Poincare Journal of Analysis & Applications
Vol. 9, No. 2 (2022), 249-273

SOFT N-TOPOLOGICAL SPACES

GIORGIO NORDO

Date of Receiving	:	15. 11. 2021
Date of Revision	:	19. 03. 2022
Date of Acceptance	:	30. 10. 2022

Abstract. Very recently, the idea of studying structures equipped with two or more soft topologies has been considered by several researchers. Soft bitopological spaces were introduced and studied, in 2014, by Ittanagi as a soft counterpart of the notion of bitopological space. and, independently, in 2015, by Naz, Shabir and Ali. In 2017, Hassan too introduced the concept of soft tritopological spaces and gave some first results. The notion of N-topological space related to ordinary topological spaces was instead introduced and studied, in 2011, by Tawfiq and Majeed. In this paper we introduce the concept of Soft N-Topological Space as generalization both of the concepts of Soft Topological Space and N-Topological Space and we investigate such class of spaces and their basic properties with particular regard to their subspaces, the parameterized families of crisp topologies generated by them and some new separation axioms called N-wise soft T_0, N-wise soft T_1, and N-wise soft T_2.

1. Introduction

Inspired by a Pawlak's idea [32], in 1999, Molodtsov [27] initiated the novel concept of Soft Sets Theory as a new mathematical tool and a completely different approach for dealing with uncertainties while modelling problems in computer science, engineering physics, economics, social sciences and medical sciences. Molodtsov defines a soft set as a parameterized family of subsets of universe set where each element is considered as a set of approximate elements of the soft set.

The absence of any restrictions on the approximate description in Soft Set Theory makes it very convenient and easy to apply respect to other existing methods as Probability Theory and Fuzzy Set Theory. In fact, we can define and use any kind of parametrization with the help of words, sentences, real numbers, real functions, mappings, etc.

In the past few years, the fundamentals of soft set theory have been studied by many researchers.

2010 *Mathematics Subject Classification.* 54A05, 54D10, 54E99, 54F65.
Key words and phrases. soft set, soft space, N-topological space, soft N-topological space, soft N-open set, N-wise soft T_i-space.
Communicated by. S. K. Kaushik

Starting from 2002, Maji, Biswas and Roy [17, 18] studied the theory of soft sets initiated by Molodstov, defining equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set and absolute soft set with examples. Soft binary operations like AND, OR and the operations of union, intersection were also defined.

In the original formulation, every soft set is defined on a own subset of the common set of parameters but, recently, Ma, Yang and Hu [16] proved that every soft set is equivalent to a soft set related to the whole set of parameters. This allow us to consider all the soft sets over the same parameter set and simplify the definitions of all the relations and operations between them. In particular, in 2014 Çağman [1] improved and simplified the definitions of operations on soft sets by using a single parameter set.

In 2011, Shabir and Naz [33] introduced the concept of soft topological spaces, also defining and investigating the notions of soft closed sets, soft closure, soft neighborhood, soft subspace and some separation axioms. In the same year Hussain and Ahmad [7] continued the study investigating the properties of soft closed sets, soft neighborhoods, soft interior, soft exterior and soft boundary. The notion of soft interior, soft neighborhood and soft continuity were also object of study by Zorlutuna, Akdag, Min and Atmaca in [40]. The neighborhood properties of a soft topological space were investigated in 2013 by Nazmul and Samanta [30].

In [26], Min pointed out some errors contained in the Shabir–Naz paper and investigated some properties of the separation axioms defined there. The class of soft Hausdorff spaces was extensively studied by Varol and Aygün in [37].

Furthermore, the notion of soft continuity between soft topological spaces was independently introduced and investigated by Hazra, Majumdar and Samanta in [6].

Soft connectedness was also studied in 2015 by Al-Khafaj [11]. In the same year, Das and Samanta [3] introduced and extensively studied the soft metric spaces.

In 2015, Hussain and Ahmad [8] redefine and explore several properties of soft T_i (with $i = 0, 1, 2, 3, 4$) separation axioms and discuss some soft invariance properties namely soft topological property and soft hereditary property.

In [38], Xie introduced the concept of soft points and prove that soft sets can be translated into soft point sets and then may conveniently deal with soft sets as same as ordinary sets.

In the same year, Matedjdes [19] and, independently, Shi and Pang [34] proved that the notion of soft topology is redundant, i.e. that a soft topology in the sense of Shabir and Naz [33] can be interpreted as a classical (crisp) topology (by two different points of view).

In 2017, Matejdes [20] studied various type of soft separation axioms and pointed out that any soft topological space is homeomorphic to a crisp topological space defined on a cartesian product and so that many soft topological notions and results can be directly derived from general topology. For such a reason, Chiney and Samanta [2] introduced a new definition of soft topology by using elementary union and elementary intersection instead of the soft ones used by Shabir and Naz and studied some basic properties of this new type of soft topological space.

Further contributions to the theory of soft sets and topology were recently added in 2019 and 2020 by Mehmood, Hanif, Nadeem, Zamir, Nordo, Park, Kelsoom, Gul, Khan and Jabeen [21, 22, 23, 24] and in 2022 by Mehmood, Ghour, Afzal, Nordo and Saleem [25].

In 1963, Kelly [10] introduced the concept of bitopological space, that is a structure with a pair of distinct topologies on the same set, and studied their pairwise separation axioms. In later years, many researchers have investigated bitopological spaces due to the richness of their structure and potential for carrying out many generalization of classical topological results in bitopological environment.

In 2003 Luay [15] gave a further generalization by introducing the notion of tripological space, i.e. a set equipped with three different topologies on it.

Furthermore, in 2013, Mukundan introduced the notion of quad topological space and studied some sets related to that space.

In the last years, the idea of studying a soft set equipped with two or more soft topologies has been considered by several researchers. Soft bitopological spaces were introduced and studied, in 2014, by Ittanagi [9] as a soft counterpart of the notion of bitopological space. and, independently, in 2015, by Naz, Shabir and Ali [29] (under the slight different name of ”bi-soft topological space”). In 2017, Hassan [5] introduced also the concept of soft tritopological spaces and gave some first results. In the same year, Khattak et al. [13] defined the notion of soft quad topological space which involves four soft topologies and focused their study on some soft separation axioms in such a space. In 2018, Khattak and some other researchers [14] continued the investigation studying the soft semi separation axioms in soft quad topological spaces.

In 2011, Tawfiq and Majeed [35] and, independently, in 2012, Khan [12] introduced the concept of N-topological space.

In the present paper we will present the notion of soft N-topological space as generalization both of the concepts of soft topological space and N-topological space.

The rest of this work is organized as follows. In the next section, concepts, notations and basic properties of soft sets and their operations are recalled. In Section 3, the main notions of the theory of soft topology and some fundamental properties are described and reviewed. In Section 4, the new definition of N-soft topology is introduced and some basic properties concerning – in particular – the subspace, the parameterized families of crisp topologies generated by it and the new separation axioms called N-wise soft T_0, N-wise soft T_1, and N-wise soft T_2 are investigated. In the final section some concluding comments are summarized.

2. Soft Sets

In this section we present some basic definitions and results on soft sets and suitably exemplify them.

Definition 2.1. [27] Let $\mathbb{U}$ be an initial universe set and $\mathbb{E}$ be a nonempty set of parameters (or abstract attributes) under consideration with respect to $\mathbb{U}$ and $A \subseteq \mathbb{E}$, we say that a pair (F, A) is a ***soft set*** over $\mathbb{U}$ if F is a set-valued mapping $F : A \to \mathbb{P}(\mathbb{U})$ which maps every parameter $e \in A$ to a subset $F(e)$ of $\mathbb{U}$.

In other words, a soft set is not a real (crisp) set but a parameterized family $\{F(e)\}_{e\in A}$ of subsets of the universe $\mathbb{U}$. For every parameter $e \in A$, $F(e)$ may be considered as the set of *e-approximate elements* of the soft set (F, A).

Remark 2.2. Let us note that when the parameter set has only one element, i.e. when $\mathbb{E} = \{\alpha\}$ any soft set (F, A) is equivalent to the ordinary (crisp) set $F(\alpha)$.

Although the Soft Sets Theory have had a great development in the past few years, many researchers pointed out that some propositions, such as are generalization of De Morgan's Laws, Distributive Laws to soft sets are affected by errors that are essentially due to some misunderstanding in the definition of the notions of soft subset and soft intersections as given in [18].

In 2010, Ma, Yang and Hu [16] proved that every soft set (F, A) is equivalent to the soft set $(F, \mathbb{E})$ related to the whole set of parameters $\mathbb{E}$, simply considering empty every approximations of parameters which are missing in A, that is extending in a trivial way its set-valued mapping, i.e. setting $F(e) = \emptyset$, for every $e \in \mathbb{E} \setminus A$.

For such a reason, in this paper we can consider all the soft sets over the same parameter set $\mathbb{E}$ as in [2] and we will redefine all the basic operations and relations between soft sets originally introduced in [27, 17, 18] as in [30], that is by considering the same parameter set.

Remark 2.3. Another way to represent a soft set $(F, \mathbb{E})$ is as the set of all the pairs $(e, F(e))$ parameter-approximation (with $e \in \mathbb{E}$), i.e the graph $Gr(F) \subseteq E \times \mathbb{P}(\mathbb{U})$ of the set-valued mapping $F : \mathbb{E} \to \mathbb{P}(\mathbb{U})$. In fact, in 2015, Matejdes [19] pointed out that there is a one-to-one correspondance from the set $\mathbf{F}(\mathbb{E}, \mathbb{P}(\mathbb{U}))$ of all the set-valued mappings from $\mathbb{E}$ to U onto the set $\mathbf{R}(\mathbb{E}, \mathbb{U})$ of all binary relations from $\mathbb{E}$ to $\mathbb{U}$ (that is a bijective mapping $\Phi : \mathbf{F}(\mathbb{E}, \mathbb{P}(\mathbb{U})) \to \mathbf{R}(\mathbb{E}, \mathbb{U})$ defined, for any $F \in \mathbf{F}(\mathbb{E}, \mathbb{P}(\mathbb{U}))$ by $\Phi(F) = \mathscr{R}_F$ where $\mathscr{R}_F = \{(e, u) \in \mathbb{E} \times \mathbb{U} : u \in F(e)\} = Gr(F)$ or, equivalently, for every $\mathscr{R} \in \mathbf{R}(\mathbb{E}, \mathbb{U})$, by $\Phi^{-1}(\mathscr{R}) = F_{\mathscr{R}})$ where $F_{\mathscr{R}} : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping which maps every $e \in \mathbb{E}$ in $F_{\mathscr{R}}(e) = \{u \in \mathbb{U} : (e, u) \in \mathscr{R}\} = \mathscr{R}(e)$) and so that a soft set $(F, \mathbb{E})$, which is substantially defined by $F \in \mathbb{P}(\mathbb{U})$, bijectivly corresponds to the graph $\Phi(F) = Gr(F)$ of its set-valued mapping.

