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COEFFICIENT BOUNDS FOR NEW FAMILIES OF BAZILEVIČ AND ϕ-PSEUDO-STARLIKE BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SAKAGUCHI TYPE FUNCTIONS

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Abstract. In this research paper, we introduce and study two new families $\mathcal{Z}_\Sigma(\theta,\eta,\phi,t;\alpha)$ and $\mathcal{Z}^*_\Sigma(\theta,\eta,\phi,t;\beta)$ of normalized holomorphic and bi-univalent functions which involve the Sakaguchi type Bazilevič functions and Sakaguchi type ϕ-pseudo-starlike functions. We establish the bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these new families. Further, certain several special cases and consequences for our results are also pointed.

1. Introduction

Let $\mathcal{A}$ be the family of holomorphic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and have the following normalized form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be Bazilevič function in $\mathbb{U}$ if (see [12])

$$\Re\left(\frac{z^{1-\eta} f'(z)}{\big(f(z)\big)^{1-\eta}}\right) > 0 \qquad (z \in \mathbb{U};\ \eta \geqq 0).$$

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On the other hand, a function $f \in \mathcal{A}$ is said to be a ϕ-pseudo-starlike function in $\mathbb{U}$ if (see [2])

$$\Re\left(\frac{z\big(f'(z)\big)^{\phi}}{f(z)}\right) > 0 \qquad (z \in \mathbb{U};\ \phi \geqq 1).$$

Frasin [7] introduced and studied the family $S(\mu, s, t)$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \mu,$$

for some $0 \leqq \mu < 1;\ s, t \in \mathbb{C}$ with $s \neq t;\ |t| \leqq 1$ and for all $z \in \mathbb{U}$.

According to the Koebe one-quarter theorem [6], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ stand for the family of normalized bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief historical account and for several interesting examples of functions in the family Σ, see the pioneering work on this subject by Srivastava *et al.* [20], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [20], we choose to recall here the following examples of functions in the family Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the family Σ is not empty. However, the Koebe function is not a member of Σ.

In a considerably large number of sequels to the aforementioned work of Srivastava *et al.* [20], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [1, 4, 5, 9, 13, 14, 15, 16, 17, 18, 21, 22, 24, 25, 26]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1.1) were obtained in many of these recent papers. The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} := \{1, 2, 3, \cdots\})$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function family Σ (see, for example, [16, 21, 22]; see also [19]).

We now recall the following lemma that will be used to prove our main results.

Lemma 1.1. [6] *If* $h \in \mathcal{P}$, *then*

$$|c_k| \leqq 2 \qquad (\forall\ k \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions h, holomorphic in $\mathbb{U}$, for which

$$\Re\big(h(z)\big) > 0 \qquad (z \in \mathbb{U})$$

with

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

2. Coefficient Estimates for the Family $\mathcal{Z}_\Sigma(\theta, \eta, \phi, t; \alpha)$

In this section, we first define the family $\mathcal{Z}_\Sigma(\theta, \eta, \phi, t; \alpha)$.

Definition 2.1. A function $f \in \Sigma$, given by (1.1), is said to be in the family $\mathcal{Z}_\Sigma(\theta, \eta, \phi, t; \alpha)$ if it satisfies the following conditions:

$$\left|\arg\left((1-\theta)\,\frac{\left((1-t)z\right)^{1-\eta} f'(z)}{\left(f(z)-f(tz)\right)^{1-\eta}} + \theta\,\frac{(1-t)z\left(f'(z)\right)^{\phi}}{f(z)-f(tz)}\right)\right| < \frac{\alpha\pi}{2} \tag{2.1}$$

and

$$\left|\arg\left((1-\theta)\,\frac{\left((1-t)w\right)^{1-\eta} g'(w)}{\left(g(w)-g(tw)\right)^{1-\eta}} + \theta\,\frac{(1-t)w\left(g'(w)\right)^{\phi}}{g(w)-g(tw)}\right)\right| < \frac{\alpha\pi}{2}, \tag{2.2}$$

where

$$z, w \in \mathbb{U},\;\; 0 < \alpha \leqq 1,\;\; 0 \leqq \theta \leqq 1,\;\; \eta \geqq 0,\;\; \phi \geqq 1,\;\; t \in \mathbb{C} \quad \text{and} \quad |t| \leqq 1,$$

and $g = f^{-1}$ is given by (1.2).

