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NEVANLINNA METHOD OF OBTAINING SOLUTION OF A SYSTEM OF q-SHIFT COMPLEX DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. The Nevanlinna theory of value distribution is used to investigate the existence of a meromorphic solution to a general system of q-shift complex differential-difference equations whose coefficients are small functions of the meromorphic function. Several limiting cases are reported in the study and an example is given to complement our result. A note is added about problems for which the method does not work.

1. Background

Ordinary differential-difference equations are widely used as models to describe many problems of dynamics in various fields of science, particularly in fluid mechanics, solid state physics, plasma physics, and so on. Many methods have been developed by mathematicians and physicists to find solutions of ordinary differential-difference equations but there is no unified technique that can be employed to handle all types of differential-difference equations.

The motivation of the present work is to investigate the existence of a meromorphic solution for general systems of such types of differential-difference equations by using the Nevanlinna theory. Our results show that the method provides a powerful mathematical tool for solving many differential-difference equations appearing in mathematical physics and also several results are reported as corollaries of our result. Some works have reported solutions for certain types of complex differential equations ([20], [21], [26]).

The non-autonomous Schröder q-difference equation

$$f(qz) = R(z, f(z)),$$

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where the right-hand side is rational in both arguments, has been widely studied during the last decades. Recently, the Nevanlinna theory involving q-difference has been developed to study q-difference equations. There are many articles focusing on complex difference equations ([8], [9], [30]) and complex q-shift difference equations([3], [15], [25], [31]) using the Nevanlinna theory. In this paper we use standard notions such as $T(r, f)$, $m(r, f)$, $N(r, f)$, etc., and the results of Nevanlinna theory([10], [28], [29]) in obtaining the desired meromorphic solution.

2. Preliminary Lemmas

The following three lemmas are required to prove our results on the desired meromorphic solution.

Lemma 2.1. *([18]) Let Ψ be a transcendental meromorphic function, then*

$$T(r, \Psi^{(k)}) \leq (k+1)T(r, \Psi) + S(r, \Psi).$$

Lemma 2.2. *([7]) Let Ψ be a transcendental meromorphic function, and $P(z) = a_k z^k + a_{k-1}z^{k-1} + ... + a_1 z + a_0$, $a_k \neq 0$ be a nonconstant polynomial of degree k. Given $0 < \delta < |a_k|$, denote $\alpha = |a_k| + \delta$ and $\beta = |a_k| - \delta$. Then given $\varepsilon > 0$ and $a \,\epsilon\, \mathbb{C} \cup \{\infty\}$, one has*

$$kn(\beta r^k, a, \Psi) \leq n(r, a, \Psi(p(z))) \leq kn(\alpha r^k, a, \Psi),$$
$$N(\beta r^k, a, \Psi) + O(logr) \leq N(r, a, \Psi(p(z))) \leq N(\alpha r^k, a, \Psi) + O(logr),$$
$$(1-\varepsilon)T(\beta r^k, \Psi) \leq T(r, \Psi(p(z))) \leq (1+\varepsilon)T(\alpha r^k, \Psi),$$

for all large enough r.

Lemma 2.3. *([6]) Let $\Psi : (r, +\infty) \to (0, +\infty)$ be positive and bounded in every finite interval, and suppose that $\Psi(\alpha r^k) \leqslant A\Psi(r) + B$ holds for all large enough r, where $\alpha > 0, k > 1, A > 1$ and B are real constants. Then $\Psi(r) = O((logr)^{\beta})$, where $\beta = \dfrac{logA}{logk}$.*

3. Main Results

In this research paper, inspired by ideas of Li and Gao [16] and Luo and Zheng [17], we consider system of q-shift differential-difference equations which is of the form:

$$\Omega(\Psi_i) = R_I(z, \Psi_I), \ for \ i = 1, 2, ..., n, \tag{3.1}$$

where

$$\begin{aligned}\Omega(\Psi_i) \ &= \ \sum_{h=1}^{n_1} A_{ih}(z)\Psi_i^{(\lambda_{ih})}(qz+\eta_h) + \sum_{j=1}^{n_2} B_{ij}(z)\Psi_i^{\lambda_{ij}}(qz+\eta_j) + \sum_{l=1}^{n_3} C_{il}(z)\Psi_i^{(\lambda_{il})}(z) \\ &+ \ \sum_{m=1}^{n_4} D_{im}(z)\Psi_i^{\lambda_{im}}(z).\end{aligned}$$