Example 2.4. Here we present some examples of soft sets commonly used in the literature.

(1) [1] Assume that there are six houses in the universe $\mathbb{U} = \{h_l, h_2, h_3, h_4, h_5, h_6\}$ under consideration, and that $\mathbb{E} = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of decision parameters, where the e_i parameter (with $i = 1, \dots 5$) stands for "*expensive*", "*beautiful*", "*wooden*", "*cheap*" and "*in green surroundings*" respectively. Consider the subset of parameters $A = \{e_1, e_3, e_4\}$ and define the set-valued mapping "*house with some characteristic*" $F : A \to \mathbb{P}(\mathbb{U})$ by setting $F(e_1) = \{h_2, h_4\}$, $F(e_3) = \mathbb{U}$ and $F(e_4) = \{h_1, h_3, h_5\}$. Then the pair (F, A) is a soft set which espresses the fact that the houses h_2 and h_4 are expensive, all the houses are wooden and the house h_1, h_3, h_5 are cheap and it can also be represented by the graph of F consisting of the following family of approximations $Gr(F) = \{(e_1, \{h_2, h_4\}), (e_3, \mathbb{U}), (e_4, \{h_1, h_3, h_5\})\}$.

(2) [27] Let $(X, \mathcal{T})$ be a topological space on a nonempty set X. If, for every $x \in X$, we consider the family $\mathcal{N}_x^o = \{N \in \mathcal{T} : x \in \mathcal{T}\}$ of all open neighborhoods of x and define a set-valued mapping $T : X \to \mathbb{P}(X)$ by setting $T(x) = \mathcal{N}_x^o$ (for any $x \in X$), then the pair (X, T) is a soft set over the universe X.

(3) [27] Let A be a fuzzy set over a universe $\mathbb{U}$ and let $\mu_a : \mathbb{U} \to [0, 1]$ be its membership function (see [39]). For every $\alpha \in [0, 1]$, consider the α-level set

$A^{\geq\alpha} = \{x \in \mathbb{U} : \mu_A(x) \geq \alpha\}$ and define a set-valued mapping $F : [0,1] \to \mathbb{P}(\mathbb{U})$ by setting $F(\alpha) = A^{\geq\alpha}$ for any $\alpha \in [0,1]$. Since, for every $x \in A$, we know that $\mu_A(x) = \sup\{\alpha \in [0,1],\, x \in A^{\geq\alpha}\} = \sup\{\alpha \in [0,1],\, x \in F(\alpha)\}$, ii follows that every Zadeh's fuzzy set A may be considered as a special case of soft set and more precisely the soft set $(F, [0,1])$ over the same universe $\mathbb{U}$.

Definition 2.5. [40] The set of all the soft sets over a universe $\mathbb{U}$ with respect to a set of parameters $\mathbb{E}$ will be denoted by $\mathcal{SS}(\mathbb{U})_{\mathbb{E}}$.

Definition 2.6. [30] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$ and a common set of parameters $\mathbb{E}$, we say that $(F,\mathbb{E})$ is a ***soft subset*** of $(G,\mathbb{E})$ and we write $(F,\mathbb{E})\tilde{\subseteq}(G,\mathbb{E})$ if $F(e) \subseteq G(e)$ for every $e \in \mathbb{E}$.

Definition 2.7. [30] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$ and a common set of parameters $\mathbb{E}$, we say that $(F,\mathbb{E})$ is a ***soft super set*** of $(G,\mathbb{E})$ and we write $(F,\mathbb{E})\tilde{\supseteq}(G,\mathbb{E})$ if $(G,\mathbb{E})$ is a soft subset of $(F,\mathbb{E})$, i.e. if $(G,\mathbb{E})\tilde{\subseteq}(F,\mathbb{E})$

Definition 2.8. [30] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$, we say that $(F,\mathbb{E})$ and $(G,\mathbb{E})$ are ***soft equal*** and we write $(F,\mathbb{E})\tilde{=}(G,\mathbb{E})$ if $(F,\mathbb{E})\tilde{\subseteq}(G,\mathbb{E})$ and $(G,\mathbb{E})\tilde{\subseteq}(F,\mathbb{E})$.

Definition 2.9. [30] A soft set $(F,\mathbb{E})$ over a universe $\mathbb{U}$ is said to be ***null soft set*** and it is denoted by $(\tilde{\emptyset},\mathbb{E})$ if $F(e) = \emptyset$ for every $e \in \mathbb{E}$.

Definition 2.10. [30] A soft set $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ over a universe $\mathbb{U}$ is said to be a ***absolute soft set*** and it is denoted by $(\tilde{\mathbb{U}},\mathbb{E})$ if $F(e) = \mathbb{U}$ for every $e \in \mathbb{E}$.

Clearly, for every soft set $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, we have $(\tilde{\emptyset},\mathbb{E})\tilde{\subseteq}(F,\mathbb{E})\tilde{\subseteq}(\tilde{\mathbb{U}},\mathbb{E})$.

Definition 2.11. Let $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$ and V be a nonempty subset of U, the ***constant soft set*** of V, denoted by by $(\tilde{V},\mathbb{E}))$ (or, sometimes, by $\tilde{V}$), is the soft set $(\underline{V},\mathbb{E})$, where $\underline{V} : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the constant set-valued mapping defined by $\underline{V}(e) = V$, for every $e \in \mathbb{E}$.

Proposition 2.12. [1] For every triplet $(F,\mathbb{E}),(G,\mathbb{E}),(H,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ of soft sets, we have tht $(F,\mathbb{E})\tilde{\subseteq}(G,\mathbb{E})$ and $(G,\mathbb{E})\tilde{\subseteq}(H,\mathbb{E})$ imply $(F,\mathbb{E})\tilde{\subseteq}(H,\mathbb{E})$.

Definition 2.13. [30] Let $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$, the ***soft complement*** (or more exactly the *soft relative complement*) of $(F,\mathbb{E})$, denoted with $(F,\mathbb{E})^{\complement}$, is the soft set $\left(F^{\complement}, E\right)$ where $F^{\complement} : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $F^{\complement}(e) = F(e)^{\complement} = \mathbb{U} \setminus F(e)$, for every $e \in \mathbb{E}$.

It is routine to show that the soft complement of the null soft set is soft equal to the absolute soft set, i.e. $(\tilde{\emptyset},\mathbb{E})^{\complement} \tilde{=} (\tilde{\mathbb{U}},\mathbb{E})$, that the soft complement of the absolute soft set is soft equal to the null soft set, i.e. $(\tilde{\mathbb{U}},\mathbb{E})^{\complement} \tilde{=} (\tilde{\emptyset},\mathbb{E})$, and that the soft complement of the soft complement of any soft set $(F,\mathbb{E})$ is soft equal to the soft set itself, i.e. $\left((F,\mathbb{E})^{\complement}\right)^{\complement} \tilde{=} (F,\mathbb{E})$.

Definition 2.14. [30] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a common universe $\mathbb{U}$, the ***soft difference*** of $(F,\mathbb{E})$ and $(G,\mathbb{E})$, denoted by $(F,\mathbb{E})\widetilde{\setminus}(G,\mathbb{E})$, is the soft set $(F \setminus G, E)$ where $F \setminus G : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $(F \setminus G)(e) = F(e) \setminus G(e)$, for every $e \in \mathbb{E}$.

Clearly, for every soft set $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, it results $(F, \mathbb{E})^{\complement} \cong (\tilde{\mathbb{U}}, \mathbb{E}) \widetilde{\setminus} (F, \mathbb{E})$.

Definition 2.15. [30] Let $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, the ***soft union*** of $(F, \mathbb{E})$ and $(G, \mathbb{E})$, denoted with $(F, \mathbb{E}) \tilde{\cup} (G, \mathbb{E})$, is the soft set $(F \cup G, E)$ where $F \cup G : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $(F \cup G)(e) = F(e) \cup G(e)$, for every $e \in \mathbb{E}$.

Definition 2.16. [30] Let $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, the ***soft intersection*** of $(F, \mathbb{E})$ and $(G, \mathbb{E})$, denoted with $(F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E})$, is the soft set $(F \cap G, E)$ where $F \cap G : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $(F \cap G)(e) = F(e) \cap G(e)$, for every $e \in \mathbb{E}$.

Definition 2.17. [11] Two soft sets $(F, \mathbb{E})$ and $(G, \mathbb{E})$ over a common universe $\mathbb{U}$ are said to be ***soft disjoint*** if their soft intersection is the soft null set, i.e. if $(F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E}) \cong (\tilde{\emptyset}, \mathbb{E})$.

Remark 2.18. It is a simple matter to verify that two soft sets $(F, \mathbb{E})$ and $(G, \mathbb{E})$ are soft disjoint according to Definition 2.17, if and only if every their corresponding approximations is disjoint, that is $F(e) \cap G(e) = \emptyset$, for every $e \in \mathbb{E}$.
Let us note that in some paper (see, for example, [13]) the last equivalent expression is assumed as definition.

The soft operators of union, intersection and complement satisfy relations similar to those of (crisp) set theory, such as the well-known commutative, associative, distributive, exclusion, contradiction and De Morgan's Laws.