Theorem 2.2. *Let $f \in \mathcal{Z}_\Sigma(\theta, \eta, \phi, t; \alpha)$ ($0 < \alpha \leqq 1;\ 0 \leqq \theta \leqq 1;\ \eta \geqq 0;\ \phi \geqq 1;\ t \in \mathbb{C};\ |t| \leqq 1$) be given by* (1.1). *Then*

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left|\alpha\left[(1-\theta)\Phi(\eta,t) + 2\theta\left(\phi\left(2\phi - 2t - 1\right) + t\right)\right] + (1-\alpha)\Psi(\eta,\theta,\phi,t)\right|}}$$

and

$$|a_3| \leqq \frac{4\alpha^2}{\left[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)\right]^2} + \frac{2\alpha}{\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right) + \theta\left(3\phi - t^2 - t - 1\right)\right]},$$

where

$$\Phi(\eta, t) = (1-\eta)\left((2-\eta)\,(t+1)^2 - 6(t+1) - 2t^2\right) + 6, \tag{2.3}$$

and

$$\Psi(\eta,\theta,\phi,t) = \left[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)\right]^2. \tag{2.4}$$

Proof. In light of the conditions (2.1) and (2.2), we have

$$(1-\theta)\,\frac{\left((1-t)z\right)^{1-\eta} f'(z)}{\left(f(z)-f(tz)\right)^{1-\eta}} + \theta\,\frac{(1-t)z\left(f'(z)\right)^{\phi}}{f(z)-f(tz)} = [p(z)]^{\alpha} \tag{2.5}$$

and

$$(1-\theta)\,\frac{\left((1-t)w\right)^{1-\eta} g'(w)}{\left(g(w)-g(tw)\right)^{1-\eta}} + \theta\,\frac{(1-t)w\left(g'(w)\right)^{\phi}}{g(w)-g(tw)} = [q(w)]^{\alpha}, \tag{2.6}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.8}$$

By comparing the corresponding coefficients of (2.5) and (2.6), we find that

$$[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)]\, a_2 = \alpha p_1, \tag{2.9}$$

$$\begin{aligned}&\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right) + \theta\left(3\phi - t^2 - t - 1\right)\right] a_3 \\ &+ \left[(1-\theta)(1-\eta)(t+1)\left(\frac{1}{2}(2-\eta)(t+1) - 2\right) + \theta\left((t+1)^2 - 2\phi(t-\phi+2)\right)\right] a_2^2 \\ &= \alpha p_2 + \frac{\alpha(\alpha-1)}{2}\, p_1^2,\end{aligned} \tag{2.10}$$

$$-\left[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)\right] a_2 = \alpha q_1 \tag{2.11}$$

and

$$\begin{aligned}&\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right) + \theta\left(3\phi - t^2 - t - 1\right)\right]\left(2a_2^2 - a_3\right) \\ &+ \left[(1-\theta)(1-\eta)(t+1)\left(\frac{1}{2}(2-\eta)(t+1) - 2\right) + \theta\left((t+1)^2 - 2\phi(t-\phi+2)\right)\right] a_2^2 \\ &= \alpha q_2 + \frac{\alpha(\alpha-1)}{2}\, q_1^2.\end{aligned} \tag{2.12}$$

Thus, by using (2.9) and (2.11), we conclude that

$$p_1 = -q_1 \tag{2.13}$$

and

$$2\left[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)\right]^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.14}$$

If we add (2.10) to (2.12), we obtain

$$[(1-\theta)\Phi(\eta,t) + 2\theta\left(\phi\left(2\phi - 2t - 1\right) + t\right)] a_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 + q_1^2\right), \tag{2.15}$$

where $\Phi(\eta,t)$ is given by (2.3). Substituting the value of $p_1^2 + q_1^2$ from (2.14) into the right-hand side of (2.15), and after some computations, we deduce that

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{\alpha\left[(1-\theta)\Phi(\eta,t) + 2\theta\left(\phi\left(2\phi - 2t - 1\right) + t\right)\right] + (1-\alpha)\Psi(\eta,\theta,\phi,t)}, \tag{2.16}$$

where $\Psi(\eta,\theta,\phi,t)$ is given by (2.4). By taking the moduli of both sides of (2.16) and applying the Lemma of Section 1 for the coefficients p_2 and q_2, we have

