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GENERALIZED ROUMIEU TEMPERED ULTRADISTRIBUTIONS

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Abstract. The aim of this paper is to introduce and to study a new classes of generalized functions of Colombeau type containing the space of tempered Roumieu ultradistributions.

1. Introduction

It is fundamental to provide a more general framework of a differential algebra of generalized functions containing the spaces of ultradistributions as well as their tempered ones, because of the great importance they have in studying the quantum field theory, convolution equations, harmonic analysis, pseudo-differential theory, time-frequency analysis, and other fields of analysis. For more details see [18] and [25].

Generalized Gevrey ultradistributions of Colombeau type have been defined, but as a side-theme, in [15]. The first paper aiming to introduce differential algebras containing ultradistributions is [21] and even the interesting approach of the article [11]. However, a Colombeau type theory of generalized Gevrey ultradistributions has been addressed in [4], where a kernel of a complete theory was developed and has also introduced a new way of defining the differential algebras of generalized Gevrey ultradistributions which makes such a complete theory possible. In [3] and [2] a general construction of generalized Gevrey ultradistribution algebras was given and has shown why in [4] different Gevrey exponents occurred in the embedding of Gevrey ultradistribution spaces. The authors in [1] introduce new general algebras of generalized functions containing Roumieu ultradistributions and study the embedding of these spaces and in [5] authors develop a microlocal analysis suitable for them by introducing a notion of generalized regularity which coincides with ultradifferentiability.

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In [10] authors show that it is possible to embed the spaces of M_p-ultradistributions into differentiable algebras in such a way that the multiplication of all M_p-ultradifferentiable functions is preserved. As for the recent developments regarding the theory of ultradistributions, it is worth mentioning in this context, the developments and analyses presented by authors in [8] based on [7]. Authors in [20] have developed the theory of wavelet transform using ultraditribution theory of Beurling and Björck and in [27], authors have defined wavelet transform of ultraditribution in $\mathcal{Z}'_w$ and established the corresponding inversion formula by interpreting convergence in the weak distributional sense.

The classical theory of Fourier analysis requires the notion of tempered distribution space and the space of rapidly decreasing functions. In the same way, to introduce a global theory of generalized Roumieu ultradistributoins one needs to introduce the generalized tempered ultradistributions which is the objective of this paper.

First, we introduce new differential algebras of generalized tempered ultardistributions, denoted as $\mathcal{G}_\tau^M(\Omega)$, and defined as quotient algebra:

$$\mathcal{G}_\tau^M(\Omega) = \frac{\mathcal{E}_{m.\tau}^M(\Omega)}{\mathcal{N}_\tau^M(\Omega)},$$

where $\mathcal{E}_{m.\tau}^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying $\forall\alpha \in Z_+^n$, $\exists k > 0$, $\exists p > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^{-p} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(M\left(\frac{k}{\varepsilon}\right)\right),$$

and $\mathcal{N}_\tau^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying $\forall\alpha \in Z_+^n$, $\forall k > 0$, $\exists p > 0$ $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^{-p} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(-M\left(\frac{k}{\varepsilon}\right)\right).$$

Then we show that $\mathcal{G}_\tau^M(\Omega)$ contains the space of tempered ultradistributions of class M_pN_p, and the following diagram of the embeddings is commutative

$$\begin{array}{ccc} O_{\mathcal{M}}^{\frac{M}{N}p!}(\Omega) & \rightarrow & \mathcal{G}_\tau^M(\Omega) \\ & \searrow & \uparrow \\ & & S'^{MN}(\Omega) \end{array}$$

2. Preliminaries

In this paper, we will study only the case of Roumieu, let $(M_p)_{p\in\mathbb{Z}_+}$ be a sequence of positive numbers, recall the following properties :

$(H1)$ Logarithmic convexity :

$$M_p^2 \leq M_{p-1}M_{p+1}$$

$(H2)$ Stability under ultradifferentiation :

$$\exists A > 0; \exists H > 0; M_{p+q} \leq AH^{p+q}M_pM_q; \forall p \geq 0; \forall q \geq 0$$

$(H2)'$ Stability under differentiation :

$$\exists A > 0; \exists H > 0; M_{p+1} \leq AH^{p+1} M_p; \forall p \geq 0;$$

$(H3)'$ Non-quasi-analyticity :

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty.$$

The associated function of the sequence $(M_p)_{p\in\mathbb{Z}_+}$ is the function defined by

$$M(t) = \sup_p \log \frac{M_0 t^p}{M_p}; t \in \mathbb{R}_+^*$$

A differential operator of infinite order $P(D) = \sum\limits_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma$ is called an ultradifferential operator of class M_p, if for every $h > 0$ there exist $c > 0$ such that $\forall \gamma \in Z_+^n$,

$$|a_\gamma| \leq c \frac{h^{|\gamma|}}{M_{|\gamma|}}. \tag{2.1}$$

An infinitely differentiable function f on an open set Ω in $\mathbb{R}^n$ will be called an ultradifferentiable function of class M_p if for every compact subset K of Ω, $\exists c > 0$, $\forall \alpha \in Z_+^n$,

$$\sup_{x\in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} M_{|\alpha|}. \tag{2.2}$$

This space is also called the space of Denjoy-Carleman.

Let $E^M(\Omega)$ be the space of all ultradifferentiable function of class M_p on Ω and let $D^M(\Omega)$ be the subspace of $E^M(\Omega)$ composed of all functions with compact support. see for more detail [17, 26]

If $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H1)$ and $(H3)'$ then by the Denjoy-Carleman-Mandelbrojt theorem there are sufficiently many functions in $D^M(\Omega)$.