Here the coefficients $A_{ih}(z)$, $B_{ij}(z)$, $C_{il}(z)$, $D_{im}(z)$ denote small functions relative to $\Psi_i(z)$ respectively and λ_{ih}, λ_{ij}, λ_{il}, λ_{im} are finite non-negative integers. We consider $R_I(z, \Psi_I(z)) = \dfrac{a_{i0}(z) + a_{i1}(z)\Psi_I(z) + ... + a_{is_I}(z)\Psi_I^{s_I}(z)}{b_{i0}(z) + b_{i1}(z)\Psi_I(z) + ... + b_{it_I}(z)\Psi_I^{t_I}(z)}$ which is rational in $\Psi_I(z)$ with coefficients $a_{i0}(z)$, $a_{i1}(z), ..., a_{is_I}(z)$ and $b_{i0}(z)$, $b_{i1}(z), ..., b_{it_I}(z)$ being small functions of $\Psi_i(z)$ and for $i = 1,\ 2, ..., n$ we choose $I = n,\ 1,\ 2, ..., n-1$. Let $q \in \mathbb{C} \setminus \{0\}$ and $\eta_h,\ \eta_j$ be nonzero complex numbers.

Theorem 3.1. Let $\left(\Psi_1(z),\ \ \Psi_2(z), ..., \Psi_n(z)\right)$ be a zero-order transcendental meromorphic solution of the system of equation (3.1). We denote

$$d_i = deg_{\Psi_I} R_I(z, \Psi_I) = max\{s_I, t_I\},$$

and

$$\lambda_i = \left[\sum_{h=1}^{n_1}(\lambda_{ih}+1) + \sum_{j=1}^{n_2}\lambda_{ij} + \sum_{l=1}^{n_3}(\lambda_{il}+1) + \sum_{m=1}^{n_4}\lambda_{im}\right],$$

then we get $(d_i + o(1))T(r, \Psi_I) \leq \lambda_i T(r, \Psi_i)$ for $i = 1,\ 2, ..., n$.

Proof. Applying the Nevanlinna characteristic function on both sides of equation (3.1) and using the Theorem 2.2.5 of Laine[13], we get

$$\begin{aligned} d_i T(r, \Psi_I) &= T(r, R_I(z, \Psi_I)) + S(r, \Psi_I), \\ &= T(r, \Omega(\Psi_i)) + S(r, \Psi_I), \\ &= T\Bigg(r, \sum_{h=1}^{n_1} A_{ih}(z)\Psi_i^{(\lambda_{ih})}(qz+\eta_h) + \sum_{j=1}^{n_2} B_{ij}(z)\Psi_i^{\lambda_{ij}}(qz+\eta_j) \\ &+ \sum_{l=1}^{n_3} C_{il}(z)\Psi_i^{(\lambda_{il})}(z) + \sum_{m=1}^{n_4} D_{im}(z)\Psi_i^{\lambda_{im}}(z)\Bigg) + S(r, \Psi_I),\ for\ i = 1, 2, ..., n. \end{aligned} \tag{3.2}$$

Since the coefficients $A_{ih}(z)$, $B_{ij}(z)$, $C_{il}(z)$, $D_{im}(z)$, $a_{i0},\ a_{i1}, ..., a_{is_I}$ and $b_{i0},\ b_{i1}, ..., b_{it_I}$ in system (3.1) are small functions, we have

$$T(r, A_{ih}) = S(r, \Psi_i),\ T(r, B_{ij}) = S(r, \Psi_i),\ T(r, C_{il}) = S(r, \Psi_i),\ T(r, D_{im}) = S(r, \Psi_i),$$

$$T(r, a_{i0}) = T(r, a_{i1}) = ... = T(r, a_{is_I}) = S(r, \Psi_i),$$

$$T(r, b_{i0}) = T(r, b_{i1}) = ... = T(r, b_{is_I}) = S(r, \Psi_i),$$

which hold for all r outside the set with a finite logarithmic measure. Let $\left(\Psi_1(z),\ \ \Psi_2(z), ..., \Psi_n(z)\right)$ be a zero-order meromorphic solution of (3.1). By using

Lemmas 2.1 and Remark 5 of Xu and Zhang[27] in (3.2), we obtain

$$\begin{aligned}
d_iT(r,\Psi_I) &\le \sum_{h=1}^{n_1} T(r,\Psi_i^{(\lambda_{ih})}(qz+\eta_h)) + \sum_{j=1}^{n_2} T(r,\Psi_i^{\lambda_{ij}}(qz+\eta_j)) \\
&+ \sum_{l=1}^{n_3} T(r,\Psi_i^{(\lambda_{il})}(z)) + \sum_{m=1}^{n_4} T(r,\Psi_i^{\lambda_{im}}(z)) + S(r,\Psi_I), \\
&\le \sum_{h=1}^{n_1}(\lambda_{ih}+1)T(r,\Psi_i(qz+\eta_h)) + \sum_{j=1}^{n_2}\lambda_{ij}T(r,\Psi_i(qz+\eta_j)) \\
&+ \sum_{l=1}^{n_3}(\lambda_{il}+1)T(r,\Psi_i(z)) + \sum_{m=1}^{n_4}\lambda_{im}T(r,\Psi_i(z)) + S(r,\Psi_I), \\
&= \left[\sum_{h=1}^{n_1}(\lambda_{ih}+1) + \sum_{j=1}^{n_2}\lambda_{ij} + \sum_{l=1}^{n_3}(\lambda_{il}+1) + \sum_{m=1}^{n_4}\lambda_{im}\right] T(r,\Psi_i(z)) \\
&+ S(r,\Psi_I), \\
&= \lambda_i T(r,\Psi_i) + S(r,\Psi_I),\ for\ i=1,2,...,n.
\end{aligned} \tag{3.3}$$

From equation (3.3) we arrive at the following expression

$$(d_i+o(1))T(r,\Psi_I) \le \lambda_i T(r,\Psi_i)\ for\ i=1,2,...,n. \tag{3.4}$$

Hence the proof of Theorem 3.1 .