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left|\alpha\left[(1-\theta)\Phi(\eta,t) + 2\theta\left(\phi\left(2\phi - 2t - 1\right) + t\right)\right] + (1-\alpha)\Psi(\eta,\theta,\phi,t)\right|}}.$$

Next, in order to determinate the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$\begin{aligned}&2\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right) + \theta\left(3\phi - t^2 - t - 1\right)\right]\left(a_3 - a_2^2\right) \\ &\qquad = \alpha\left(p_2 - q_2\right) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 - q_1^2\right).\end{aligned} \tag{2.17}$$

Now, upon substituting the value of a_2^2 from (2.14) into (2.17) and using (2.13), we deduce that

$$a_3 = \frac{\alpha^2\left(p_1^2+q_1^2\right)}{2\left[(1-\theta)(2-(1-\eta)(t+1))+\theta(2\phi-t-1)\right]^2} + \frac{\alpha\left(p_2-q_2\right)}{2\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)\right]}. \tag{2.18}$$

Finally, by taking the moduli on both sides of (2.18) and applying the Lemma of Section 1 once again for the coefficients p_1, p_2, q_1 and q_2, it follows that

$$|a_3| \leqq \frac{4\alpha^2}{\left[(1-\theta)(2-(1-\eta)(t+1))+\theta(2\phi-t-1)\right]^2} + \frac{2\alpha}{\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)\right]}.$$

This completes the proof of Theorem 2.2. $\square$

3. Coefficient Estimates for the Family $\mathcal{Z}_\Sigma^*(\theta,\eta,\phi,t;\beta)$

In this section, we first define the family $\mathcal{Z}_\Sigma^*(\theta,\eta,\phi,t;\beta)$.

Definition 3.1. A function $f \in \Sigma$, given by (1.1), is said to be in the family $\mathcal{Z}_\Sigma^*(\theta,\eta,\phi,t;\beta)$ if it satisfies the following conditions:

$$\Re\left((1-\theta)\frac{((1-t)z)^{1-\eta}f'(z)}{(f(z)-f(tz))^{1-\eta}}+\theta\,\frac{(1-t)z\,(f'(z))^{\phi}}{f(z)-f(tz)}\right)>\beta \tag{3.1}$$

and

$$\Re\left((1-\theta)\frac{((1-t)w)^{1-\eta}g'(w)}{(g(w)-g(tw))^{1-\eta}}+\theta\,\frac{(1-t)w\,(g'(w))^{\phi}}{g(w)-g(tw)}\right)>\beta, \tag{3.2}$$

where

$$z,w\in\mathbb{U},\ \ 0\leqq\beta<1,\ \ 0\leqq\theta\leqq1,\ \ \eta\geqq0,\ \ \phi\geqq1,\ \ t\in\mathbb{C}\quad\text{and}\quad|t|\leqq1,$$

and $g = f^{-1}$ is given by (1.2).

Theorem 3.2. *Let* $f \in \mathcal{Z}_\Sigma^*(\theta,\eta,\phi,t;\beta)$ $(0 \leqq \beta < 1;\ 0 \leqq \theta \leqq 1;\ \eta \geqq 0;\ \phi \geqq 1;\ t \in \mathbb{C};\ |t| \leqq 1)$ *be given by* (1.1). *Then*

$$|a_2| \leqq 2\sqrt{\frac{1-\beta}{(1-\theta)\left(6+(1-\eta)\left((2-\eta)(t+1)^2-6(t+1)-2t^2\right)\right)+2\theta\Upsilon(\phi,t)}}$$

and

$$|a_3| \leqq \frac{4(1-\beta)^2}{\left[(1-\theta)(2-(1-\eta)(t+1))+\theta(2\phi-t-1)\right]^2} + \frac{2(1-\beta)}{(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)},$$

where

$$\Upsilon(\phi,t)=\phi\,(2\phi-2t-1)+t. \tag{3.3}$$

Proof. In view of the conditions (3.1) and (3.2), there exist the functions $p, q \in \mathcal{P}$ such that

$$(1-\theta)\frac{((1-t)z)^{1-\eta} f'(z)}{(f(z)-f(tz))^{1-\eta}} + \theta\,\frac{(1-t)z\,(f'(z))^{\phi}}{f(z)-f(tz)} = \beta + (1-\beta)p(z) \tag{3.4}$$