Definition 2.1. The strong dual of $D^M(\Omega)$, denoted as $D'^M(\Omega)$, is called the space of Roumieu ultradistributions.

Proposition 2.2. Let the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfy condition (H1), then it satisfies (H2) if and only if $\exists A > 0, \exists H > 0, \forall t > 0$,

$$2M(t) \leq M(Ht) + \ln(AM_0).$$

Below, we list the most important ones, then we need next. For more details see[6, 13, 14, 16, 19, 22, 23] and [24].

Definition 2.3. The space of smooth functions ϕ on $\mathbb{R}^n$ such that, for all $h > 0$,

$$\sigma_h(\phi) = \left(\sum_{\alpha,\beta} \int | \frac{h^{\alpha+\beta}}{M_{|\alpha|} M_{|\beta|}} (1 + |x|^2)^{\frac{\beta}{2}} \partial^\alpha \phi(x)|^2 \right)^{\frac{1}{2}} < \infty,$$

equipped with the topology induced by the norm $\sigma_h(\phi)$ is denoted by $S^{M,h}$. The strong duals of

$$S^M = \varinjlim_{h\to\infty} S^{M,h}$$

is called the space of tempered Roumieu ultraditributions, denoted S'^M.

Remark 2.4. The family of norms $\{\sigma_h, h > 0\}$ is equivalent to the family of norms $\{\|.\|_m, m > 0\}$, where

$$\|\phi\|_m = \sup_{\alpha,x}\{\frac{m^{|\alpha|}}{M_{|\alpha|}}|\partial^\alpha\phi(x)|\exp M(m|x|)\}.$$

Definition 2.5. The space of the convolutors of S'^M, denoted as O'^M_c, is the space of all $T \in S'^M$ such that the convolution $T \star \phi$ is in S^M, for every $\phi \in \mathcal{S}^M$, and the mapping

$$\begin{array}{cccc} : & S^M & \to & S^M \\ & \phi & \to & T \star \phi \end{array}$$

is continuous.

Definition 2.6. The space of multipliers denoted as $O^M_{\mathcal{M}}$ is the space of functions φ from $E^M(\Omega)$ such that $\varphi \in O^M_{\mathcal{M}}$ if and only if for every $\phi \in \mathcal{S}^M$, $\varphi\phi \in \mathcal{S}^M$ and the mapping

$$\begin{array}{cccc} : & S^M & \to & S^M \\ & \phi & \to & \varphi\phi \end{array}$$

is continuous.

Proposition 2.7. Let $\phi \in C^\infty$, then $\phi \in O^M_{\mathcal{M}}$ if and only if $\forall m > 0, \exists r > 0$, such that

$$\sup_{\alpha\in\mathbb{Z}^n_+}\left\{\frac{r^{|\alpha|}\|\exp\left(-M(m|.|)\right)\partial^\alpha(\phi)\|_\infty}{M_{|\alpha|}}\right\} < \infty.$$

Proof. see [9, 14]. □

Proposition 2.8. Assume that the sequence M_p satisfes conditions $(H1), (H2)$ and $(H3)$, then $T \in S'^M$ if, and only if, $T = P(D)F$, where P is an operator of class M_p and F is continuous functions on $\mathbb{R}^n$ such that for every $k > 0$ there is a $c > 0$:

$$|F(x)| \leq c\exp\left(M(k|x|)\right); x \in \mathbb{R}^n$$

Proof. see [6]. □

3. Generalized tempered Roumieu ultradistributions

For our needs, we recall the essential definitions and results on algebra of generalized Roumieu ultradistributions of class M_p, denoted as $\mathcal{G}^M(\Omega)$, for more details, we send the reader to [1] .

Let Ω be a non void open subset of $\mathbb{R}^n$ and $I =]0, 1]$.

Definition 3.1. The space of moderate elements, denoted as $\mathcal{E}^M_m(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in Z^n_+$, $\exists k > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K}|\partial^\alpha f_\varepsilon(x)| \leq c\exp\left(M\left(\frac{k}{\varepsilon}\right)\right). \tag{3.1}$$

The space of null elements, denoted as $\mathcal{N}^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in Z^n_+$, $\forall k > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K}|\partial^\alpha f_\varepsilon(x)| \leq c\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right). \tag{3.2}$$

Proposition 3.2. 1) The space of moderate elements $\mathcal{E}_m^M(\Omega)$ is an algebra stable by derivation.
2) The space $\mathcal{N}^M(\Omega)$ is an ideal of $\mathcal{E}_m^M(\Omega)$.

Definition 3.3. The algebra of generalized Roumieu ultradistributions of class M_p, denoted as $\mathcal{G}^M(\Omega)$, is the quotient algebra:

$$\mathcal{G}^M(\Omega) = \frac{\mathcal{E}_m^M(\Omega)}{\mathcal{N}^M(\Omega)}.$$

In order to introduce tempered ones, we need to study the elements which are missing the property of compact support but they have the same behavior at infinity. Therefore, a polynomial must be added and re-study of all characteristics, which is the focus of our aim.