Proposition 2.19. [1] For every soft set $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, we have:

(1) $(F, \mathbb{E}) \tilde{\cup} (F, \mathbb{E}) \cong (F, \mathbb{E})$
(2) $(F, \mathbb{E}) \tilde{\cup} (\tilde{\emptyset}, \mathbb{E}) \cong (F, \mathbb{E})$
(3) $(F, \mathbb{E}) \tilde{\cup} (\tilde{\mathbb{U}}, \mathbb{E}) \cong (\tilde{\mathbb{U}}, \mathbb{E}))$
(4) $(F, \mathbb{E}) \tilde{\cap} (F, \mathbb{E}) \cong (F, \mathbb{E})$
(5) $(F, \mathbb{E}) \tilde{\cap} (\tilde{\emptyset}, \mathbb{E}) \cong (\tilde{\emptyset}, \mathbb{E})$
(6) $(F, \mathbb{E}) \tilde{\cap} (\tilde{\mathbb{U}}, \mathbb{E}) \cong (F, \mathbb{E})$

Proposition 2.20. [1] For every pair $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ of soft sets, we have:

(1) $(F, \mathbb{E}) \tilde{\cup} (G, \mathbb{E}) \cong (G, \mathbb{E}) \tilde{\cup} (F, \mathbb{E})$
(2) $(F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E}) \cong (G, \mathbb{E}) \tilde{\cap} (F, \mathbb{E})$

Proposition 2.21. [1] For every triplet $(F, \mathbb{E}), (G, \mathbb{E}), (H, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ of soft sets, we have:

(1) $(F, \mathbb{E}) \tilde{\cap} ((G, \mathbb{E}) \tilde{\cap} (H, \mathbb{E})) \cong ((F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E})) \tilde{\cap} (H, \mathbb{E})$
(2) $(F, \mathbb{E}) \tilde{\cup} ((G, \mathbb{E}) \tilde{\cup} (H, \mathbb{E})) \cong ((F, \mathbb{E}) \tilde{\cup} (G, \mathbb{E})) \tilde{\cup} (H, \mathbb{E})$
(3) $(F, \mathbb{E}) \tilde{\cap} ((G, \mathbb{E}) \tilde{\cup} (H, \mathbb{E})) \cong ((F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E})) \tilde{\cup} ((F, \mathbb{E}) \tilde{\cap} (H, \mathbb{E}))$
(4) $(F, \mathbb{E}) \tilde{\cup} ((G, \mathbb{E}) \tilde{\cap} (H, \mathbb{E})) \cong ((F, \mathbb{E}) \tilde{\cup} (G, \mathbb{E})) \tilde{\cap} ((F, \mathbb{E}) \tilde{\cup} (H, \mathbb{E}))$

Proposition 2.22. [31] For every soft set $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, we have:

(1) $(F, \mathbb{E}) \tilde{\cup} (F, \mathbb{E})^{\complement} \cong (\tilde{\mathbb{U}}, \mathbb{E})$,

(2) $(F,\mathbb{E})\tilde{\cap}(F,\mathbb{E})^{\complement} \tilde{=} (\tilde{\emptyset},\mathbb{E})$

Proposition 2.23. [40] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, then:

(1) $(F,\mathbb{E})\tilde{\subseteq}(G,\mathbb{E})$ iff $(F,\mathbb{E})\tilde{\cap}(G,\mathbb{E})\tilde{=}(F,\mathbb{E})$

(2) $(F,\mathbb{E})\tilde{\subseteq}(G,\mathbb{E})$ iff $(F,\mathbb{E})\tilde{\cup}(G,\mathbb{E})\tilde{=}(G,\mathbb{E})$

Proposition 2.24. [33] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, then:

(1) $((F,\mathbb{E})\tilde{\cup}(G,\mathbb{E}))^{\complement} \tilde{=} (F,\mathbb{E})^{\complement} \tilde{\cap} (G,\mathbb{E})^{\complement}$

(2) $((F,\mathbb{E})\tilde{\cap}(G,\mathbb{E}))^{\complement} \tilde{=} (F,\mathbb{E})^{\complement} \tilde{\cup} (G,\mathbb{E})^{\complement}$

Proposition 2.25. [33] Let $(F,\mathbb{E}),(G,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be two soft sets over a universe $\mathbb{U}$, we have that $(F,\mathbb{E})\widetilde{\setminus}(G,\mathbb{E})\tilde{=}(F,\mathbb{E})\tilde{\cap}(G,\mathbb{E})^{\complement}$.

Proposition 2.26. [1] For every soft set $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$, we have:

(1) $(F,\mathbb{E})\widetilde{\setminus}(F,\mathbb{E})\tilde{=}(\tilde{\emptyset},\mathbb{E})$

(2) $(F,\mathbb{E})\widetilde{\setminus}(\tilde{\emptyset},\mathbb{E})\tilde{=}(F,\mathbb{E})$

(3) $(\tilde{\emptyset},\mathbb{E})\widetilde{\setminus}(F,\mathbb{E})\tilde{=}(\tilde{\emptyset},\mathbb{E})$

(4) $(F,\mathbb{E})\widetilde{\setminus}(\tilde{\mathbb{U}},\mathbb{E})\tilde{=}(\tilde{\emptyset},\mathbb{E})$

The notions of soft union and intersection admits a obvious generalization to a family with any number of soft sets.

Definition 2.27. [30] Let $\{(F_i,\mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, the (generalized) ***soft union*** of $\{(F_i,\mathbb{E})\}_{i\in I}$, denoted with $\widetilde{\bigcup}_{i\in I}(F_i,\mathbb{E})$, is defined by $\left(\bigcup_{i\in I} F_i,\mathbb{E}\right)$ where $\bigcup_{i\in I} F_i : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $\left(\bigcup_{i\in I} F_i\right)(e) = \bigcup_{i\in I} F_i(e)$, for every $e \in \mathbb{E}$.

Definition 2.28. [30] Let $\{(F_i,\mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, the (generalized) ***soft intersection*** of $\{(F_i,\mathbb{E})\}_{i\in I}$, denoted with $\widetilde{\bigcap}_{i\in I}(F_i,\mathbb{E})$, is defined by $\left(\bigcap_{i\in I} F_i, E\right)$ where $\bigcap_{i\in I} F_i : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by $\left(\bigcap_{i\in I} F_i\right)(e) = \bigcap_{i\in I} F_i(e)$, for every $e \in \mathbb{E}$.

Proposition 2.29. Let $\{(F_i,\mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, the for every $i \in I$, we have that: $\widetilde{\bigcap}_{i\in I}(F_i,\mathbb{E}) \tilde{\subseteq} (F_i,\mathbb{E}) \tilde{\subseteq} \widetilde{\bigcup}_{i\in I}(F_i,\mathbb{E})$.

Propositions 2.21(iii)-(iv) and 2.24 can be easily extended to arbitrary union and arbitrary intersection.

Proposition 2.30. Let respectively $(F,\mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set and $\{(G_i,\mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a common universe $\mathbb{U}$, we have:

(1) $(F,\mathbb{E})\tilde{\cap}\left(\widetilde{\bigcup}_{i\in I}(G_i,\mathbb{E})\right) \tilde{=} \widetilde{\bigcup}_{i\in I}\left((F,\mathbb{E})\tilde{\cap}(G_i,\mathbb{E})\right)$

(2) $(F,\mathbb{E})\tilde{\cup}\left(\widetilde{\bigcap}_{i\in I}(G_i,\mathbb{E})\right) \tilde{=} \widetilde{\bigcap}_{i\in I}\left((F,\mathbb{E})\tilde{\cup}(G_i,\mathbb{E})\right)$

Proposition 2.31. Let $\{(F_i,\mathbb{E})\}_{i\in I} \subseteq \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a nonempty subfamily of soft sets over a universe $\mathbb{U}$, it results:

(1) $\left(\widetilde{\bigcup}_{i\in I}(F_i,\mathbb{E})\right)^{\complement} \tilde{=} \widetilde{\bigcap}_{i\in I}(F_i,\mathbb{E})^{\complement}$

(2) $\left(\widetilde{\bigcap}_{i\in I}(F_i,\mathbb{E})\right)^{\complement} \tilde{=} \widetilde{\bigcup}_{i\in I}(F_i,\mathbb{E})^{\complement}$

Despite the name, soft sets are not real sets since they defined by means of a set-valued mapping. For such a reason, their original definition lacked the concept of point. In 2015, Xie [38] introduced the notion of soft point and study some relationships between soft points and soft sets, finding in particular that soft sets can be converted into ordinary sets of soft points so that we may conveniently deal with soft relations, soft operations and so on.

Definition 2.32. [38] A soft set $(F,\mathbb{E})\in\mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ over a universe $\mathbb{U}$ is said to be a ***soft point*** over U if it has only one non-empty approximation and it is a singleton, i.e. if there exists some parameter $\alpha\in E$ and an element $p\in\mathbb{U}$ such that $F(\alpha)=\{p\}$ and $F(e)=\emptyset$ for every $e\in E\setminus\{\alpha\}$. Such a soft point is usually denoted with $(p_\alpha,\mathbb{E})$. The singleton $\{p\}$ is called the *support set* of the soft point and α is called the *expressive parameter* of $(p_\alpha,\mathbb{E})$.

Remark 2.33. Let us observe that the soft point notation $(p_\alpha,\mathbb{E})$ maintains and makes immediately recognizable both the salient information, that is the value of the parameter and that of the point itself. Every reference to the set-valued mapping from which it derives is completely superfluous since it has only one non-empty value $\{p\}$ corresponding to the parameter α.
In other words, a soft point $(p_\alpha,\mathbb{E})$ is a soft set corresponding to the set-valued mapping $p_\alpha:\mathbb{E}\to(U)$ that, for any $e\in\mathbb{E}$, is defined by:

$$p_\alpha(e)=\begin{cases}\{p\} & \text{if } e=\alpha\\ \emptyset & \text{if } e\in\mathbb{E}\setminus\{\alpha\}\end{cases}$$

Example 2.34. Let $\mathbb{U}=\{h_l,h_2,h_3,h_4,h_5,h_6\}$ the universe set and $\mathbb{E}=\{e_1,e_2,e_3,e_4,e_5\}$ the set of decision parameters as in the Example 2.4(iii). Then, if we consider the fixed value $p=h_5\in\mathbb{U}$, the fixed value $\alpha=e_2\in\mathbb{E}$ and the set-valued mapping $p_\alpha:\mathbb{E}\to\mathbb{P}(\mathbb{U})$ defined by setting $p_\alpha(e_1)=\emptyset$, $p_\alpha(e_2)=\{p\}=\{h_5\}$, $p_\alpha(e_3)=\emptyset$, $p_\alpha(e_4)=\emptyset$ and $p_\alpha(e_5)=\emptyset$, the soft set $(p_\alpha,\mathbb{E})$ (or more precisely $(h_{5_{e_2}},\mathbb{E})$) is a soft point.

Definition 2.35. [38] The set of all the soft points over a universe $\mathbb{U}$ with respect to a set of parameters $\mathbb{E}$ will be denoted by $\mathcal{SP}(\mathbb{U})_{\mathbb{E}}$.

Definition 2.36. [38] Let $(p_\alpha,\mathbb{E})\in\mathcal{SP}(\mathbb{U})_{\mathbb{E}}$ and $(F,\mathbb{E})\in\mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ respectively be a soft point and a softset over a common universe $\mathbb{U}$. We say that ***the soft point*** $(p_\alpha,\mathbb{E})$ ***soft belongs to the soft set*** $(F,\mathbb{E})$ and we write $(p_\alpha,\mathbb{E})\tilde{\in}(F,\mathbb{E})$, if the soft point is a soft subset of the soft set, i.e. if $(p_\alpha,\mathbb{E})\tilde{\subseteq}(F,\mathbb{E})$ and hence if $p\in F(\alpha)$.