and

$$(1-\theta)\frac{((1-t)w)^{1-\eta} g'(w)}{(g(w)-g(tw))^{1-\eta}} + \theta\,\frac{(1-t)w\,(g'(w))^{\phi}}{g(w)-g(tw)} = \beta + (1-\beta)q(w), \tag{3.5}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the series expansions given by (2.7) and (2.8), respectively. Thus, by comparing the corresponding coefficients in (3.4) and (3.5), we get

$$[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)]\,a_2 = (1-\beta)p_1, \tag{3.6}$$

$$\begin{aligned}
&\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right) + \theta\left(3\phi - t^2 - t - 1\right)\right] a_3 \\
&+ \left[(1-\theta)(1-\eta)(t+1)\left(\frac{1}{2}(2-\eta)(t+1)-2\right) + \theta\left((t+1)^2 - 2\phi(t-\phi+2)\right)\right] a_2^2 \\
&= (1-\beta)p_2,
\end{aligned} \tag{3.7}$$

$$-\left[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)\right] a_2 = (1-\beta)q_1 \tag{3.8}$$

and

$$\begin{aligned}
&\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right) + \theta\left(3\phi - t^2 - t - 1\right)\right] \left(2a_2^2 - a_3\right) \\
&+ \left[(1-\theta)(1-\eta)(t+1)\left(\frac{1}{2}(2-\eta)(t+1)-2\right) + \theta\left((t+1)^2 - 2\phi(t-\phi+2)\right)\right] a_2^2 \\
&= (1-\beta)q_2.
\end{aligned} \tag{3.9}$$

We now find from (3.6) and (3.8) that

$$p_1 = -q_1 \tag{3.10}$$

and

$$2\left[(1-\theta)(2-(1-\eta)(t+1)) + \theta(2\phi - t - 1)\right]^2 a_2^2 = (1-\beta)^2 (p_1^2 + q_1^2). \tag{3.11}$$

By adding (3.7) and (3.9), we obtain

$$\begin{aligned}
&\left[(1-\theta)\left(6+(1-\eta)\left((2-\eta)(t+1)^2 - 6(t+1) - 2t^2\right)\right) + 2\theta\left(\phi\left(2\phi - 2t - 1\right) + t\right)\right] a_2^2 \\
&\qquad = (1-\beta)(p_2+q_2).
\end{aligned} \tag{3.12}$$

Consequently, we have

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{(1-\theta)\left(6+(1-\eta)\left((2-\eta)(t+1)^2 - 6(t+1) - 2t^2\right)\right) + 2\theta\Upsilon(\phi,t)},$$

where $\Upsilon(\phi, t)$ is given by (3.3).

Next, by applying the Lemma of Section 1 for the coefficients p_2 and q_2, we deduce that

$$|a_2| \leqq 2\sqrt{\frac{1-\beta}{(1-\theta)\left(6+(1-\eta)\left((2-\eta)(t+1)^2 - 6(t+1) - 2t^2\right)\right) + 2\theta\Upsilon(\phi,t)}}.$$

In order to determinate the bound on $|a_3|$, by subtracting (3.9) from (3.7), we get

$$2\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)\right]\left(a_3-a_2^2\right)=(1-\beta)\left(p_2-q_2\right)$$

or, equivalently,

$$a_3=a_2^2+\frac{(1-\beta)\left(p_2-q_2\right)}{2\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)\right]}. \tag{3.13}$$

Substituting the value of a_2^2 from (3.11) into (3.13), it follows that

$$a_3=\frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{2\left[(1-\theta)(2-(1-\eta)(t+1))+\theta(2\phi-t-1)\right]^2} + \frac{(1-\beta)\left(p_2-q_2\right)}{2\left[(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)\right]}.$$

Finally, by applying the Lemma of Section 1 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leqq \frac{4(1-\beta)^2}{\left[(1-\theta)(2-(1-\eta)(t+1))+\theta(2\phi-t-1)\right]^2} + \frac{2(1-\beta)}{(1-\theta)\left(3-(1-\eta)\left(t^2+t+1\right)\right)+\theta\left(3\phi-t^2-t-1\right)}.$$

We have thus completed the proof of Theorem 3.2. □

4. Special Cases and Consequences

This section is devoted to the presentation of some special cases and consequences of our main results (Theorem 2.2 and Theorem 3.2).