Definition 3.4. The space of tempered moderate elements, denoted as $\mathcal{E}_{m.\tau}^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying $\forall\alpha \in Z_+^n$, $\exists k > 0$, $\exists p > 0$, $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^{-p}|\partial^\alpha f_\varepsilon(x)| \leq c\exp\left(M\left(\frac{k}{\varepsilon}\right)\right). \tag{3.3}$$

The space of tempered null elements, denoted as $\mathcal{N}_\tau^M(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying $\forall\alpha \in Z_+^n$, $\forall k > 0$, $\exists p > 0$ $\exists c > 0$, $\exists\varepsilon_0 \in I$, $\forall\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^{-p}|\partial^\alpha f_\varepsilon(x)| \leq c\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right). \tag{3.4}$$

The following proposition gives properties of the spaces $\mathcal{E}_{m.\tau}^M(\Omega)$ and $\mathcal{N}_\tau^M(\Omega)$.

Proposition 3.5. 1) The space of tempered moderate elements $\mathcal{E}_{m.\tau}^M(\Omega)$ is an algebra stable by derivation.
2) The space $\mathcal{N}_\tau^M(\Omega)$ is an ideal of $\mathcal{E}_{m.\tau}^M(\Omega)$.

Proof.
1) Let $(f_\epsilon)_\epsilon, (g_\epsilon)_\epsilon \in \mathcal{E}_{m.\tau}^M(\Omega)$, then $\forall\beta \in Z_+^n, \exists k_1 > 0, \exists p_1 > 0, \exists c_1 > 0, \exists\epsilon_{1\beta} \in I, \forall\epsilon \leq \epsilon_{1\beta}$

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^{-p_1}|\partial^\beta f_\varepsilon(x)| \leq c_1\exp\left(M\left(\frac{k_1}{\varepsilon}\right)\right), \tag{3.5}$$

$\forall\beta \in Z_+^n, \exists k_2 > 0, \exists p_2 > 0, \exists c_2 > 0, \exists\epsilon_{2\beta} \in I, \forall\epsilon \leq \epsilon_{2\beta}$

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^{-p_2}|\partial^\beta g_\varepsilon(x)| \leq c_2\exp\left(M\left(\frac{k_2}{\varepsilon}\right)\right). \tag{3.6}$$

Let $\alpha \in Z_+^n$, then

$$|\partial^\alpha(f_\epsilon g_\epsilon)(x)| \leq \sum_{\beta=0}^{\alpha} C_\alpha^\beta|\partial^{\alpha-\beta}(f_\epsilon)(x)||\partial^\beta(g_\epsilon)(x)|.$$

From the Proposition 2.2, we have $\exists A > 0, \exists H > 0, \forall t > 0$

$$2M(t) \leq M(Ht) + \ln(AM_0).$$

For $k = H(max\{k_1, k_2, \}), \epsilon \leq \min\{\epsilon_{1\beta}, \epsilon_{2\beta}, |\beta| \leq |\alpha|\}$, we have for $t = \frac{k}{\epsilon}$

$$\begin{aligned}
\exp\left(-M\left(\frac{k}{\epsilon}\right)\right)|\partial^{\alpha}(f_\epsilon g_\epsilon)(x)| &\leq AM_0\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}\exp\left(-M\left(\frac{k_1}{\epsilon}\right)\right)|\partial^{\alpha-\beta}(f_\epsilon)(x)| \\
&\quad \times \exp\left(-M\left(\frac{k_2}{\epsilon}\right)\right)|\partial^{\beta}(g_\epsilon)(x)| \\
&\leq AM_0\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}(1+|x|)^{p_1}c_1(1+|x|)^{p_2}c_2 \\
&\leq c_3(1+|x|)^{p_1+p_2},
\end{aligned}$$

for $p = p_1 + p_2$, we have

$$(1+|x|)^{-p}\exp\left(-M\left(\frac{k}{\epsilon}\right)\right)|\partial^{\alpha}(f_\epsilon g_\epsilon)(x)| \leq c_3,$$

which means that $(f_\epsilon g_\epsilon)_\epsilon \in \mathcal{E}_{m.\tau}^{M}(\Omega)$.

We have $\forall\beta \in Z_+^n, \exists k_1 = k_1(\beta+1) > 0, \exists p_1 = p_1(\beta+1) > 0, \exists c_1 = c_1(\beta+1) > 0, \exists \epsilon_{1\beta} \in I, \forall\epsilon \leq \epsilon_{1\beta}$

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^{-p_1}|\partial^{\beta}(\partial f_\varepsilon)(x)| \leq c_1\exp\left(M\left(\frac{k_1}{\epsilon}\right)\right).$$

So, we have $(\partial f_\varepsilon)_\epsilon \in \mathcal{E}_{m.\tau}^{M}(\Omega)$.

2) If $(g_\epsilon)_\epsilon \in \mathcal{N}_{\tau}^{M}(\Omega); \forall\beta \in Z_+^n, \forall k_2 > 0, \exists p_2 > 0, \exists c_2 > 0, \exists\epsilon_{2\beta} \in I, \forall\epsilon \leq \epsilon_{2\beta}$

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^{-p_2}|\partial^{\beta}(g_\varepsilon)(x)| \leq c_2\exp\left(-M\left(\frac{k_2}{\epsilon}\right)\right).$$

Let $\alpha \in Z_+^n$ and $k > 0$, then

$$\exp\left(M\left(\frac{k}{\epsilon}\right)\right)|\partial^{\alpha}(f_\epsilon g_\epsilon)(x)| \leq \exp\left(M\left(\frac{k}{\epsilon}\right)\right)\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}|\partial^{\alpha-\beta}(f_\epsilon)(x)||\partial^{\beta}(g_\epsilon)(x)|.$$