Definition 2.37. [3] Let $(p_\alpha,\mathbb{E}),(q_\beta,\mathbb{E})\in\mathcal{SP}(\mathbb{U})_{\mathbb{E}}$ be two soft points over a common universe $\mathbb{U}$, we say that $(p_\alpha,\mathbb{E})$ and $(q_\beta,\mathbb{E})$ are ***soft equal***, and we write $(p_\alpha,\mathbb{E})\tilde{=}(q_\beta,\mathbb{E})$, if they are equals as soft sets and hence if $p=q$ and $\alpha=\beta$.

Definition 2.38. [3] We say that two soft points $(p_\alpha,\mathbb{E})$ and $(q_\beta,\mathbb{E})$ are ***soft distinct***, and we write $(p_\alpha,\mathbb{E})\tilde{\neq}(q_\beta,\mathbb{E})$, if and only if $p\neq q$ or $\alpha\neq\beta$.

In the special case in which the soft points $p\in F(\alpha)$ and $q\in F(\alpha)$ are defined respect to the same expressive parameter α it is obvious that they are soft equal if and only if $p=q$ and soft distinct if and only if $p\neq q$.

Remark 2.39. In some papers (see, for example [13] and [14]), using a different notation, two soft points $(p_\alpha, \mathbb{E})$ and $(q_\beta, \mathbb{E})$ satisfying Definition 2.38 are said "disjoint" but for uniformity of language, it seems to us more appropriated to define them as distinct.

Proposition 2.40. [3] Any soft set $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ over a universe $\mathbb{U}$ can be represented as soft union of all its soft points, i.e.

$$(F, \mathbb{E}) = \widetilde{\bigcup} \{(p_\alpha, \mathbb{E}) : (p_\alpha, \mathbb{E}) \tilde{\in} (F, \mathbb{E})\}.$$

A different notion of membership, used in particular, for defining soft separation axioms is given in [33] by Shabir and Naz.

Definition 2.41. [33] Let $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$ and $p \in \mathbb{U}$. We say that ***the point*** p ***belongs to the soft set*** $(F, \mathbb{E})$ and we write that $p \in (F, \mathbb{U})$ if $p \in F(e)$, for every $e \in \mathbb{E}$.

By the above Definition, it is immediately clear that $p \notin (F, \mathbb{E})$ if there exists some $\alpha \in E$ such that $p \notin F(\alpha)$.

Remark 2.42. Let us note that a (ordinary) point $p \in \mathbb{U}$ belongs to a soft set $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ if and only if every corresponding soft point $(p_e, \mathbb{E})$ relative to the same support set $\{p\}$ and any expressive parameter $e \in \mathbb{E}$ soft belongs to the soft set, i.e. that $p \in (F, \mathbb{E})$ if and only if $(p_e, \mathbb{E}) \tilde{\in} (F, \mathbb{E})$ for every $e \in \mathbb{E}$. Thus, the soft membership $(p_\alpha, \mathbb{E}) \tilde{\in} (F, \mathbb{E})$ is a weaker condition than to the ordinary membership $p \in (F, \mathbb{E})$.

Definition 2.43. [7] Let $(F, \mathbb{E}) \in \mathcal{SS}(\mathbb{U})_{\mathbb{E}}$ be a soft set over a universe $\mathbb{U}$ and V be a nonempty subset of $\mathbb{U}$, the ***sub soft set*** of $(F, \mathbb{E})$ over V, is the soft set $({}^V F, \mathbb{E})$, where ${}^V F : \mathbb{E} \to \mathbb{P}(\mathbb{U})$ is the set-valued mapping defined by ${}^V F(e) = F(e) \cap V$, for every $e \in \mathbb{E}$.

Remark 2.44. Using Definitions 2.11 and 2.16, it is a trivial matter to verify that a sub soft set of $(F, \mathbb{E})$ over V can also be expressed as $({}^V F, \mathbb{E}) = (F, \mathbb{E}) \tilde{\cap} (\tilde{V}, \mathbb{E})$.

Example 2.45. [7] Let $\mathbb{U} = \{h_l, h_2, h_3, h_4, h_5, h_6\}$ the universe set, $\mathbb{E} = \{e_1, e_2, e_3, e_4, e_5\}$ the set of decision parameters as in the Example 2.4(i) and consider the soft set $(F, \mathbb{E})$ defined by:

$$\begin{array}{lll} F(e_1) = \{h_2, h_4\}, & F(e_2) = \{h_1, h_3, h_4, h_6\}, & F(e_3) = \{h_2, h_3, h_6\}, \\ F(e_4) = \{h_1, h_3, h_5, h_6\}, & F(e_5) = \{h_2, h_3, h_4, h_6\}. & \end{array}$$

Now, if we consider the subset $V = \{h_1, h_2, h_3, h_4\}$ of the universe U, the sub soft set $({}^V F, \mathbb{E})$ of $(F, \mathbb{E})$ over V results defined by:

$$\begin{array}{lll} {}^V F(e_1) = \{h_2, h_4\}, & {}^V F(e_2) = \{h_1, h_3, h_4\}, & {}^V F(e_3) = \{h_2, h_3\}, \\ {}^V F(e_4) = \{h_1, h_3\}, & {}^V F(e_5) = \{h_2, h_3, h_4\}. & \end{array}$$

3. Soft Topological Spaces

The notion of soft topological spaces as topological spaces defined over a initial universe with a fixed set of parameters was introduced in 2011 by Shabir and Naz [33].

Definition 3.1. [33] Let X be an initial universe set, $\mathbb{E}$ be a nonempty set of parameters with respect to X and $\mathcal{T} \subseteq \mathcal{SS}(X)_{\mathbb{E}}$ be a family of soft sets over X, we say that $\mathcal{T}$ is a ***soft topology*** on X with respect to $\mathbb{E}$ if the following four conditions are satisfied:

(1) the null soft set belongs to $\mathcal{T}$, i.e. $(\tilde{\emptyset}, \mathbb{E}) \in \mathcal{T}$
(2) the absolute soft set belongs to $\mathcal{T}$, i.e. $(\tilde{X}, \mathbb{E}) \in \mathcal{T}$
(3) the soft intersection of any two soft sets of $\mathcal{T}$ belongs to $\mathcal{T}$, i.e. for every $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{T}$ then $(F, \mathbb{E}) \tilde{\cap} (G, \mathbb{E}) \in \mathcal{T}$.
(4) the soft union of any subfamily of soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$, i.e. for every $\{(F_i, \mathbb{E})\}_{i \in I} \subseteq \mathcal{T}$ then $\widetilde{\bigcup}_{i \in I} (F_i, \mathbb{E}) \in \mathcal{T}$

The triplet $(X, \mathcal{T}, \mathbb{E})$ is called a ***soft topological space*** over X with respect to $\mathbb{E}$.
In some case, when it is necessary to better specify the universal set and the set of parameters, the topology will be denoted by $\mathcal{T}(X, \mathbb{E})$.

It is a trivial fact to verify that condition *(iii)* of Definition 3.1 is equivalent to state that the soft intersection of any finite number of soft sets of $\mathcal{T}$ belongs to $\mathcal{T}$, i.e. that for every $\{(F_i, \mathbb{E})\}_{i=1,\dots n} \subseteq \mathcal{T}$ then $\widetilde{\bigcap}_{i=1}^{n} (F_i, \mathbb{E}) \in \mathcal{T}$.

Definition 3.2. [33] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X with respect to $\mathbb{E}$, then the members of $\mathcal{T}$ are said to be ***soft open set*** in X.

Example 3.3. For the sake of clarity, here are some examples of soft set families that form a soft topology or not.

(1) [33] Let X be an initial universe set, $\mathbb{E}$ be the set of parameters. If we consider the family of soft sets defined by: $\mathcal{T}_b = \left\{ (\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}) \right\}$, then $\mathcal{T}_b$ is a soft topology. We say that $\mathcal{T}_b$ is the ***soft indiscrete topology*** (or soft trivial topology) on X and we call $(X, \mathcal{T}_b, \mathbb{E})$ the ***soft indiscrete topological space*** (or soft trivial topological space) over X.
(2) [33] Let X be an initial universe set, $\mathbb{E}$ be the set of parameters. If we consider the family $\mathcal{T}_d = \mathcal{SS}(X)_{\mathbb{E}}$ of all soft sets over X, then $\mathcal{T}_d$ is a soft topology. We say that $\mathcal{T}_d$ is the ***soft discrete topology*** on X and we call $(X, \mathcal{T}_d, \mathbb{E})$ the ***soft discrete topological space*** over X.
(3) Let $X = \{h_1, h_2, h_3\}$ be the universe set and $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters. Consider the family of soft sets

$$\mathcal{T} = \left\{ (\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E}) \right\}$$

where the soft sets $(F_i, \mathbb{E})$ (with $i = 1, \dots 4$) over X are defined by setting:

$$\begin{array}{ll} F_1(e_1) = \{h_2\}, & F_1(e_2) = \{h_1\}, \\ F_2(e_1) = \{h_1, h_2\}, & F_2(e_2) = \{h_1\}, \\ F_3(e_1) = \{h_1, h_2\}, & F_3(e_2) = \{h_1, h_3\}, \\ F_4(e_1) = \{h_1, h_2\}, & F_4(e_2) = X. \end{array}$$

Since, it is a simple routine to verify that all non-trivial unions and intersections of the members of the family $\mathcal{T}$ still belong to $\mathcal{T}$ and, more exactly that

$$\begin{array}{lll} (F_1, \mathbb{E}) \tilde{\cup} (F_2, \mathbb{E}) = (F_2, \mathbb{E}), & (F_1, \mathbb{E}) \tilde{\cup} (F_3, \mathbb{E}) = (F_3, \mathbb{E}), & (F_1, \mathbb{E}) \tilde{\cup} (F_4, \mathbb{E}) = (F_4, \mathbb{E}) \\ (F_2, \mathbb{E}) \tilde{\cup} (F_3, \mathbb{E}) = (F_3, \mathbb{E}), & (F_2, \mathbb{E}) \tilde{\cup} (F_4, \mathbb{E}) = (F_4, \mathbb{E}), & (F_3, \mathbb{E}) \tilde{\cup} (F_4, \mathbb{E}) = (F_4, \mathbb{E}), \\ (F_1, \mathbb{E}) \tilde{\cap} (F_2, \mathbb{E}) = (F_1, \mathbb{E}), & (F_1, \mathbb{E}) \tilde{\cap} (F_3, \mathbb{E}) = (F_1, \mathbb{E}), & (F_1, \mathbb{E}) \tilde{\cap} (F_4, \mathbb{E}) = (F_1, \mathbb{E}), \\ (F_2, \mathbb{E}) \tilde{\cap} (F_3, \mathbb{E}) = (F_2, \mathbb{E}), & (F_2, \mathbb{E}) \tilde{\cap} (F_4, \mathbb{E}) = (F_2, \mathbb{E}), & (F_3, \mathbb{E}) \tilde{\cap} (F_4, \mathbb{E}) = (F_3, \mathbb{E}). \end{array}$$

by Definition 3.1 it follows $\mathcal{T}$ is a soft topology on X and hence that $(X, \mathcal{T}, \mathbb{E})$ is a (finite) soft topological space over X.

