set for $(c, \alpha, \beta, \gamma)$=(0.74-0.7i, 0.99, 0.99, 0.99)

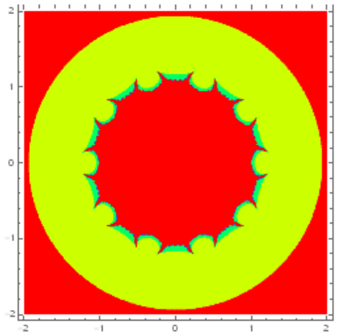

FIGURE 37. Sixth Julia set for $(c, \alpha, \beta, \gamma)$ =(0.04+0.01i, 0.524, 0.01, 0.1)

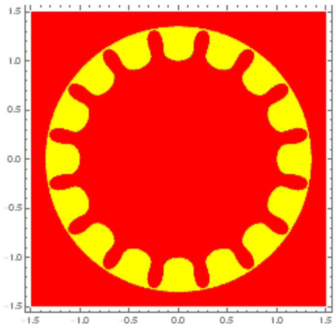

FIGURE 38. Tenth order Julia set for $(c, \alpha, \beta, \gamma)$=(0.01+0.01i, 0.99, 0.99, 0.03)

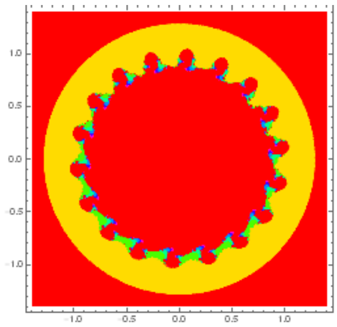

FIGURE 39. Thirteenth order Julia set for $(c, \alpha, \beta, \gamma)$ =(0.3+0.91i, 0.51, 0.83, 0.08)

9. Conclusion

In this paper, escape criteria of the complex polynomials have been derived to generate fractals via Thakur iteration. We obtained different patterns of fractals and antifractals with the change of involved parameters α, β and γ. We observed that with the change of parameters α, β and γ, there are distinguished variations in the shapes of fractals and anti-fractals.

References

[1] Ashish, M. Rani and R. Chugh, Julia sets and Mandelbrot sets in Noor orbit, Appl. Math. Comput., 228(2014), 615-631.

[2] E. B. Burger and M. Starbird, *The Heart Mathematics: An Invitation to Effective Thinking*, New York, NY, USA: Springer, 2004.

[3] Y. S. Chauhan, R. Rana and A. Negi, New Julia sets of Ishikawa iterates, Int. J. Comput. Appl., 7(2010), no. 13, 34-42.

[4] S. Y. Cho, A. A. Shahid, W. Nazeer and S. M. Kang, Fixed point results for fractal generation in Noor orbit and s-convexity, Springer Plus, 5(2016), no. 1, 18-43.

[5] N. Cohen, Fractal antenna applications in wireless telecommunications, Proc. Prof. Program Electron. Ind. Forum New England, 1997, 43-49.

[6] A. J. Crilly, R. A. Earnshaw and H. Jones, *Fractals and Chaos*, Springer-Verlag, New York 1991.

[7] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Second Ed., Westview Press, USA, 2003.

[8] K. Gdawiec and A. A. Shahid, Fixed point results for the complex fractal generation in the Thakur-iteration orbit with s-convexity, Open J. Math. Sci, 2(2018), no. 1, 56-72.

[9] S. R. Holtzman and D. Mantras, *The Languages of Abstract and Virtual Worlds*, Cambridge, MA, USA: Mit Press, 1995.

[10] S. Kang, A. Raq, A. Latif, A. Shahid and F. Alif, Fractals through modified iteration scheme, Filomat, 30(2016), no. 1, 3033-3046.

[11] N. Kenkel and D. Walker, Fractals in the biological sciences, Proc. Coenoses, 1996, 77-100.

[12] W. J. Krzysztofik, Fractals in antennas and metamaterials applications, Fractal Analysis: Applications in Physics, Engineering and Technology, F. Brambila, Eds. Rijeka, Croatia: in Tech, 2017, 953-978.

[13] M. Kumari, Ashish and R. Chugh, New Julia and Mandelbrot sets for a new faster iterative process, International Journal of Pure and Applied Mathematics, 107(2016), no. 1, 161-177.

[14] S. Kumari, M. Kumari and R. Chugh, Dynamics of superior fractals via Jungck SP orbit with s-convexity, Annals of the University of Craiova - Mathematics and Computer Science Series 46(2019), 344-365.

[15] S. Kumari, M. Kumari and R. Chugh, Graphics for complex polynomials in Jungck SP orbit, IAENG International Journal of Applied Mathematics, 49(2019), 568-576.

[16] Y. C. Kwun, A. A. Shahid, W. Nazeer, M. Abbas and S. M. Kang, Fractal generation via CR iteration scheme with s-convexity, IEEE Access, 7(2019), 69986-69997.

[17] Y. C. Kwun, M. Tanveer, W. Nazeer, K. Gdawiec and S. M. Kang, Mandelbrot and Julia sets via Jungck CR iteration with s-convexity, IEEE Access, 7(2019), 12167-12176.

[18] D. Li, M. Tanveer, W. Nazeer and X. Guo, Boundaries of filled Julia sets in generalized Jungck Mann orbit, IEEE Access, 7(2019), 76859-76867.

[19] B. B. Mandelbrot, *The Fractal Geometry Nature*, Vol. 2, W. H. Freeman, New York 1982.

[20] W. Nazeer, S. M. Kang, M. Tanveer and A. A. Shahid, Fixed point results in the generation of Julia and Mandelbrot sets, J. Inequalities Appl., 2015(2015), Paper No. 298, 16 pp.

[21] F. Orsucci, *Complexity Science, Living Systems, and Reflexing Interfaces: New Models and Perspectives: New Models and Perspectives*, Hershey, PA, USA: IGI Global, 2012.

[22] H. O. Peitgen, H. Jurgens and D. Saupe, *Chaos and Fractals*, Springer-Verlag, New York, Inc., 1992.

[23] M. Rani, Iterative procedures in fractals and chaos, PhD Thesis, Department of Computer Science, Gurukula Kangri Vishwavidyalaya, Hardwar, India, 2002.

[24] M. Rani and V. Kumar, Superior Julia set, J. Korea Soc. Math. Edu. Ser. D, Res. Math. Edu., 8(2004), no. 4, 261-277.

[25] M. Rani and V. Kumar, Superior Mandelbrot set, J. Korea Soc. Math. Edu. Ser. D, Res. Math. Edu., 8(2004), no. 4, 279-291.

[26] D. Thakur, B. S. Thakur and M. Postolache, New iteration scheme for numerical reckoning fixed points of nonexpansive mappings, Journal of Inequalities and Applications, 2014(2014), Paper No. 328, 15 pp.

(Amit Kumar and Renu Chugh) Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, India

E-mail address, Amit Kumar: amit.math2018@gmail.com

E-mail address, Renu Chugh: chugh.r1@gmail.com

(Sudesh Kumari) Department of Mathematics, Government College for Girls Sector 14, Gurugram-122001, India

E-mail address: tanwarsudesh10@gmail.com