Take $k_2 = H\max\{k_1, k\}$ and $\epsilon \leq \min\{\epsilon_{1\beta}, \epsilon_{2\beta}, |\beta| \leq |\alpha|\}$, then for $t = \frac{k_2}{\epsilon}$ in 2.2, we obtain

$$\begin{aligned}
\exp\left(M\left(\frac{k}{\epsilon}\right)\right)|\partial^{\alpha}(f_\epsilon g_\epsilon)(x)| &\leq A\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}\exp\left(-M\left(\frac{k_1}{\epsilon}\right)\right)|\partial^{\alpha-\beta}(f_\epsilon)(x)| \\
&\quad \times \exp\left(M\left(\frac{k_2}{\epsilon}\right)\right)|\partial^{\beta}(g_\epsilon)(x)| \\
&\leq A\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}c_1(1+|x|)^{p_1} \\
&\quad \times(1+|x|)^{p_2}c_2 \\
&\leq c_1c_2A(1+|x|)^{p}\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta} \\
&\leq c(1+|x|)^{p},
\end{aligned}$$

which means also that $(f_\epsilon g_\epsilon)_\epsilon \in \mathcal{N}_\tau^M(\Omega)$. □

Definition 3.6. The algebra of generalized tempered Roumieu ultradistributions of class M_p, denoted as $\mathcal{G}_\tau^M(\Omega)$, is the quotient algebra:

$$\mathcal{G}_\tau^M(\Omega) = \frac{\mathcal{E}_{m.\tau}^M(\Omega)}{\mathcal{N}_\tau^M(\Omega)}.$$

Remark 3.7. In the same way, as a generalized function, we can introduce the notion of the point value.

4. Embedding of Roumieu tempered ultradistributions

As the same way as [1] for embedding of Roumieu ultradistributions, it remains for us just to adapt the evidences.

Let $N = (N_p)_{p\in\mathbb{Z}_+}$ be a sequence satisfying the conditions $(H1)$, $(H2)$, $(H3)'$ and $N_0 = 1$, the space $S^N(R^n)$ is the space of functions $\varphi \in C^\infty(R^n)$ such that $\forall m > 0$, we have

$$\|\varphi\|_m = \sup_{\alpha,x}\{\frac{m^{|\alpha|}}{N_{|\alpha|}}|\partial^\alpha\varphi(x)| \exp N(m|x|)\} < \infty.$$

Define Σ^N as the set of functions $\phi \in S^N(R^n)$ satisfying

$$\int \phi(x)dx = 1 \ and \int x^\alpha\phi(x)dx = 0, \quad \forall\alpha \in \mathbb{Z}_+^n\diagdown\{0\}.$$

Definition 4.1. The net $\phi_\varepsilon = \varepsilon^{-n}\phi(./\varepsilon)$, $\varepsilon \in I$, where $\phi \in \Sigma^N$ is called a $N-$ mollifier net.

Let $(L_p)_{p\in\mathbb{Z}_+}$ be a sequence satisfying $(H1)$, $(H2)$, $(H3)'$, and $\forall p \in Z_+, L_p \leq M_p$ the space $O_{\mathcal{M}}^L(\Omega)$ is embedded into $\mathcal{G}_\tau^M(\Omega)$ by the standard canonical injection

$$\begin{array}{rlll} I: & O_{\mathcal{M}}^L(\Omega) & \to & \mathcal{G}_\tau^M(\Omega) \\ & f & \to & [f] = cl(f_\varepsilon) \end{array} \tag{4.1}$$

where $f_\varepsilon = f$, $\forall\varepsilon \in I$.

The following proposition gives the embedding of Roumieu tempered ultradistributions into $\mathcal{G}_\tau^M(\Omega)$. Let M and N two sequences satisfying $(H1)$, $(H2)$ and $(H3)'$ with $M_0 = N_0 = 1$ and $\phi \in \Sigma^N$.

Theorem 4.2. *The map*

$$\begin{array}{rlll} J_0: & S'^{MN}(\Omega) & \to & \mathcal{G}_\tau^M(\Omega) \\ & f & \to & [f] = cl((f * \phi_\varepsilon)_{/\Omega}) \end{array}$$

is an embedding.

For the proof, we need the following lemma, see [14].

Lemma 4.3. *For any $t, s > 0$ it holds:*

$$e^{M(t+s)} \leq 2e^{M(2t)}e^{M(2s)}.$$

Proof. Let $T \in S'^{MN}$ and $\phi \in S^N$, then by Proposition 2.8 there exists an operator of class M_pN_p, and continuous functions f wich for every $m > 0$ there is a $c > 0$, such that

$$|f(x)| \leq c \exp\left(MN(m|x|)\right); x \in \mathbb{R}^n$$

and $T = P(D)f$

Let $\alpha \in Z_+^n$, then

$$|\partial^\alpha (T \star \phi_\varepsilon)(x)| \leq \sum_{\gamma \in \mathbb{Z}_+^n} C_\gamma \frac{1}{\epsilon^{|\alpha+\gamma|}} \int |f(x+\epsilon y)||\partial^{\alpha+\gamma}\phi(y)|dy.$$