(4) [7] Let $X = \{h_1, h_2, h_3\}$ be the universe set and $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters. Consider the family of soft sets $\mathcal{A} = \Big\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E})\Big\}$ where the soft sets $(F_i, \mathbb{E})$ (with $i = 1, \ldots 4$) over X are defined by setting:

$$\begin{array}{ll} F_1(e_1) = \{h_2\}, & F_1(e_2) = \{h_1\}, \\ F_2(e_1) = \{h_2, h_3\}, & F_2(e_2) = \{h_1, h_2\}, \\ F_3(e_1) = \{h_1, h_2\}, & F_3(e_2) = \{h_1, h_2\}, \\ F_4(e_1) = \{h_2\}, & F_4(e_2) = \{h_1, h_3\}. \end{array}$$

Since, it can be easily verified that the soft union $(F_2, \mathbb{E}) \tilde{\cup} (F_3, \mathbb{E})$ is the soft set $(H, \mathbb{E})$ defined by $H(e_1) = X$ and $H(e_2) = \{h_1, h_2\}$ and so that $(H, \mathbb{E}) \notin \mathcal{A}$ it follows that $\mathcal{A}$ is not a soft topology on X.

Definition 3.4. [33] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X and be $(F, \mathbb{E})$ be a soft set over X. We say that $(F, \mathbb{E})$ is ***soft closed set*** in X if its complement $(F, \mathbb{E})^{\complement}$ is a soft open set, i.e. if $(F^{\complement}, \mathbb{E}) \in \mathcal{T}$.

Notation 3.1. The family of all soft closed sets of a soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X with respect to $\mathbb{E}$ will be denoted by σ, or more precisely with $\sigma(X, \mathbb{E})$ when it is necessary to specify the universal set X and the set of parameters $\mathbb{E}$.

Using Proposition 2.24 and Proposition 2.31 with Definition 3.1, the following statement is immediately proved.

Proposition 3.5. [33] Let σ be the family of soft closed sets of a soft topological space $(X, \mathcal{T}, \mathbb{E})$, the following hold:

(1) the null soft set is a soft closed set, i.e. $(\tilde{\emptyset}, \mathbb{E}) \in \sigma$
(2) the absolute soft set is a soft closed set, i.e. $(\tilde{X}, \mathbb{E}) \in \sigma$
(3) the soft union of any two soft closed sets is still a soft closed set, i.e. for every $(C, \mathbb{E}), (D, \mathbb{E}) \in \sigma$ then $(C, \mathbb{E}) \tilde{\cup} (D, \mathbb{E}) \in \sigma$.
(4) the soft intersection of any subfamily of soft closed sets is still a soft closed set, i.e. for every $\{(C_i, \mathbb{E})\}_{i \in I} \subseteq \sigma$ then $\widetilde{\bigcap}_{i \in I} (C_i, \mathbb{E}) \in \sigma$

As a consequence of Proposition 3.5, the family $\sigma(X, \mathbb{E})$ is sometimes called the soft closed topology over X.

The following is a trivial generalization of a proposition given in [33].

Proposition 3.6. Let $N \in \mathbb{N}$ be an integer number greater than 0 and $\{(X, \mathcal{T}_i, \mathbb{E})\}_{i=1,\ldots N}$ be a finite family of N soft topological spaces over the same universe X with respect to $\mathbb{E}$, then the (crisp) intersection $\bigcap_{i=1}^{N} \mathcal{T}_i$ of all its soft topologies $\mathcal{T}_i$ (with $i = 1, \ldots N$) is still a soft topology on X and so $\left(X, \bigcap_{i=1}^{N} \mathcal{T}_i, \mathbb{E}\right)$ is a soft topological space over X.

Remark 3.7. The following example shows that, unlike intersections, the (crisp) union of (two) soft topologies is not necessarily a soft topology.

Example 3.8. Let $X = \{h_1, h_2, h_3\}$ be the universe set and $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters. Consider the family of soft sets

$$\mathcal{T}_1 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E})\right\}$$

where the soft sets $(F_i, \mathbb{E})$ (with $i = 1, \ldots 4$) over X are defined by setting:

$$\begin{array}{ll} F_1(e_1) = \{h_2\}, & F_1(e_2) = \{h_1\}, \\ F_2(e_1) = \{h_1, h_2\}, & F_2(e_2) = \{h_1\}, \\ F_3(e_1) = \{h_1, h_2\}, & F_3(e_2) = \{h_1, h_3\}, \\ F_4(e_1) = \{h_1, h_2\}, & F_4(e_2) = X. \end{array}$$

and $\mathcal{T}_2 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (G_1, \mathbb{E}), (G_2, \mathbb{E}), (G_3, \mathbb{E})\right\}$ where the soft sets $(G_i, \mathbb{E})$ (with $i = 1, \ldots 3$) over X are defined by setting:

$$\begin{array}{ll} G_1(e_1) = \{h_2\}, & G_1(e_2) = \{h_1\}, \\ G_2(e_1) = \{h_2\}, & G_2(e_2) = \{h_1, h_3\}, \\ G_3(e_1) = \{h_2, h_3\}, & G_3(e_2) = \{h_1, h_3\}. \end{array}$$

It is a simple routine to verify that $\mathcal{T}_1$ and $\mathcal{T}_2$ verify the axioms from Definition 3.1 (in particular $\mathcal{T}_1$ coincides with the soft topology $\mathcal{T}$ of the Example 3.3(iii)) and so that they are soft topologies on X.

It is worth noting that, according to Proposition 3.6, the crisp intersection of the two soft topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, that is

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E})\right\}$$

is indeed a soft topology over the same universe X.

However, if we consider the union

$$\mathcal{A} = \mathcal{T}_1 \cup \mathcal{T}_2 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E}), (G_2, \mathbb{E}), (G_3, \mathbb{E})\right\}$$

we can observe that $(F_3, \mathbb{E}) \tilde{\cup} (G_2, \mathbb{E})$ is the soft set $(H, \mathbb{E})$ defined by $H(e_1) = X$ and $H(e_2) = \{h_1, h_3\}$ and so that $(H, \mathbb{E}) \notin \mathcal{A}$. This proves that the union $\mathcal{A} = \mathcal{T}_1 \cup \mathcal{T}_2$ of the two soft topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ is a family of soft sets which is not a soft topology on X.

Although the union of soft topologies is not in general a soft topology, it will be useful to give the following definition.

Definition 3.9. Let $\{(X, \mathcal{T}_i, \mathbb{E})\}_{i=1,\ldots N}$ be a finite family of soft topological spaces over the same universe X with respect to $\mathbb{E}$. The ***supremum soft topology*** of all soft topologies $\mathcal{T}_i$ (with $i = 1, \ldots N$), denoted by $\bigvee_{i=1}^{N} \mathcal{T}_i$, is the smallest soft topology on X containing the (crisp) union $\bigcup_{i=1}^{N} \mathcal{T}_i$. The soft topological space $(\left(X, \bigvee_{i=1}^{N} \mathcal{T}_i, \mathbb{E}\right)$ is called the ***supremum soft topological space***.

Example 3.10. Let $X = \{h_1, h_2, h_3\}$ be the universe set, $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters and consider the two soft topologies

$$\mathcal{T}_1 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E})\right\}$$

and

$$\mathcal{T}_2 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (G_1, \mathbb{E}), (G_2, \mathbb{E}), (G_3, \mathbb{E})\right\}$$

defined in the Example 3.8.

The supremum soft topology $\mathcal{T}_1 \vee \mathcal{T}_2$ is the smallest soft topology over X which contain the crisp union $\mathcal{T}_1 \cup \mathcal{T}_2$. Thus, after noticing that $(F_1, \mathbb{E})\tilde{\cup}(G_2, \mathbb{E}) = (F_4, \mathbb{E})\tilde{\cup}(G_2, \mathbb{E}) = (G_2, \mathbb{E})$, $(F_2, \mathbb{E})\tilde{\cup}(G_2, \mathbb{E}) = (F_3, \mathbb{E})\tilde{\cup}(G_2, \mathbb{E}) = (F_3, \mathbb{E})$, $(F_1, \mathbb{E})\tilde{\cup}(G_3, \mathbb{E}) = (G_3, \mathbb{E})$ and that $(F_1, \mathbb{E})\tilde{\cap}(G_2, \mathbb{E}) = (F_1, \mathbb{E})\tilde{\cap}(G_3, \mathbb{E}) = (F_2, \mathbb{E})\tilde{\cap}(G_3, \mathbb{E}) = (F_1, \mathbb{E})$, $(F_2, \mathbb{E})\tilde{\cap}(G_2, \mathbb{E}) = (F_3, \mathbb{E})\tilde{\cap}(G_2, \mathbb{E}) = (G_2, \mathbb{E}) = (F_3, \mathbb{E})\tilde{\cap}(G_3, \mathbb{E}) = (F_4, \mathbb{E})\tilde{\cap}(G_2, \mathbb{E}) = (F_4, \mathbb{E})\tilde{\cap}(G_3, \mathbb{E}) = (G_2, \mathbb{E})$, it follows that:

$$\mathcal{T}_1 \vee \mathcal{T}_2 = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E}), (G_2, \mathbb{E}), (G_3, \mathbb{E}), (H, \mathbb{E})\right\}$$

where the unique new soft open set $(H, \mathbb{E}) = (F_2, \mathbb{E})\tilde{\cup}(G_3, \mathbb{E}) = (F_3, \mathbb{E})\tilde{\cup}(G_3, \mathbb{E}) = (F_4, \mathbb{E})\tilde{\cup}(G_3, \mathbb{E})$ is defined by $H(e_1) = X$ and $H(e_2) = \{h_1, h_3\}$.

Notation 3.2. The unique soft topology $\bigcap_{i=1}^{N} \mathcal{T}_i$ of Proposition 3.6 is evidently the ***infimum*** of all the soft topologies $\mathcal{T}_i$ (with $i = 1, \dots N$) and it is generally denoted by $\bigwedge_{i=1}^{N} \mathcal{T}_i$.

Thus, the set $\mathcal{S}\mathrm{Top}(X)_{\mathbb{E}}$ of all soft topologies over a universe X with respect to the set of parameters $\mathbb{E}$, equipped with the two binary operations $\wedge$ and $\vee$, that is $(\mathcal{S}\mathrm{Top}(X)_{\mathbb{E}}, \wedge, \vee)$ is a lattice.