Example 3.1: Let $q=1$, $i=1,\ 2$, $\eta_h=\pm1$, $\eta_j=0$, $A=2e+e^{-1}+2$, $B=e+2e^{-1}+2$ and $\Big(\Psi_1(z),\ \Psi_2(z)\Big)=\Big(e^z,\ e^{-z}\Big)$ which satisfies the following system of equations

$$\Psi_1'(z+1)+\Psi_1''(z-1)+\Psi_1(z+1)+\Psi_1'(z)+\Psi_1(z)=\frac{A}{\Psi_2}, \tag{3.5}$$

$$\Psi_2'(z+1)-\Psi_2''(z-1)-\Psi_2(z+1)+\Psi_2'(z)-\Psi_2(z)=\frac{-B}{\Psi_1}. \tag{3.6}$$

From this system we have $d_1=d_2=1$, $\lambda_1=\lambda_2=9$. This shows that $d_1d_2=1<\lambda_1\lambda_2=81$ which appears in Theorem 3.1 . Similarly other examples can be added for integer values of q other than 1.

Remark 3.1. If $i=1,2$ then from equation (3.4) we get the following inequality

$$(d_1+o(1))T(r,\Psi_2) \le \lambda_1 T(r,\Psi_1), \tag{3.7}$$

$$(d_2+o(1))T(r,\Psi_1) \le \lambda_2 T(r,\Psi_2). \tag{3.8}$$

Taking $(d_2+o(1))$ times equation (3.7) and λ_1 times equation (3.8), we obtain respectively

$$(d_1+o(1))(d_2+o(1))T(r,\Psi_2) \le \lambda_1(d_2+o(1))T(r,\Psi_1), \tag{3.9}$$

$$\lambda_1(d_2+o(1))T(r,\Psi_1) \le \lambda_1\lambda_2 T(r,\Psi_2). \tag{3.10}$$

Therefore, from equations (3.9) and (3.10), we can get the following inequality:

$$(d_1+o(1))(d_2+o(1))T(r,\Psi_2) \le \lambda_1\lambda_2 T(r,\Psi_2). \tag{3.11}$$

From equation (3.11), we get

$$d_1d_2T(r,\Psi_2) \le \lambda_1\lambda_2 T(r,\Psi_2) + S(r,\Psi_2). \tag{3.12}$$

Noting that $T(r, \Psi_2) > 0$ and dividing throughout by $T(r, \Psi_2)$ on both sides of the inequality and letting $r \to \infty$ outside of a possible exceptional set E with finite logarithmic measure, we get $d_1 d_2 \leq \lambda_1 \lambda_2$. This inequality is same as that obtained in Theorem 2.1 of [16].

Example 3.2: Applying the Nevanlinna characteristic function on both sides of equations (3.5) and (3.6) respectively we get

$$9\ T(r, \Psi_1) + S(r, \Psi_1) > T(r, \Psi_2). \tag{3.13}$$

$$9\ T(r, \Psi_2) + S(r, \Psi_2) > T(r, \Psi_1). \tag{3.14}$$

Multiplying the equation (3.14) by 9 throughout, we get

$$81\ T(r, \Psi_2) + S(r, \Psi_2) > 9\ T(r, \Psi_1). \tag{3.15}$$

From equations (3.13) and (3.15), we can now get the following inequality:

$$81\ T(r, \Psi_2) + S(r, \Psi_2) > T(r, \Psi_2),$$

which implies $\lambda_1 \lambda_2\, T(r, \Psi_2) + S(r, \Psi_2) > d_1 d_2\, T(r, \Psi_1)$.