Remark 4.1. Each of the following families:

$$\mathcal{Z}_\Sigma(\theta,\eta,\phi,t;\alpha) \qquad \text{and} \qquad \mathcal{Z}_\Sigma^*(\theta,\eta,\phi,t;\beta)$$

is a generalization of several known families considered in earlier investigations which are being recalled below.

1. For $t=0$, we have

$$\mathcal{Z}_\Sigma(\theta,\eta,\phi,0;\alpha) =: \mathcal{T}_\Sigma(\theta,\eta,\phi;\alpha)$$

and

$$\mathcal{Z}_\Sigma^*(\theta,\eta,\phi,0;\beta) =: \mathcal{T}_\Sigma^*(\theta,\eta,\phi;\beta),$$

where the families $\mathcal{T}_\Sigma(\theta,\eta,\phi;\alpha)$ and $\mathcal{T}_\Sigma^*(\theta,\eta,\phi;\beta)$ were defined recently by Srivastava *et al.* [23].

2. For $\theta=t=0$, we have

$$\mathcal{Z}_\Sigma(0,\eta,\phi,0;\alpha) =: P_\Sigma(\alpha,\eta)$$

and

$$\mathcal{Z}_\Sigma^*(0,\eta,\phi,0;\beta) =: P_\Sigma(\beta,\eta),$$

where the families $P_\Sigma(\alpha,\eta)$ and $P_\Sigma(\beta,\eta)$ were studied recently by Prema and Keerthi [11].

3. For $\theta = 1$ and $t = 0$, we have

$$\mathcal{Z}_\Sigma(1, \eta, \phi, 0; \alpha) =: \mathcal{L}B^{\phi}_{\Sigma}(\alpha)$$

and

$$\mathcal{Z}^*_\Sigma(1, \eta, \phi, 0; \beta) =: \mathcal{L}B_\Sigma(\phi, \beta),$$

where the families $\mathcal{L}B^{\phi}_{\Sigma}(\alpha)$ and $\mathcal{L}B_\Sigma(\phi, \beta)$ were studied by Joshi *et al.* [8].

4. For $\theta = \eta = t = 0$, we have

$$\mathcal{Z}_\Sigma(0, 0, \phi, 0; \alpha) =: S^*_\Sigma(\alpha)$$

and

$$\mathcal{Z}^*_\Sigma(0, 0, \phi, 0; \beta) =: S^*_\Sigma(\beta),$$

where the families $S^*_\Sigma(\alpha)$ and $S^*_\Sigma(\beta)$ were introduced by Brannan and Taha [3].

5. For $\theta = t = 0$ and $\eta = 1$, we have

$$\mathcal{Z}_\Sigma(0, 1, \phi, 0; \alpha) =: \mathcal{H}^{\alpha}_{\Sigma}$$

and

$$\mathcal{Z}^*_\Sigma(0, 1, \phi, 0; \beta) =: \mathcal{H}_\Sigma(\beta),$$

where the families $\mathcal{H}^{\alpha}_{\Sigma}$ and $\mathcal{H}_\Sigma(\beta)$ were investigated in the aforementioned pioneering work by Srivastava *et al.* [20].

Remark 4.2. By selecting particular values of θ, η and t in our main results (Theorem 2.2 and Theorem 3.2), we can derive a number of known results. Some of these special cases are recorded below.

1. Taking $t = 0$ in Theorems 2.2 and 3.2, we obtain the results which were obtained by Srivastava *et al.* [23, Theorems 1 and 2].

2. Taking $\theta = t = 0$ in Theorems 2.2 and 3.2, we obtain the results which were proven by Prema and Keerthi [11, Theorems 2.2 and 3.2].

3. Taking $\theta = 1$ and $t = 0$ in Theorems 2.2 and 3.2, we obtain the results which were given by Joshi *et al.* [8, Theorems 1 and 2].

4. Taking $\theta = \eta = t = 0$ in Theorems 2.2 and 3.2, we get the results which were derived by Murugusundaramoorthy *et al.* [10, Corollaries 6 and 7].

5. Taking $\theta = t = 0$ and $\eta = 1$ in Theorems 2.2 and 3.2, we obtain the results which were obtained by Srivastava *et al.* [20, Theorems 1 and 2].

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