Proposition 3.11. [33] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X. Then, for every $e \in \mathbb{E}$, the family $\mathcal{T}_e = \{F(e) : (F, \mathbb{E}) \in \mathcal{T}\}$ is a (crisp) topology on X.

Remark 3.12. Proposition 3.11 shows that every soft topological space $(X, \mathcal{T}, \mathbb{E})$ gives a parameterized family $\{(X, \mathcal{T}_e)\}_{e \in \mathbb{E}}$ of ordinary topological spaces. By way, as shown in the following counterexample, the converse does not hold, i.e. a family $\mathcal{A} \subseteq \mathcal{SS}(X)_{\mathbb{E}}$ of soft sets is not in general a soft topology even if every family $\mathcal{A}_e = \{F(e) : (F, \mathbb{E}) \in \mathcal{A}\}$ of sets corresponding to each parameter $e \in \mathbb{E}$ defines a topology.

Example 3.13. Let $X = \{h_1, h_2, h_3\}$ be the universe set and $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters. Consider the family $\mathcal{A} = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E})\right\}$ of soft sets $(F_i, \mathbb{E})$ (with $i = 1, \dots 4$) over X defined by:

$$\begin{array}{ll} F_1(e_1) = \{h_2\}, & F_1(e_2) = \{h_1\}, \\ F_2(e_1) = \{h_2, h_3\}, & F_2(e_2) = \{h_1, h_2\}, \\ F_3(e_1) = \{h_1, h_2\}, & F_3(e_2) = \{h_1, h_2\}, \\ F_4(e_1) = \{h_2\}, & F_4(e_2) = \{h_1, h_3\}. \end{array}$$

that, in Example 3.3(iv), we have already proved to do not form a soft topology over X. Also, the crisp families corresponding to the parameters e_1 and e_2 respectively, i.e.

$$\mathcal{A}_{e_1} = \{\emptyset, X, \{h_2\}, \{h_1, h_2\}, \{h_2, h_3\}\}$$

and

$$\mathcal{A}_{e_1} = \{\emptyset, X, \{h_1\}, \{h_1, h_2\}, \{h_1, h_3\}\}$$

are topologies on the set X.

Let us also observe that in [6] the point of view is reversed and the previous Proposition is taken as a different definition for a topology of soft subsets over X. So, Proposition 3.11 says that every soft topological space in the sense of Shabir–Naz [33] is also a soft topological space in the sense of Hazra–Mujumdar–Samanta [6].

Definition 3.14. [40] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space, $(N, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ be a soft set and $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ be a soft point over a common universe X. We say that $(N, \mathbb{E})$ is a ***soft neighborhood*** of the soft point $(x_\alpha, \mathbb{E})$ if there is some soft open set soft containing the soft point and soft contained in the soft set, that is if there exists some soft open set $(A, \mathbb{E}) \in \mathcal{T}$ such that $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E}) \tilde{\subseteq} (N, \mathbb{E})$.

Notation 3.3. The family of all soft neighborhoods of a soft point $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ in a soft topological space $(X, \mathcal{T}, \mathbb{E})$ will be denoted by $\mathcal{N}_{(x_\alpha, \mathbb{E})}$ (or more precisely with $\mathcal{N}^{\mathcal{T}}_{(x_\alpha, \mathbb{E})}$ if it is necessary to specify the topology).

Proposition 3.15. [40] The family $\mathcal{N}_{(x_\alpha, \mathbb{E})}$ of all the soft neighborhoods of a soft point $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ in a soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X satisfies the following four properties:

(1) for any $(N, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$, $(x_\alpha, \mathbb{E}) \tilde{\in} (N, \mathbb{E})$
(2) for any $(M, \mathbb{E}), (N, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$, $(M, \mathbb{E}) \tilde{\cap} (N, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$
(3) for any $(N, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$ and every soft set $(F, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ such that $(N, \mathbb{E}) \tilde{\subseteq} (F, \mathbb{E})$ then $(F, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$
(4) for any $(N, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$ there exists some $(M, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$ such that for every soft point $(y_\beta, \mathbb{E}) \tilde{\in} (M, \mathbb{E})$ then $(N, \mathbb{E}) \in \mathcal{N}_{(y_\beta, \mathbb{E})}$

Proposition 3.16. [4] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over the universe X. Then a soft set $(F, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ is a soft open set if and only if for every soft point $(x_\alpha, \mathbb{E}) \in (F, \mathbb{E})$ there exists a soft open neighborhood $(N, \mathbb{E}) \in \mathcal{N}_{(x_\alpha, \mathbb{E})}$ such that $(x_\alpha, \mathbb{E}) \tilde{\in} (N, \mathbb{E}) \tilde{\subseteq} (F, \mathbb{E})$.

Corollary 3.17. Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X. Then a soft set $(F, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ is a soft open set if and only if it is a soft open neighborhood of every its soft point $(x_\alpha, \mathbb{E}) \in (F, \mathbb{E})$.

Definition 3.18. [33] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X and $(F, \mathbb{E})$ be a soft set over X. Then the ***soft closure*** of the soft set $(F, \mathbb{E})$, denoted by $s\text{-}cl_X((F, \mathbb{E}))$, is the soft intersection of all soft closed set over X soft containing $(F, \mathbb{E})$, that is

$$s\text{-}cl_X((F, \mathbb{E})) \tilde{=} \widetilde{\bigcap} \left\{ (C, \mathbb{E}) \in \sigma(X, \mathbb{E}) : (F, \mathbb{E}) \tilde{\subseteq} (C, \mathbb{E}) \right\}$$

Clearly, $s\text{-}cl_X((F, \mathbb{E}))$ is the smallest soft closed set over X which soft contains $(F, \mathbb{E})$.

Proposition 3.19. [33] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, and $(F, \mathbb{E})$ be a soft set over X. Then the following hold:

(1) $s\text{-}cl_X\left((\tilde{\emptyset}, \mathbb{E})\right) \tilde{=} (\tilde{\emptyset}, \mathbb{E})$
(2) $s\text{-}cl_X\left((\tilde{X}, \mathbb{E})\right) \tilde{=} (\tilde{X}, \mathbb{E})$
(3) $(F, \mathbb{E}) \tilde{\subseteq} s\text{-}cl_X((F, \mathbb{E}))$
(4) $(F, \mathbb{E})$ is a soft closed set over X if and only if $s\text{-}cl_X((F, \mathbb{E})) \tilde{=} (F, \mathbb{E})$
(5) $s\text{-}cl_X(s\text{-}cl_X((F, \mathbb{E}))) \tilde{=} s\text{-}cl_X((F, \mathbb{E}))$

Proposition 3.20. [33] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space and $(F, \mathbb{E}), (G, \mathbb{E}) \in \mathcal{SS}(X)_{\mathbb{E}}$ be two soft sets over a common universe X. Then the following hold:

(1) $(F, \mathbb{E}) \tilde{\subseteq} (G, \mathbb{E})$ implies $s\text{-}cl_X((F, \mathbb{E})) \tilde{\subseteq} s\text{-}cl_X((G, \mathbb{E}))$

(2) $s\text{-}cl_X\left((F,\mathbb{E})\tilde{\cup}(G,\mathbb{E})\right) \tilde{=} s\text{-}cl_X\left((F,\mathbb{E})\right) \tilde{\cup}\, s\text{-}cl_X\left((G,\mathbb{E})\right)$
(3) $s\text{-}cl_X\left((F,\mathbb{E})\tilde{\cap}(G,\mathbb{E})\right) \tilde{\subseteq} s\text{-}cl_X\left((F,\mathbb{E})\right) \tilde{\cap}\, s\text{-}cl_X\left((G,\mathbb{E})\right)$

Having in mind the Definition 2.43 we can recall the following proposition.

Proposition 3.21. [7] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, and Y be a nonempty subset of X, then the family $\mathcal{T}_Y$ of all sub soft sets of $\mathcal{T}$ over Y, i.e.

$$\mathcal{T}_Y = \left\{({}^{Y}F, \mathbb{E}) : (F, \mathbb{E}) \in \mathcal{T}\right\}$$

is a soft topology on Y.

Definition 3.22. [7] Let $(X, \mathcal{T}, \mathbb{E})$ be a soft topological space over X, and Y be a nonempty subsets of X, the soft topology $\mathcal{T}_Y = \left\{({}^{Y}F, \mathbb{E}) : (F, \mathbb{E}) \in \mathcal{T}\right\}$ is said to be the ***soft relative topology*** of $\mathcal{T}$ on Y and $(Y, \mathcal{T}_Y, \mathbb{E})$ is called a ***soft topological subspace*** of $(X, \mathcal{T}, \mathbb{E})$ on Y.

Example 3.23. Let $X = \{h_1, h_2, h_3\}$ be the universe set and $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters. Consider the family of soft sets

$$\mathcal{T} = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E})\right\}$$

where the soft sets $(F_i, \mathbb{E})$ (with $i = 1, \ldots 4$) over X are defined by setting:

$$\begin{array}{ll} F_1(e_1) = \{h_2\}, & F_1(e_2) = \{h_1\}, \\ F_2(e_1) = \{h_1, h_2\}, & F_2(e_2) = \{h_1\}, \\ F_3(e_1) = \{h_1, h_2\}, & F_3(e_2) = \{h_1, h_3\}, \\ F_4(e_1) = \{h_1, h_2\}, & F_4(e_2) = X. \end{array}$$

that in Example 3.3(iii) we have already proved to be a soft topology on X. Now, if we consider the subset $Y = \{h_1, h_2\}$ of X, the sub soft sets $({}^{Y}F_i, \mathbb{E})$ (with $i = 1, \ldots 4$) of the soft open set $(F_i, \mathbb{E})$ over Y results to be defined by:

$$\begin{array}{ll} {}^{Y}F_1(e_1) = \{h_2\}, & {}^{Y}F_1(e_2) = \{h_1\}, \\ {}^{Y}F_2(e_1) = Y, & {}^{Y}F_2(e_2) = \{h_1\}, \\ {}^{Y}F_3(e_1) = Y, & {}^{Y}F_3(e_2) = \{h_1\}, \\ {}^{Y}F_4(e_1) = Y, & {}^{Y}F_4(e_2) = Y. \end{array}$$

and so, being $({}^{Y}F_2, \mathbb{E}) \tilde{=} ({}^{Y}F_3, \mathbb{E})$ and $({}^{Y}F_4, \mathbb{E}) = (\tilde{Y}, \mathbb{E})$, the soft relative topology $\mathcal{T}_Y$ of $\mathcal{T}$ on Y is:

$$\mathcal{T}_Y = \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{Y}, \mathbb{E}), ({}^{Y}F_1, \mathbb{E}), ({}^{Y}F_2, \mathbb{E})\right\}.$$

Definition 3.24. [8] A soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X is called a ***soft*** T_0***-space*** if for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ there exists a soft open set which soft contains exactly one of these soft points, i.e. there is some $(A, \mathbb{E}) \in \mathcal{T}$ such that $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E})$ and $(y_\beta, \mathbb{E}) \tilde{\notin} (A, \mathbb{E})$, or there is some $(B, \mathbb{E}) \in \mathcal{T}$ such that $(y_\beta, \mathbb{E}) \tilde{\in} (B, \mathbb{E})$ and $(x_\alpha, \mathbb{E}) \tilde{\notin} (B, \mathbb{E})$.