Remark 3.2. If $i = 1, 2, 3$, then from equation (3.4) we get the following inequality:

$$(d_3 + o(1))T(r, \Psi_3) \leq \lambda_1 T(r, \Psi_1), \tag{3.16}$$

$$(d_1 + o(1))T(r, \Psi_1) \leq \lambda_2 T(r, \Psi_2), \tag{3.17}$$

$$(d_2 + o(1))T(r, \Psi_2) \leq \lambda_3 T(r, \Psi_3). \tag{3.18}$$

Taking $(d_1 + o(1))$ times equation (3.16) and λ_1 times equation (3.17), we obtain respectively

$$(d_1 + o(1))(d_3 + o(1))T(r, \Psi_3) \leq \lambda_1 (d_1 + o(1))T(r, \Psi_1), \tag{3.19}$$

$$\lambda_1 (d_1 + o(1))T(r, \Psi_1) \leq \lambda_1 \lambda_2 T(r, \Psi_2). \tag{3.20}$$

Now from equations (3.19) and (3.20), we obtain the following:

$$(d_1 + o(1))(d_3 + o(1))T(r, \Psi_3) \leq \lambda_1 \lambda_2 T(r, \Psi_2). \tag{3.21}$$

Taking $(d_1 + o(1))(d_3 + o(1))$ times equation (3.18) and λ_3 times equation (3.21), we obtain respectively

$$(d_1 + o(1))(d_2 + o(1))(d_3 + o(1))T(r, \Psi_2) \leq \lambda_3 (d_1 + o(1))(d_3 + o(1))T(r, \Psi_3), \tag{3.22}$$

$$\lambda_3 (d_1 + o(1))(d_3 + o(1))T(r, \Psi_3) \leq \lambda_1 \lambda_2 \lambda_3 T(r, \Psi_2). \tag{3.23}$$

From equations (3.22) and (3.23), we obtain the following:

$$(d_1 + o(1))(d_2 + o(1))(d_3 + o(1))T(r, \Psi_2) \leq \lambda_1 \lambda_2 \lambda_3 T(r, \Psi_2). \tag{3.24}$$

Now, from equation (3.24), we get

$$d_1 d_2 d_3 T(r, \Psi_2) \leq \lambda_1 \lambda_2 \lambda_3 T(r, \Psi_2) + S(r, \Psi_2). \tag{3.25}$$

Noting that $T(r, \Psi_2) > 0$ and dividing throughout by $T(r, \Psi_2)$ on both sides of the inequality and letting $r \to \infty$ outside of a possible exceptional set E with finite logarithmic measure, we get $d_1 d_2 d_3 \leq \lambda_1 \lambda_2 \lambda_3$. Proceeding in the same manner for $i = 1, 2, ..., n$ we can prove that $\prod_{i=1}^{n} d_i \leq \prod_{i=1}^{n} \lambda_i$.

Example 3.3 : Let $q = 1$, $i = 1,\ 2,\ 3$, $\eta_h = \pm 1$, $\eta_j = 0$, $A = 2e+e^{-1}+2$, $B = e+2e^{-1}+2$, $C = e + \frac{e^{-1}}{2} + 1$ and $\Big(\Psi_1(z),\ \Psi_2(z),\ \Psi_3(z)\Big) = \Big(e^z,\ e^{-z},\ e^z+4\Big)$ which satisfies the following system of equations:

$$\Psi_3'(z+1) + \Psi_3''(z-1) + \Psi_3(z+1) + \Psi_3'(z) + \Psi_3(z) = 2C\Psi_1 + 8, \tag{3.26}$$

$$\Psi_1'(z+1) + \Psi_1''(z-1) + \Psi_1(z+1) + \Psi_1'(z) + \Psi_1(z) = \frac{A}{\Psi_2}, \tag{3.27}$$

$$\Psi_2'(z+1) - \Psi_2''(z-1) - \Psi_2(z+1) + \Psi_2'(z) - \Psi_2(z) = \frac{B}{4-\Psi_1}. \tag{3.28}$$

For this system we have $d_1 = d_2 = d_3 = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 9$. Applying the Nevanlinna characteristic function on both sides of equations (3.26), (3.27) and (3.28) respectively, we get

$$9\,T(r,\Psi_3) + S(r,\Psi_3) > T(r,\Psi_1). \tag{3.29}$$

$$9\,T(r,\Psi_1) + S(r,\Psi_1) > T(r,\Psi_2). \tag{3.30}$$

$$9\,T(r,\Psi_2) + S(r,\Psi_2) > T(r,\Psi_3). \tag{3.31}$$

Multiplying the equation (3.31) by 9 throughout, we get

$$81\,T(r,\Psi_2) + S(r,\Psi_2) > 9\,T(r,\Psi_3). \tag{3.32}$$

From equations (3.29) and (3.32), we can get the following inequality

$$81\,T(r,\Psi_2) + S(r,\Psi_2) > T(r,\Psi_1). \tag{3.33}$$

Now from equations (3.33) and (3.30), we obtain the following:

$$729\,T(r,\Psi_2) + S(r,\Psi_2) > T(r,\Psi_2),$$

which implies that $\lambda_1\lambda_2\lambda_3\,T(r,\Psi_2) + S(r,\Psi_2) > d_1d_2d_3\,T(r,\Psi_2)$.