From (2.1) and condition $(H2)$, we have $\exists A > 0, \exists H > 0, \forall h > 0, \exists c > 0$, such that

$$\begin{aligned}
|\partial^\alpha (T \star \phi_\varepsilon)(x)| &\leq C \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|}}{\epsilon^{|\alpha+\gamma|} M_{|\gamma|} N_{|\gamma|}} \int |f(x+\epsilon y)||\partial^{\alpha+\gamma}\phi(y)|dy \\
&\leq C \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|} N_{|\gamma+\alpha|}}{\epsilon^{|\alpha+\gamma|} n^{|\alpha+\gamma|} M_{|\gamma|} N_{|\gamma|}} \int |f(x+\epsilon y)| \frac{n^{|\alpha+\gamma|}|\partial^{\alpha+\gamma}\phi(y)|}{N_{|\gamma+\alpha|}} dy \\
&\leq CA \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|} M_{|\alpha|} N_{|\alpha|} H^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|} n^{|\alpha+\gamma|} M_{|\gamma+\alpha|}} \int |f(x+\epsilon y)| \frac{n^{|\alpha+\gamma|}|\partial^{\alpha+\gamma}\phi(y)|}{N_{|\gamma+\alpha|}} dy
\end{aligned}$$

$$\begin{aligned}
|\partial^\alpha (T \star \phi_\varepsilon)(x)| &\leq CA \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|} M_{|\alpha|} N_{|\alpha|} H^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|} n^{|\alpha+\gamma|} M_{|\gamma+\alpha|}} \\
&\quad \times \int C \exp\left(MN(m|x+\epsilon y|)\right) \frac{n^{|\alpha+\gamma|}|\partial^{\alpha+\gamma}\phi(y)|}{N_{|\gamma+\alpha|}} dy \\
&\leq CA \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|} M_{|\alpha|} N_{|\alpha|} H^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|} n^{|\alpha+\gamma|} M_{|\gamma+\alpha|}} \\
&\quad \times \int \exp\left(MN(2m|x|)\right) \exp\left(MN(2m|\epsilon y|)\right) \frac{n^{|\alpha+\gamma|}|\partial^{\alpha+\gamma}\phi(y)|}{N_{|\gamma+\alpha|}} dy \\
&\leq CA \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|} M_{|\alpha|} N_{|\alpha|} H^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|} n^{|\alpha+\gamma|} M_{|\gamma+\alpha|}} \\
&\quad \times \int \exp\left(MN(2m|x|)\right) \exp\left(N(2m|\epsilon y|)\right) \frac{n^{|\alpha+\gamma|}|\partial^{\alpha+\gamma}\phi(y)|}{N_{|\gamma+\alpha|}} dy \\
&\leq CA \sum_{\gamma \in \mathbb{Z}_+^n} \frac{h^{|\gamma|} M_{|\alpha|} N_{|\alpha|} H^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|} n^{|\alpha+\gamma|} M_{|\gamma+\alpha|}} \exp\left(MN(2m|x|)\right) \\
&\quad \times \int \exp\left(N(2m|y|)\right) \frac{|n^{|\alpha+\gamma|}\partial^{\alpha+\gamma}\phi(y)|}{N_{|\gamma+\alpha|}} dy,
\end{aligned}$$

then

$$\begin{aligned}\frac{(2h)^{|\alpha|}}{M_{|\alpha|}N_{|\alpha|}}|\partial^{\alpha}(T\star\phi_{\varepsilon})(x)| &\leq CA\sum_{\gamma\in\mathbb{Z}_{+}^{n}}2^{-|\gamma|}\frac{(\frac{2Hh}{n})^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|}M_{|\gamma+\alpha|}}\exp\left(MN(2m|x|)\right) \\ &\quad\times\int\exp\left(N(2m|y|)\right)\frac{\exp\left(N(n|y|)\right)n^{|\alpha+\gamma|}|\partial^{\alpha+\gamma}\phi(y)|}{\exp\left(N(n|y|)\right)N_{|\gamma+\alpha|}}dy.\end{aligned}$$

Take m such that

$$\int\frac{\exp\left(N(2m|y|)\right)}{\exp\left(N(n|y|)\right)}dy<\infty.$$

$$\begin{aligned}\frac{(2h)^{|\alpha|}}{M_{|\alpha|}N_{|\alpha|}}|\partial^{\alpha}(T\star\phi_{\varepsilon})(x)| &\leq CA\sum_{\gamma\in\mathbb{Z}_{+}^{n}}2^{-|\gamma|}\frac{(\frac{2Hh}{n})^{|\alpha+\gamma|}}{\epsilon^{|\alpha+\gamma|}M_{|\gamma+\alpha|}}\exp\left(MN(2m|x|)\right)\|\phi\|_{n} \\ &\leq C\exp\left(M(\frac{k}{\epsilon})\right)\exp\left(MN(2m|x|)\right) \\ &\leq C\exp\left(M(\frac{k}{\epsilon})\right)\exp\left(MN(m'|x|)\right),\end{aligned}$$

where $m'=2m$, we have

$$\exp\left(MN(m'|x|)\right)\leq\sup_{p\in\mathbb{Z}_{+}}\frac{m'^{p}(1+|x|)^{p}}{M_{p}N_{p}},$$

then $\exists p_0>0$, such that

$$\frac{(2h)^{|\alpha|}}{M_{|\alpha|}N_{|\alpha|}}|\partial^{\alpha}(T\star\phi_{\varepsilon})(x)|\leq C\exp\left(M\left(\frac{k}{\epsilon}\right)\right)\frac{m'^{p_0}(1+|x|)^{p_0}}{M_{p_0}N_{p_0}},$$

which means

$$\sup_{x\in\mathbb{R}^{n}}(1+|x|)^{-p_0}|\partial^{\alpha}(T\star\phi_{\varepsilon})(x)|\leqslant C(\alpha)\exp\left(M\left(\frac{k}{\epsilon}\right)\right),$$

where $k=(\frac{2hH}{n})$. □

Proposition 4.4. The following diagram

$$\begin{array}{ccc} O_{\mathcal{M}}^{\frac{M}{N}p!}(\Omega) & \to & \mathcal{G}_{\tau}^{M}(\Omega) \\ & \searrow & \uparrow \\ & & S'^{MN}(\Omega)\end{array}$$

is commutative.