Definition 3.25. [8] A soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X is called a ***soft*** T_1***-space*** if for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ there exists a soft open set $(A, \mathbb{E}) \in \mathcal{T}$ which soft contains one soft point but not the other one, that is $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E})$ and $(y_\beta, \mathbb{E}) \tilde{\notin} (A, \mathbb{E})$.

Remark 3.26. Let us note that some other authors (see, for example, [4], [30] and [33]) defined the soft separation axioms using ordinary points and not soft points, that is referring to the Definition 2.41 given by Xie in [38] instead of the Definition 2.36, which – in our opinion – appears more proper and correct.

Proposition 3.27. [8] A soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X is a soft T_1-space if and only if every its soft point $(x_\alpha, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ is a soft closed set.

Definition 3.28. [8] A soft topological space $(X, \mathcal{T}, \mathbb{E})$ over X is called a ***soft*** T_2***-space*** (or a ***soft Hausdorff space***) if for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ there exist two soft open sets $(A, \mathbb{E}), (B, \mathbb{E}) \in \mathcal{T}$ which are soft disjoints and soft contain the two soft points respectively, that is $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E})$, $(y_\beta, \mathbb{E}) \tilde{\in} (B, \mathbb{E})$ and $(A, \mathbb{E}) \tilde{\cap} (B, \mathbb{E}) = (\tilde{\emptyset}, \mathbb{E})$.

Remark 3.29. Evidently, every soft T_2-space is a soft T_1-space and every soft T_1-space is a soft T_0-space.

In 2015, Matejdes [19], after having noted that every soft set $(F, \mathbb{E})$ bijectively corresponds to the graph $Gr(F)$ of its set-valued mapping (see Remark 2.3), proved that $(X, \mathcal{T}, \mathbb{E})$ is a soft topological space if and only if the set $\mathcal{T}_{\mathbb{E}\times X} = \{Gr(F) : (F, \mathbb{E}) \in \mathcal{T}\}$ of the graphs corresponding to the set-valued mappings of all the soft open sets of $\mathcal{T}$ forms an ordinary topology on the cartesian product $\mathbb{E} \times X$, i.e. if $(\mathbb{E} \times X, \mathcal{T}_{\mathbb{E}\times X})$ is a (crisp) topological space.
Matejdes also proved that a soft set $(F, \mathbb{E})$ is a soft open set in a soft topological space $(X, \mathcal{T}, \mathbb{E})$ if and only if the graph $Gr(F)$ corresponding to the set-valued mapping is an open set in the topological space $(\mathbb{E} \times X, \mathcal{T}_{\mathbb{E}\times X})$ defined on the cartesian product. Hence, every topological notion can be introduced for a soft topological space $(X, \mathcal{T}, \mathbb{E})$ by direct reformulation of that notion in the topological space $(\mathbb{E} \times X, \mathcal{T}_{\mathbb{E}\times X})$.

4. Soft N-Topological Spaces

Very recently, the idea of studying structures equipped with two or more soft topologies has been considered by several researchers. Soft bitopological spaces were introduced and studied, in 2014, by Ittanagi [9] as a soft counterpart of the notion of bitopological space. and, independently, in 2015, by Naz, Shabir and Ali [29] (under the slight different name of "bi-soft topological space"). In 2017, Hassan [5] introduced also the concept of soft tritopological spaces and gave some first results, while Khattak et al. [13] defined the notion of soft quad topological space whose study continued in [14].

The concept of N-topological space related to ordinary topological spaces was introduced and studied, in 2011, by Tawfiq and Majeed [35] and, independently, in 2012, by Khan [12].

In this section we initiate the study of soft N-Topological Spaces as a natural soft counterpart of the notion above, in order to extends and generalizes the results on soft bitopological and soft tritopological spaces.

Definition 4.1. Let X be an initial universe set, $\mathbb{E}$ be a nonempty set of parameters with respect to X, $N \in \mathbb{N}$ be an integer number greater than 0 and $(X, \mathcal{T}_i, \mathbb{E})$ (with $i = 1, \dots N$) be N different soft topological spaces over the same universe X, then the soft set X equipped with all these topologies will be said ***soft*** N***-topological space***

over X and will be denoted by $(X, \mathcal{T}_i, N, \mathbb{E})$. For any $i = 1, \dots N$, a member of $\mathcal{T}_i$ is said to be a $\mathcal{T}_i$ soft open set. A complement of a $\mathcal{T}_i$ soft open set is called a $\mathcal{T}_i$ soft closed set.

Remark 4.2. Evidently Definition 4.1 generalizes several already studied classes of soft topological spaces since for $N = 2$ we have the notion of soft bitopological spaces introduced and studied in 2014, by Ittanagi [9] and, independently, in 2015, by Naz, Shabir and Ali [29]; for $N = 3$ we obtain the definition of soft tripological spaces defined in [5] by Hassan, and for $N = 4$ we have the notion of soft quad topological space introduced by Khattak et al. in [13].

Definition 4.3. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X. A soft set $(F, \mathbb{E})$ over X is said to be a ***soft N-open set*** if it is a $\mathcal{T}_j$ soft open set for some $j = 1, \dots N$, i.e. if there exists some $j \in \{1, \dots N\}$ such that $(F, \mathbb{E}) \in \mathcal{T}_j$.

Remark 4.4. Let us note that Definition 4.3 is equivalent to say that $(F, \mathbb{E}) \in \bigcup_{i=1}^{N} \mathcal{T}_i$ where the union operator is the usual set-union and not a soft union as defined in Definition 2.27.

Furthermore, recalling Remark 3.7, it is clear that $\bigcup_{i=1}^{N} \mathcal{T}_i$ is not, in general, a soft topology.

Definition 4.5. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X. A soft set $(G, \mathbb{E})$ over X is said to be a ***soft N-closed set*** if its soft complement is $(G, \mathbb{E})^{\complement}$ is a soft N-open set.

Evidently, a soft set $(G, \mathbb{E})$ is a soft N-closed set if it is at least a $\mathcal{T}_j$ soft closed set for some $j = 1, \dots N$.

Example 4.6. Let $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the universe set and $\mathbb{E} = \{e_1, e_2, e_3\}$ be the set of parameters. Consider the following $N = 4$ soft topologies over X:

$$\begin{aligned}
\mathcal{T}_1 &= \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E})\right\}, \\
\mathcal{T}_2 &= \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_3, \mathbb{E})\right\}, \\
\mathcal{T}_3 &= \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_4, \mathbb{E}), (F_5, \mathbb{E}), (F_6, \mathbb{E})\right\}, \\
\mathcal{T}_4 &= \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_7, \mathbb{E}), (F_8, \mathbb{E})\right\}
\end{aligned}$$

where the soft sets $(F_i, \mathbb{E})$ (with $i = 1, \dots 8$) over X are respectively defined by setting:

$$\begin{array}{lll}
F_1(e_1) = \{h_1\}, & F_1(e_2) = \{h_2, h_4\}, & F_1(e_3) = \{h_3\}, \\
F_2(e_1) = \{h_1, h_2\}, & F_2(e_2) = \{h_2, h_4, h_6\}, & F_2(e_3) = \{h_2, h_3\}, \\
F_3(e_1) = \{h_3\}, & F_3(e_2) = \{h_4\}, & F_3(e_3) = \{h_5\}, \\
F_4(e_1) = \{h_4\}, & F_4(e_2) = \{h_4\}, & F_4(e_3) = \{h_6\}, \\
F_5(e_1) = \{h_4, h_5\}, & F_5(e_2) = \{h_4, h_6\}, & F_5(e_3) = \{h_6, h_8\}, \\
F_6(e_1) = X, & F_6(e_2) = \{h_4, h_6\}, & F_6(e_3) = \{h_5, h_6, h_8\}, \\
F_7(e_1) = \{h_5, h_7\}, & F_7(e_2) = \{h_7\}, & F_7(e_3) = \{h_6, h_8\}, \\
F_8(e_1) = \{h_5, h_7, h_8\}, & F_8(e_2) = X, & F_8(e_3) = \{h_6, h_7, h_8\}.
\end{array}$$

Then $(X, \mathcal{T}_i, 4, \mathbb{E})$ is a soft 4-topological space over X. Let us note that $\bigcup_{i=1}^{4} \mathcal{T}_i$ is not a soft topology since, for example, $(F_4, \mathbb{E}) \tilde{\cup} (F_7, \mathbb{E})$ is the soft set $(H, \mathbb{E})$ defined by $H(e_1) = \{h_4, h_5, h_7\}, H(e_2) = \{h_4, h_7\}, H(e_3) = \{h_6, h_8\}$ and $(H, \mathbb{E}) \notin \bigcup_{i=1}^{4} \mathcal{T}_i$.

From Definitions 4.1 and 4.3 immediately derives the following proposition.

Proposition 4.7. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X, then we have that:

(1) every $\mathcal{T}_i$ soft open set (with $i = 1, \dots N$) is a soft N-open set
(2) every $\mathcal{T}_i$ soft closed set (with $i = 1, \dots N$) is a soft N-closed set

Let us note that the converse of the above statements does not hold in general since can exists some soft N-open (closed) set which is not a $\mathcal{T}_i$ soft open (closed) set for every $i = 1, \dots N$.

Proposition 4.8. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X. Then, for every parameter $e \in \mathbb{E}$, the structure $(X, \mathcal{T}_{1e}, \dots \mathcal{T}_{Ne})$ where $\mathcal{T}_{ie} = \{F(e) : (F, \mathbb{E}) \in \mathcal{T}_i\}$ (with $i = 1, \dots N$) is an N-topological space.

Proof. In fact, for every $i = 1, \dots N$ and every parameter $e \in \mathbb{E}$, by Proposition 3.11, any $\mathcal{T}_{ie}$ is a crisp topology on X and hence, by Definition 4.1, $(X, \mathcal{T}_{1e}, \dots \mathcal{T}_{Ne})$ is a (crisp) N-topological space on X in the sense of definition given in [35]. □

In other words, Proposition 4.8 states that every soft N-topological space over X gives a parameterized family of (crisp) N-topological space over the same set X.