We now consider the system of q-shift differential-difference equations which is of the form:

$$\Omega(\Psi_i) = \Psi_I(P(z)),\ for\ i = 1,2,...,n, \tag{3.34}$$

where $I = i - (-1)^i$ with $i = 1,\ 2,...,n$ and $\Omega(\Psi_i) = \sum_{h=1}^{n_1} A_{ih}(z)\Psi_i^{(\lambda_{ih})}(qz+\eta_h) + \sum_{j=1}^{n_2} B_{ij}(z)\Psi_i^{\lambda_{ij}}(qz+\eta_j) + \sum_{l=1}^{n_3} C_{il}(z)\Psi_i^{\lambda_{(il)}}(z) + \sum_{m=1}^{n_4} D_{im}(z)\Psi_i^{\lambda_{im}}(z)$. Here $A_{ih}(z)$, $B_{ij}(z)$, $C_{il}(z)$, $D_{im}(z)$ denote small functions relative to $\Psi_i(z)$ respectively and λ_{ih}, λ_{ij}, λ_{il}, λ_{im} are finite non-negative integers. Let $P(z) = a_kz^k + a_{k-1}z^{k-1} + ... + a_1z + a_0, a_k \neq 0$ be a nonconstant polynomial of degree $k \geq n$. Let $q\ \epsilon\ \mathbb{C} \setminus \{0\}$ and η_h, η_j be nonzero complex numbers.

Theorem 3.2. Suppose $\Big(\Psi_1(z), \Psi_2(z), ..., \Psi_n(z)\Big)$ is a zero-order transcendental meromorphic solution of the system of equation (3.34). Then, given $\varepsilon > 0$, we have

$$T(r,\Psi_1) + T(r,\Psi_2) + ... + T(r,\Psi_n) = O((logr)^{\alpha+\varepsilon}),$$

where $\alpha = \dfrac{log\chi}{logk}$, $\chi = \dfrac{(1+\varepsilon)\left[\sum_{h=1}^{n_1}(\lambda_{ih}+1) + \sum_{j=1}^{n_2}\lambda_{ij}\right]}{(1-\varepsilon) - \sum_{l=1}^{n_3}(\lambda_{il}+1) - \sum_{m=1}^{n_4}\lambda_{im}}$.

Proof. Let $\left(\Psi_1(z), \Psi_2(z), ..., \Psi_n(z)\right)$ be a zero-order transcendental meromorphic solution of the system (3.34). Given $0 < \delta < |a_k|$, denote $\alpha = |a_k| + \delta$ and $\beta = |a_k| - \delta$. From the equation (3.34), we may write

$$\begin{aligned}
T(r, \Psi_I(P(z))) \leq T\Bigg(r, &\sum_{h=1}^{n_1} A_{ih}(z)\Psi_i^{(\lambda_{ih})}(qz+\eta_h) + \sum_{j=1}^{n_2} B_{ij}(z)\Psi_i^{\lambda_{ij}}(qz+\eta_j) \\
&+ \sum_{l=1}^{n_3} C_{il}(z)\Psi_i^{(\lambda_{il})}(z) + \sum_{m=1}^{n_4} D_{im}(z)\Psi_i^{\lambda_{im}}(z)\Bigg), \\
&\leq \sum_{h=1}^{n_1}(\lambda_{ih}+1)T(r, \Psi_i(qz+\eta_h)) + \sum_{j=1}^{n_2}\lambda_{ij}T(r, \Psi_i(qz+\eta_j)) \\
&+ \sum_{l=1}^{n_3}(\lambda_{il}+1)T(r, \Psi_i(z)) + \sum_{m=1}^{n_4}\lambda_{im}T(r, \Psi_i(z)) + S(r, \Psi_i).
\end{aligned} \tag{3.35}$$

From equation (3.35), using Lemma 1 of Ablowitz et al.[1] and the last result of Lemma 2.2 in the paper, for given $\varepsilon > 0$, we have

$$\begin{aligned}
(1-\varepsilon)T(\beta r^k, \Psi_i) &\leq T(r, \Psi_I(p(z))), \\
&\leq (1+\varepsilon)\left[\sum_{h=1}^{n_1}(\lambda_{ih}+1) + \sum_{j=1}^{n_2}\lambda_{ij}\right] T(r+\eta, \Psi_i(z)) \\
&+ \left[\sum_{l=1}^{n_3}(\lambda_{il}+1) + \sum_{m=1}^{n_4}\lambda_{im}\right] T(r, \Psi_i(z)) + S(r, \Psi_i).
\end{aligned} \tag{3.36}$$

Since $T(r+\eta, \Psi_i) \leq T(\beta r, \Psi_i)$ holds for large enough r and for $\beta > 1$, in order to remove the exceptional set, we know that whenever $\mu > 1$, the following holds:

$$\begin{aligned}
(1-\varepsilon)T(\beta r^k, \Psi_I) \leq (1+\varepsilon)&\left[\sum_{h=1}^{n_1}(\lambda_{ih}+1) + \sum_{j=1}^{n_2}\lambda_{ij}\right] T(\mu\beta r, \Psi_i) \\
+ &\left[\sum_{l=1}^{n_3}(\lambda_{il}+1) + \sum_{m=1}^{n_4}\lambda_{im}\right] T(r, \Psi_i).
\end{aligned} \tag{3.37}$$