Proof. Let $f\in O_{\mathcal{M}}^{MN^{-1}p!}$ we prove that $(f-(f*\phi_{\epsilon}))_{\epsilon}\in\mathcal{N}_{\tau}^{M}(\Omega)$.

By Taylors formula and the fact that $\phi\in\Sigma^{N}$, we obtain

$$|\partial^{\alpha}(f*\phi_{\epsilon}-f)(x)|\leq\epsilon^{L}\sum_{|\beta|=L}\int\frac{|\partial^{\alpha+\beta}f(\xi)|}{B!}|y|^{|\beta|}|\phi(y)|dy,\ \text{where } x\leq\xi\leq x+\epsilon y.$$

$$\begin{aligned}
|\partial^\alpha(f*\phi_\epsilon - f)(x)| &\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}(\alpha+\beta)!}{N_{|\alpha+\beta|}\beta! r^{|\alpha+\beta|}} \\
&\quad \times \int \exp(MN^{-1}p!(m|\xi|))|y|^{|\beta|}|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \\
&\quad \times \int \exp(MN^{-1}p!(m|\xi|))|y|^{|\beta|}|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \\
&\quad \times \int \exp(MN^{-1}p!(m|x+\epsilon y|))|y|^{|\beta|}|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x|)) \\
&\quad \times \int \exp(MN^{-1}p!(2m|y|))|y|^{|\beta|}|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x|)) \\
&\quad \times \int \exp(N(2m|y|))|y|^{|\beta|}|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x|)) \\
&\quad \times \int \exp(N(2m|y|))N_{|\beta|}\exp(N(|y|))|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}N_{|\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x|)) \\
&\quad \times \int \exp(N(2m|y|))\exp(N(|y|))|\phi(y)|dy.
\end{aligned}$$

$\forall n \geq 1$, we have

$$\begin{aligned}
|\partial^\alpha(f*\phi_\epsilon - f)(x)| &\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}N_{|\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x| \\
&\quad \times \int \exp(N(2m|y|))\exp(N(n|y|))|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}N_{|\beta|}\alpha!}{N_{|\alpha+\beta|} r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x|)) \\
&\quad \times \int \frac{\exp(N(2m|y|))}{\exp(N(n|y|))}\exp(2N(n|y|))|\phi(y)|dy,
\end{aligned}$$

by the Proposition 2.2, we have $\exists A>0, \exists H>0$, such that

$$\begin{aligned}
|\partial^\alpha(f*\phi_\epsilon - f)(x)| &\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}N_{|\beta|}\alpha!}{N_{|\alpha+\beta|}r^{|\alpha+\beta|}} \exp(MN^{-1}p!(2m|x| \\
&\quad \times \int \frac{\exp(N(2m|y|))}{\exp(N(n|y|))} \exp(N(nH|y|))A|\phi(y)|dy \\
&\leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}N_{|\beta|}\alpha!}{N_{|\alpha+\beta|}r^{|\alpha+\beta|}} A\exp(MN^{-1}p!(2m|x|))\|\phi\|_{(nH)} \\
&\quad \times \int \frac{\exp(N(2m|y|))}{\exp(N(n|y|))}dy,
\end{aligned}$$

take m such that

$$\int \frac{\exp(N(2m|y|))}{\exp(N(n|y|))}dy < \infty,$$

then

$$|\partial^\alpha(f*\phi_\epsilon - f)(x)| \leq \epsilon^L \sum_{|\beta|=L} \frac{M_{|\alpha+\beta|}N_{|\beta|}\alpha!}{N_{|\alpha+\beta|}r^{|\alpha+\beta|}} A\exp(MN^{-1}p!(2m|x|))\|\phi\|_{(nH)}.$$

From condition $(H2)$, and in fact that $\forall p \geq 0, N_p^2 \leq M_p p!$ we have $\exists A^{'} > 0, \exists H > 0$, such that

$$\begin{aligned}
|\partial^\alpha(f*\phi_\epsilon - f)(x)| &\leq \epsilon^L A^{'} \sum_{|\beta|=L} \frac{H^{|\alpha+\beta|}M_{|\alpha|}M_{|\beta|}\alpha!}{N_{|\alpha|}r^{|\alpha+\beta|}} A\exp(N(2m|x|))\|\phi\|_{(nH)} \\
&\leq \exp(N(2m|x|))\epsilon^L AA^{'}\|\phi\|_{(nH)} \frac{H^{|\alpha|}M_{|\alpha|}\alpha!}{N_{|\alpha|}r^{|\alpha|}} \sum_{|\beta|=L} \frac{H^{|\beta|}M_{|\beta|}}{r^{|\beta|}},
\end{aligned}$$

for $m' = 2m$, we have

$$\exp\left(N(m'|x|)\right) \leq \sup_{p\in\mathbb{Z}_+} \frac{m'^p(1+|x|)^p}{N_p} = \frac{m'^{p_0}(1+|x|)^{p_0}}{N_{p_0}}.$$