Putting $t = \mu\beta r$, the inequality (3.37) can be written as

$$\begin{aligned}
(1-\varepsilon)T\left(\frac{\beta t^k}{(\mu\beta t)^k}, \Psi_I\right) \leq (1+\varepsilon)&\left[\sum_{h=1}^{n_1}(\lambda_{ih}+1) + \sum_{j=1}^{n_2}\lambda_{ij}\right] T(t, \Psi_i) \\
+ &\left[\sum_{l=1}^{n_3}(\lambda_{il}+1) + \sum_{m=1}^{n_4}\lambda_{im}\right] T\left(\frac{\beta t^k}{(\mu\beta t)^k}, \Psi_I\right).
\end{aligned} \tag{3.38}$$

The inequality (3.38) can now be written as

$$T\left(\frac{\beta t^k}{(\mu\beta t)^k}, \Psi_i\right) \leq \frac{(1+\varepsilon)\left[\sum_{h=1}^{n_1}(\lambda_{ih}+1)+\sum_{j=1}^{n_2}\lambda_{ij}\right]}{(1-\varepsilon)-\sum_{l=1}^{n_3}(\lambda_{il}+1)-\sum_{m=1}^{n_4}\lambda_{im}} T(t,\Psi_i),\ for\ i=1,2,...,n. \tag{3.39}$$

On using Lemma 2.3 and equation (3.37) in equation (3.39), we get

$$T(r,\Psi_i) = O((logr)^{\gamma}),$$

where

$$\gamma = \frac{log\left(\frac{(1+\varepsilon)\left[\sum_{h=1}^{n_1}(\lambda_{ih}+1)+\sum_{j=1}^{n_2}\lambda_{ij}\right]}{(1-\varepsilon)-\sum_{l=1}^{n_3}(\lambda_{il}+1)-\sum_{m=1}^{n_4}\lambda_{im}}\right)}{logk} = \frac{log\chi}{logk} + o(1).$$

Now denoting $\alpha = \dfrac{log\chi}{logk}$, we can get the required form of Theorem 3.2 . Hence, the Theorem 3.2 is proved.

Counter Example 3.1: This method of solution works for all q-shift differential-difference equations in which Ψ_i is entire or meromorphic function. When a function is other then entire or meromorphic function this method will not work.

Counter Example 3.2: The results holds for Linear differential-difference equations but not for nonlinear type equations.

4. Utility of the problem

In this research paper, we have explored the possibility of a transcendental meromorphic solution of a certain class of q-shift differential-difference equations in the complex domain. The problems involving these type of equations arise in studies of control theory [12], in determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in dendrites [23], in the modelling of activation of a neuron [24] and many more. This type of equation has been studied by several authors and the same is documented below:

1) Sirisha et. al. [22] obtained a numerical solution to the equation of the form:

$$b(x)y(x-\delta)+d(x)y(x+\eta)+a(x)y'(x)+\varepsilon y''(x)+c(x)y(x)=f(x), \tag{4.1}$$

by using the mixed finite-difference method.
2) Duressa and Reddy [5] obtained a numerical solution to the equation of the form

$$\alpha(x)y(x-\delta)+\beta(x)y(x+\eta)+a(x)y'(x)+\varepsilon y''(x)+w(x)y(x)=f(x), \tag{4.2}$$

by using the domain decomposition method.
3) Andargie and Reddy [2] obtained a numerical solution to the equation of the form

$$q(x)y(x-\delta)+s(x)y(x+\gamma)+p(x)y'(x)+\varepsilon y''(x)+r(x)y(x)=f(x), \tag{4.3}$$

by using the fitted method.

4) Prasad and Reddy [19] obtained a numerical solution to the equation of the form

$$\alpha(x)y(x-\delta)+\beta(x)y(x+\eta)+\varepsilon^2 y''(x)+w(x)y(x)=f(x), \tag{4.4}$$

by using the differential quadrature method.

In a recent work[11], a tropical version of the Nevanlinna theory is described where they have considered continuous piecewise linear functions of a real variable. Analogues of the Nevanlinna characteristic, proximity and counting functions are defined and versions of Nevanlinna's first main theorem, the lemma on the logarithmic derivative and Clunie's lemma are proved. Based on this, in taking z has a real variable x in equation (3.1), we get a real meromorphic function, $\Psi(z)$, having a real pole. This leads us to conclude that there exists transcendental real meromorphic solution for the following differential-difference equations:

a) In equation (3.1), taking $i=1$, $q=1$, $A_{ih}=0$, $m=1$, $j=1,2$, $l=1,2$, $\eta_1=-\delta$, $\eta_2=\eta$, $B_{11}(x)=b(x)$, $B_{12}(x)=d(x)$, $C_{11}(x)=a(x)$, $C_{12}(x)=\varepsilon$, $D_{11}(x)=c(x)$ and $R_I(z,\Psi_I)=\Psi(x)$, we get

$$b(x)\Psi(x-\delta)+d(x)\Psi(x+\eta)+a(x)\Psi'(x)+\varepsilon\Psi''(x)+C(x)\Psi(x)=\Psi(x). \tag{4.5}$$

b) In equation (3.1), taking $i=1$, $q=1$, $A_{ih}=0$, $m=1$, $j=1,2$, $l=1,2$, $\eta_1=-\delta$, $\eta_2=\eta$, $B_{11}(x)=\alpha(x)$, $B_{12}(x)=\beta(x)$, $C_{11}(x)=0$, $C_{12}(x)=\varepsilon^2$, $D_{11}(x)=w(x)$ and $R_I(z,\Psi_I)=\Psi(x)$, we get

$$\alpha(x)\Psi(x-\delta)+\beta(x)\Psi(x+\eta)+\varepsilon^2\Psi''(x)+w(x)\Psi(x)=\Psi(x). \tag{4.6}$$

Clearly equation (4.5) is the same as equations (4.1), (4.2), (4.3) and also equation (4.6) is the same as equation (4.4). Therefore, we may conclude that the Nevanlinna theory is an alternative method of solving differential-difference equations involving delay or advance parameters.

5. Results and Discussion

Using the much simpler proofs, this article provides two main results on the operator $\Omega(\Psi)$ by which extending the main results that were derived in ([1], [4], [9], [16], [17]). Getting, our results from more general hypotheses and without any complicated calculations is probably the most interesting feature of this article. In equation (3.1) by fixing the values of coefficients and power of Ψ_i in $\Omega(\Psi_i)$ with $q=1$, we get equations which are tabulated in the table (1). One can easily explore many more limiting cases of the present study.

Coefficients of Ψ_i in equation (3.1)	λ values	η values	Equation obtained from equation (3.1)	Reference
$A_{ih} = C_{il} = 0$ $D_{im} = 0, B_{ij} = 1$	$\lambda_{ij} = 1$	$\eta_h = 1$ $\eta_j = -1$	$i = 1$ and $j = 1,\ 2$ $\Psi(z+1) + \Psi(z-1) = R(z, \Psi(z))$	Ablowitz M. J. et al.[1]
$A_{ih} = C_{il} = 0$ $D_{im} = 0,\ B_{ij} = 1$	$\lambda_{ij} = 1$	—	$i = 1$ $\sum_{j=1}^{n_2} \Psi(z + \eta_j) = R(z, \Psi(z))$	Heittokangas J. et al.[9]
$A_{ih} = C_{il} = 0$ $D_{im} = 0$	$\lambda_{ij} = 1$	—	$R(z, \Psi(z))$ polynomial $\sum_{j=1}^{n_2} B_{ij}(z)\Psi_i(z + \eta_j) = R(z, \Psi_2(z))$	Chen Z. X.[4]
$B_{ij} = C_{il} = 0$ $D_{im} = 0$	—	—	$i = 1,\ 2$ $\sum_{h=1}^{n_1} A_{1h}(z)\Psi_i^{(\lambda_{ih})}(z + \eta_h) = R_2(z, \Psi_2(z))$, $\sum_{h=1}^{n_1} A_{2h}(z)\Psi_2^{(\lambda_{2h})}(z + \eta_h) = R_1(z, \Psi_1(z))$	Li H. and Gao L.[16]
$B_{ij} = C_{il} = 0$ $D_{im} = 0$	—	—	$i = 1,\ 2$ $\sum_{h=1}^{n_1} A_{ih}(z)\Psi_1^{(\lambda_{ih})}(z + \eta_h) = R_2(z, \Psi_2(z))$, $\sum_{h=1}^{n_1} A_{2h}(z)\Psi_2^{(\lambda_{2h})}(z + \eta_h) = R_1(z, \Psi_1(z))$	Li H. and Gao L.[16]
$B_{ij} = C_{il} = 0$ $D_{im} = 0$	—	—	$i = 0, 1, ..., n$ $\sum_{h=0}^{n_1} A_{ih}(z)\Psi_i^{(\lambda_{ih})}(z + \eta_h) = R(z, \Psi(z))$	Luo L. Q. and Zheng X. M.[17]

Table 1: Limiting cases of the present paper

6. Open problem

Is it possible to prove the existence of meromorphic solution for system of partial differential-difference equation by using the Nevanlinna theory of meromorphic function in several complex variables?

References

[1] M. J. Ablowitz, R. Halburd and B. Herbst, On the extension of the Painlevé property to difference equations, Nonlinearity, 13(2000), no. 3, 889–905.