Let $k>0$ and $T>0$, then

$$\begin{aligned}
|\partial^\alpha(f*\phi_\epsilon - f)(x)| &\leq \exp(N(m'|x|)) \frac{AA^{'}H^{|\alpha|}\|\phi\|_{(nH)}M_{|\alpha|}\alpha!}{r^{|\alpha|}N_{|\alpha|}} \\
&\quad \times \epsilon^L M_L(kT)^{-L} \sum_{|\beta|=L} \left(\frac{HkT}{r}\right)^{|\beta|} \\
&\leq \frac{m'^{p_0}(1+|x|)^{p_0}}{N_{p_0}} \frac{AA^{'}H^{|\alpha|}\|\phi\|_{(nH)}M_{|\alpha|}\alpha!}{r^{|\alpha|}N_{|\alpha|}} \\
&\quad \times \epsilon^L M_L(kT)^{-L} \sum_{|\beta|=L} \left(\frac{HkT}{r}\right)^{|\beta|} \\
&\leq (1+|x|)^{p_0}c(\alpha)\epsilon^L M_L(kT)^{-L} \sum_{|\beta|=L} \left(\frac{HkT}{r}\right)^{|\beta|},
\end{aligned}$$

where

$$c(\alpha) = \frac{m'^{p_0} A A' H^{|\alpha|} \|\phi\|_{(nH)} M_{|\alpha|} \alpha!}{r^{|\alpha|} N_{p_0} N_{|\alpha|}}.$$

Taking

$$\frac{HkT}{r} \leq \frac{1}{2a},$$

with $a > 1$, we obtain

$$|\partial^\alpha (f * \phi_\epsilon - f)(x)| \leq (1 + |x|)^{p_0} c(\alpha) \epsilon^L M_L (kT)^{-L} a^{-L} \sum_{|\beta|=L} \left(\frac{1}{2}\right)^{|\beta|}.$$

Let $\epsilon_0 \in I$ such that $\epsilon_0 M_1 < 1$ and take $T \geq \frac{M_{p-1} M_{p+1}}{M_p^2}; \forall p \geq 1$, then, there exists $L = L(\epsilon) \in Z_+$, such that

$$1 \leq \frac{\epsilon}{k} (M_L)^{-L} \leq T,$$

which gives

$$a^{-L} \leq \exp\left(-M\left(\frac{k}{\epsilon}\right)\right) \text{ and } \epsilon^L M_L (kT)^{-L} < 1,$$

if we take $a \geq 2$, finally, we have

$$(1 + |x|)^{-p_0} |\partial^\alpha (f * \phi_\epsilon - f)(x)| \leq c \exp\left(-M\left(\frac{k}{\epsilon}\right)\right).$$

□

5. Regular generalized Roumieu tempered ultradistributions

Definition 5.1. The space of regular tempered moderate elements, denoted as $\mathcal{E}_{m.\tau}^{M,\infty}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying, $\exists k > 0$, $\exists c > 0$, $\exists p > 0$, $\exists \varepsilon_0 \in I$, $\forall \alpha \in Z_+^n, \forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^{-p} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(M\left(\frac{k}{\varepsilon}\right)\right). \tag{5.1}$$

The space of regular tempered null elements is defined as :
$\mathcal{N}_\tau^{M,\infty}(\Omega) := \mathcal{N}_\tau^M(\Omega) \cap \mathcal{E}_{m.\tau}^{M,\infty}(\Omega)$

Definition 5.2. The algebra of regular generalized tempered Roumieu ultradistributions of class M_p, denoted as $\mathcal{G}_\tau^{M,\infty}(\Omega)$, is the quotient algebra:

$$\mathcal{G}_\tau^{M,\infty}(\Omega) = \frac{\mathcal{E}_{m.\tau}^{M,\infty}(\Omega)}{\mathcal{N}_\tau^{M,\infty}(\Omega)}.$$

The basic properties of $\mathcal{G}_\tau^{M,\infty}(\Omega)$ are given by the following result.

Proposition 5.3. The space $\mathcal{G}_\tau^{M,\infty}(\Omega)$ is a sheaf subalgebra of $\mathcal{G}_\tau^M(\Omega)$.

Definition 5.4. We define the $\mathcal{G}_\tau^{M,\infty}(\Omega)$-singular support of a generalized tempered ultradistribution $f \in \mathcal{G}_\tau^M(\Omega)$, as the complement of the largest open set Ω' such that $f \in \mathcal{G}_\tau^{M,\infty}(\Omega')$.

Theorem 5.5. *We have* $\mathcal{G}_{\tau}^{M,\infty}(\Omega) \cap S'^{MN}(\Omega) = O_{\mathcal{M}}^{\frac{M}{N}p!}(\Omega)$.

Proof. As in [12]. Take $T \in S'^{MN}$ such that $J_0(T)$ is in $\mathcal{G}_{\tau}^{M,\infty}$, then $(T \star \phi_{\varepsilon})_{\epsilon}$ is in $\mathcal{E}_{m.\tau}^{M,\infty}$.

Recall that

$$T \in O_{\mathcal{M}}^{\frac{M}{N}p!} \Longleftrightarrow \widehat{T} \in O_c'^{\frac{M}{N}p!} \Longleftrightarrow \forall \Psi \in S^{\frac{M}{N}p!}, \widehat{T} \star \Psi \in S^{\frac{M}{N}p!}$$

So consider $\Psi \in S^{\frac{M}{N}p!}$, we are going to show that $\frac{m^p(1+|x|)^p}{\frac{M_p}{N_p}p!}|\widehat{T} \star \Psi(.)|$ is bounded for all $p \in \mathbb{Z}_+^n$.

We have

$$\widehat{T} \star \Psi = (\widehat{T}(1 - \widehat{\phi_{\varepsilon}})) \star \Psi + (\widehat{T\phi_{\varepsilon}}) \star \Psi \tag{5.2}$$

Recalling that $\phi_{\varepsilon} = \varepsilon^{-n}\phi(./\varepsilon)$, we easily get that $\widehat{\phi_{\varepsilon}}(.) = \widehat{\phi}(\varepsilon.)$. Note also that $\widehat{\phi}(0) = \int \phi(x)dx = 1$. Thus

$$1 - \widehat{\phi_{\varepsilon}}(x) = -\varepsilon \int_0^1 \nabla\phi(\varepsilon xt)xdt = \varepsilon B(\varepsilon, x).$$

As $\widehat{\phi}$ is rapidly decreasing, there exists $c > 0$ such that $|\nabla\phi(\varepsilon xt)x| \leq c(1 + |x|^2)^{\frac{1}{2}}$. The same holds for the derivatives with respect to x and, thus, for the function B and its derivatives. From this, for example by using the structure of elements of S'^{MN}, it can be shown that $\widehat{T}(1 - \widehat{\phi_{\varepsilon}})) \star \Psi$ satisfies

$$\forall x \in \mathbb{R}^n |((\widehat{T}(1 - \widehat{\phi_{\varepsilon}})) \star \Psi)(x)| \leq c_0\varepsilon(1 + |x|^2)^{\frac{n}{2}} \tag{5.3}$$

for some $c_0 > 0$ and n not depending on ε.

For $p = \frac{p}{2} - \frac{n}{2}$, we have

$$\begin{aligned}
\frac{m^p(1+|x|)^p}{\frac{M_p}{N_p}p!}|((\widehat{T}(1 - \widehat{\phi_{\varepsilon}})) \star \Psi)(x)| &\leq \frac{m^p(1+|x|^2)^p}{\frac{M_p}{N_p}p!}|((\widehat{T}(1 - \widehat{\phi_{\varepsilon}})) \star \Psi)(x)| \\
&\leq c_0\varepsilon \frac{m^p(1+|x|^2)^{\frac{n}{2}+p}}{\frac{M_p}{N_p}p!} \\
&\leq c_0\varepsilon \frac{m^p(1+|x|^2)^{\frac{p}{2}}}{\frac{M_p}{N_p}p!} \\
&\leq c_1\varepsilon \frac{m^p(1+|x|)^p}{\frac{M_p}{N_p}p!},
\end{aligned}$$

then

$$\frac{m^p(1+|x|)^p}{\frac{M_p}{N_p}p!}|((\widehat{T}(1 - \widehat{\phi_{\varepsilon}})) \star \Psi)(x)| \leq c_1\varepsilon \exp\left(\frac{M}{N}p!\,(m'|x|)\right), \tag{5.4}$$

we have

$$\begin{aligned}
(ix)^{\beta}[(\widehat{T\phi_{\varepsilon}}) \star \Psi](x) &= (ix)^{\beta}\mathcal{F}[(T \star \phi_{\varepsilon})\mathcal{F}^{-1}(\Psi)](x) \\
&= \mathcal{F}[\partial^{\beta}(T \star \phi_{\varepsilon})\mathcal{F}^{-1}(\Psi)](x).
\end{aligned}$$

Applying the definition of $\mathcal{E}^{M,\infty}_{m.\tau}$ for $(T\star\phi_\varepsilon)_\epsilon$, we have

$$\begin{aligned}\forall x\in\mathbb{R}^n|\partial^\beta(T\star\phi_\varepsilon)(x)\mathcal{F}^{-1}\Psi(x)| &\leq c\exp\left(M\left(\frac{k}{\epsilon}\right)\right)(1+|x|)^{p_1}|\mathcal{F}^{-1}\Psi(x)|\\ &\leq c'\exp\left(M\left(\frac{k}{\epsilon}\right)\right)\\ &\leq c_2\exp\left(\frac{M}{N}p!\left(\frac{k}{\epsilon}\right)\right).\end{aligned}$$

So

$$\begin{aligned}|(ix)^\beta[(\widehat{T\phi_\varepsilon})\star\Psi](x)| &= |(ix)^\beta[\widehat{T\star\phi_\varepsilon}\star\Psi](x)|\\ &= |\mathcal{F}[\partial^\beta(T\star\phi_\varepsilon)\mathcal{F}^{-1}\Psi](x)|\\ &\leq c_2\exp\left(\frac{M}{N}p!\left(\frac{k}{\epsilon}\right)\right),\end{aligned}$$

then

$$\frac{m^p(1+|x|)^p}{\frac{M_p}{N_p}p!}|(\widehat{T\phi_\varepsilon})\star\Psi(x)|\leq c_3\exp\left(\frac{M}{N}p!\left(\frac{k}{\epsilon}\right)\right),$$

using 5.4 for $m'|x|\leq\frac{k}{\epsilon}$ and for $\epsilon=1$, we have

$$\frac{m^p(1+|x|)^p}{\frac{M_p}{N_p}p!}|(\widehat{T}\star\Psi(x)|\leq c_4.$$

i.e. $(\widehat{T}\star\Psi)\in S^{\frac{M}{N}p!}$, then T is in $O^{\frac{M}{N}p!}_{\mathcal{M}}$. □

Remark 5.6. In further coming paper, we will study the Fourier analysis with an algebra of generalized tempered Roumieu ultradistributions and give some applications.

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