[2] A. Andargie and Y. N. Reddy, Solving singularly perturbed differential difference equations via fitted method, AAM: Intern. J., 8(2013), no. 1, 318–332.

[3] D. C. Barnett, R. G. Halburd, R. J. Korhonen and W. Morgan, Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations, Proc. R. Soc. Edin. Sect. A Math., 137(2007), 457–474.

[4] Z. X. Chen, Zeros of entire solutions to complex linear difference equations, Acta Math. Sci. Ser. B (Engl. Ed.) 32 (2012), no. 3, 1141–1148.

[5] G. F. Duressa and Y. N. Reddy, Domain decomposition method for singularly perturbed differential difference equations with layer behaviour, Int. J. Eng. Appl., 7(2015), no. 1, 86–102.

[6] R. Goldstein, On meromorphic solution of meromorphic equations, Aequat. Math., 18(1978), 112–157.

[7] R. Goldstein, Some results on factorisation of meromorphic functions, J. Lond. Math. Soc., 4(1971), no. 2, 357–364.

[8] L. Y. Gao, Estimates of N-function and m-function of meromorphic solutions of systems of complex difference equations. Acta Math. Sci. Ser. B (Engl. Ed.) 32 (2012), no. 4, 1495–1502.

[9] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and K. Tohge, Complex difference equations of Malmquist type, Comput. Methods Funct. Theory, 1(2001), no. 1, 27–39.

[10] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[11] R. G. Halburd and N. J. Southall, Tropical Nevanlinna theory and ultradiscrete equations. Int. Math. Res. Not. IMRN, 5(2009), 887–911.

[12] P. V. Kokotovic, H. K. Khalil and J. O'Reilly, *Singular perturbation methods in control, Analysis and Design*, New York: Academic Press, 1986.

[13] I. Laine, *Nevanlinna theory and complex differential equations*, Berlin: Walter de Gruyter, 1993.

[14] I. Laine, J. Rieppo and H. Silvennoinen, Remarks on complex difference equations, Comput. Methods Funct. Theory, 5(2005), no. 1, 77–88.

[15] K. Liu and X. G. Qi, Meromorphic solutions of q-shift difference equations, Ann. Polon. Math., 101(2011), 215–225.

[16] H. Li and L. Gao, Meromorphic solutions of a type of system of complex differential-difference equations. Acta Math. Sci. Ser. B (Engl. Ed.) 35 (2015), no. 1, 195–206.

[17] L. Q. Luo and X. M. Zheng, Value distribution of meromorphic solutions of homogeneous and non-homogeneous complex linear differential-difference equations, Open Math. 14(2016), 970–976.

[18] H. Milloux, *Les Function Meromorphices et Leurs Derivees*, Paris: (French) Actualites Sci. Ind., 1940.

[19] H. S. Prasad and Y. N. Reddy, Numerical solution of singularly perturbed differential-difference equations with small shifts of mixed type by differential quadrature method, American Journal of Computational and Applied Mathematics, 2(2012), no. 1, 46–52.

[20] P. G. Siddheshwar and A. Tanuja, Existence of meromorphic solution of Riccati-Abel differential equation, Appl Math Sci Comput. Springer Series, (2019), 21–28.
[21] P. G. Siddheshwar, A. Tanuja, S. S. Bhoosnurmath and B. Mahesh, Meromorphic solution of a class of non-linear differential equations with sharing one value, J. Anal, 28(2020), no. 2, 415–430.
[22] L. Sirisha, K. Phaneendra and Y. N. Reddy, Mixed finite difference method for singularly perturbed differential difference equations with mixed shifts via domain decomposition, Ain Shams Eng. J., 9(2018), 647–654.
[23] R. B. Stein, Some models of neuronal variability, Biophys J., 7(1967), no. 1, 37–68.
[24] R. B. Stein, A theoretical analysis of neuronal variability, Biophys J., 5(1965), no. 2, 173–94.
[25] A. Tanuja, Some general q-shift difference equation and their meromorphic solutions, AIP Conf. Proc., 2095(2019), no.1, 030031–030041.
[26] J. F. Tang and L. W. Liao, The transcendental meromorphic solutions of a certain type of nonlinear differential equations, J. Math. Anal. Appl., 334(2007), 517–527.
[27] J. Xu and X. Zhang, The zeros of q-shift difference polynomials of meromorphic functions, Adv. Differ. Equ., (2012), 1–10.
[28] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003; Chinese original: Science Press, Beijing, 1995.
[29] L. Yang, *Value Distribution Theory*, Springer Verlag, Berlin, 1993.
[30] Y. Y. Zhang, Z. S. Gao and J. L. Zhang, Growth of meromorphic solutions for complex difference equations of Malmquist type, Bull. Iranian Math. Soc., 6(2016), 1497–1505.
[31] X. M. Zheng and Z. X. Chen, Some properties of meromorphic solutions of q-difference equations, J. Math. Anal. Appl., 361(2010), 472–480.

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