Example 4.9. If we consider the soft 4-topological space $(X, \mathcal{T}_i, 4, \mathbb{E})$ over X defined in the Example 4.6, then, the parameterized families of crisp topologies respect to each parameter of $\mathbb{E}$ are:

$$\begin{aligned}
\mathcal{T}_{1e_1} &= \{\emptyset, X, \{h_1\}, \{h_1, h_2\}\}, \\
\mathcal{T}_{1e_2} &= \{\emptyset, X, \{h_2, h_4\}, \{h_2, h_4, h_6\}\}, \\
\mathcal{T}_{1e_3} &= \{\emptyset, X, \{h_3\}, \{h_2, h_3\}\}, \\
\mathcal{T}_{2e_1} &= \{\emptyset, X, \{h_3\}\}, \\
\mathcal{T}_{2e_2} &= \{\emptyset, X, \{h_4\}\}, \\
\mathcal{T}_{2e_3} &= \{\emptyset, X, \{h_5\}\}, \\
\mathcal{T}_{3e_1} &= \{\emptyset, X, \{h_4\}, \{h_4, h_5\}\}, \\
\mathcal{T}_{3e_2} &= \{\emptyset, X, \{h_4\}, \{h_4, h_6\}\}, \\
\mathcal{T}_{3e_3} &= \{\emptyset, X, \{h_6\}, \{h_6, h_8\}, \{h_5, h_6, h_8\}\}, \\
\mathcal{T}_{4e_1} &= \{\emptyset, X, \{h_5, h_7\}, \{h_5, h_7, h_8\}\}, \\
\mathcal{T}_{4e_2} &= \{\emptyset, X, \{h_7\}\}, \\
\mathcal{T}_{4e_3} &= \{\emptyset, X, \{h_6, h_8\}, \{h_6, h_7, h_8\}\}
\end{aligned}$$

and so $(X, \mathcal{T}_{1e_j}, \mathcal{T}_{2e_j}, \mathcal{T}_{3e_j}, \mathcal{T}_{4e_j})$ (with $j = 1, \dots 3$) are crisp N-topological spaces on X in the sense of definition given by Tawfiq and Majeed in [35].

Recalling the Definition 3.22, we can give the following proposition.

Proposition 4.10. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X and Y be a nonempty subset of X, then $(Y, \mathcal{T}_{iY}, N, \mathbb{E})$, i.e. the set Y equipped with the relative soft topologies of $\mathcal{T}_{iY}$ on Y (with $i = 1, \dots N$) is a soft N-topological space over Y.

Proof. In fact, for every $i = 1, \dots N$, by Proposition 3.21, any $\mathcal{T}_{iY}$ is a soft topology on Y and hence, by Definition 4.1, $(Y, \mathcal{T}_{iY}, N, \mathbb{E})$ is a soft N-topological space over Y. □

Definition 4.11. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X and Y be a nonempty subset of X the soft N-topological space $(Y, \mathcal{T}_{iY}, N, \mathbb{E})$ is said to be the ***relative soft N-topological space*** of $(X, \mathcal{T}_i, N, \mathbb{E})$ on Y or the ***soft N-topological subspace*** of $(X, \mathcal{T}_i, N, \mathbb{E})$ on Y.

Example 4.12. Let us consider the soft 4-topological space $(X, \mathcal{T}_i, 4, \mathbb{E})$ over X defined in the Example 4.6 and the subset $Y = \{h_1, h_3, h_4, h_5, h_8\}$ of X. Then the sub soft sets $({}^{Y}F_i, \mathbb{E})$ (with $i = 1, \ldots 8$) of the soft open set $(F_i, \mathbb{E})$ over Y results to be defined by:

$$
\begin{array}{lll}
{}^{Y}F_1(e_1) = \{h_1\}, & {}^{Y}F_1(e_2) = \{h_4\}, & {}^{Y}F_1(e_3) = \{h_3\}, \\
{}^{Y}F_2(e_1) = \{h_1\}, & {}^{Y}F_2(e_2) = \{h_4\}, & {}^{Y}F_2(e_3) = \{h_3\}, \\
{}^{Y}F_3(e_1) = \{h_3\}, & {}^{Y}F_3(e_2) = \{h_4\}, & {}^{Y}F_3(e_3) = \{h_5\}, \\
{}^{Y}F_4(e_1) = \{h_4\}, & {}^{Y}F_4(e_2) = \{h_4\}, & {}^{Y}F_4(e_3) = \emptyset, \\
{}^{Y}F_5(e_1) = \{h_4, h_5\}, & {}^{Y}F_5(e_2) = \{h_4\}, & {}^{Y}F_5(e_3) = \{h_8\}, \\
{}^{Y}F_6(e_1) = Y, & {}^{Y}F_6(e_2) = \{h_4\}, & {}^{Y}F_6(e_3) = \{h_5, h_8\}, \\
{}^{Y}F_7(e_1) = \{h_5\}, & {}^{Y}F_7(e_2) = \emptyset, & {}^{Y}F_7(e_3) = \{h_8\}, \\
{}^{Y}F_8(e_1) = \{h_5, h_8\}, & {}^{Y}F_8(e_2) = Y, & {}^{Y}F_8(e_3) = \{h_8\}
\end{array}
$$

and so, being $({}^{Y}F_1, \mathbb{E}) \tilde{=} ({}^{Y}F_2, \mathbb{E})$, the soft 4-topological subspace $(Y, \mathcal{T}_{iY}, 4, \mathbb{E})$ of $(X, \mathcal{T}_i, 4, \mathbb{E})$ on Y is formed by the following soft relative topologies $\mathcal{T}_{iY}$ of $\mathcal{T}_i$ on Y (with $i = 1, \ldots 4$):

$$
\begin{aligned}
\mathcal{T}_{1Y} &= \left\{ (\tilde{\emptyset}, \mathbb{E}), (\tilde{Y}, \mathbb{E}), ({}^{Y}F_1, \mathbb{E}) \right\}, \\
\mathcal{T}_{2Y} &= \left\{ (\tilde{\emptyset}, \mathbb{E}), (\tilde{Y}, \mathbb{E}), ({}^{Y}F_3, \mathbb{E}) \right\}, \\
\mathcal{T}_{3Y} &= \left\{ (\tilde{\emptyset}, \mathbb{E}), (\tilde{Y}, \mathbb{E}), ({}^{Y}F_4, \mathbb{E}), ({}^{Y}F_5, \mathbb{E}), ({}^{Y}F_6, \mathbb{E}) \right\}, \\
\mathcal{T}_{4Y} &= \left\{ (\tilde{\emptyset}, \mathbb{E}), (\tilde{Y}, \mathbb{E}), ({}^{Y}F_7, \mathbb{E}), ({}^{Y}F_8, \mathbb{E}) \right\}.
\end{aligned}
$$

Definition 4.13. A soft N-topological space $(X, \mathcal{T}_i, N, \mathbb{E})$ over X is called an ***N-wise soft T_0-space*** if for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ there exists a soft N-open set which soft contains exactly one of these points, i.e. there is some soft N-open set $(A, \mathbb{E})$ such that $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E})$ and $(y_\beta, \mathbb{E}) \tilde{\notin} (A, \mathbb{E})$, or there is some soft N-open set $(B, \mathbb{E})$ such that $(y_\beta, \mathbb{E}) \tilde{\in} (B, \mathbb{E})$ and $(x_\alpha, \mathbb{E}) \tilde{\notin} (B, \mathbb{E})$.

Proposition 4.14. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X. If at least one soft topological space $(X, \mathcal{T}_j, \mathbb{E})$ (for some $j = 1, \ldots N$) is a soft T_0-space, then $(X, \mathcal{T}_i, N, \mathbb{E})$ is an N-wise soft T_0-space.

Proof. Suppose that there exists some $j = 1, \ldots N$ such that $(X, \mathcal{T}_j, \mathbb{E})$ is a soft T_0-space. Then, for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$, by Definition 3.24, there is some $(A, \mathbb{E}) \in \mathcal{T}_j$ such that $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E})$ and $(y_\beta, \mathbb{E}) \tilde{\notin} (A, \mathbb{E})$, or there is some $(B, \mathbb{E}) \in \mathcal{T}_j$ such that $(y_\beta, \mathbb{E}) \tilde{\in} (B, \mathbb{E})$ and $(x_\alpha, \mathbb{E}) \tilde{\notin} (B, \mathbb{E})$. Since, by Proposition 4.7, every $\mathcal{T}_j$ soft open set is a soft N-open set, it follows that $(A, \mathbb{E})$ and $(B, \mathbb{E})$ are soft N-open sets and so that Definition 4.13 holds, i.e. that $(X, \mathcal{T}_i, N, \mathbb{E})$ is an N-wise soft T_0-space. $\square$

Remark 4.15. The converse of Proposition 4.14 is not true in general, that is can exist an N-wise soft T_0-space which soft topologies are not soft T_0 as it is shown in the following counterexample.

Example 4.16. Let $X = \{h_1, h_2, h_3\}$ be the universe set and $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters. Consider the following $N = 2$ soft topologies over X:

$$\begin{aligned} \mathcal{T}_1 &= \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_1, \mathbb{E})\right\}, \\ \mathcal{T}_2 &= \left\{(\tilde{\emptyset}, \mathbb{E}), (\tilde{X}, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E}), (F_5, \mathbb{E})\right\}, \end{aligned}$$

where the soft open sets $(F_i, \mathbb{E})$ (with $i = 1, \ldots 4$) over X are respectively defined by setting:

$$\begin{array}{ll} F_1(e_1) = \{h_1\}, & F_1(e_2) = \{h_2\}, \\ & \\ F_2(e_1) = \{h_1\}, & F_2(e_2) = \{h_3\}, \\ F_3(e_1) = \{h_1, h_3\}, & F_3(e_2) = \{h_1, h_3\}, \\ F_4(e_1) = \{h_2, h_3\}, & F_3(e_2) = \{h_2, h_3\}, \\ F_5(e_1) = (\tilde{X}, \mathbb{E}), & F_4(e_2) = \{h_2, h_3\}. \end{array}$$

and hence the soft 2-topological space $(X, \mathcal{T}_i, 2, \mathbb{E})$.

After noticing that the set $\mathcal{SP}(X)_{\mathbb{E}}$ of all soft points over X contains the following 6 members:

$$\mathcal{SP}(X)_{\mathbb{E}} = \{(h_{1_{e_1}}, \mathbb{E}), (h_{1_{e_2}}, \mathbb{E}), (h_{2_{e_1}}, \mathbb{E}), (h_{2_{e_2}}, \mathbb{E}), (h_{3_{e_1}}, \mathbb{E}), (h_{3_{e_2}}, \mathbb{E})\}$$

we can easily verify that, for every pair of distinct soft points (of the $\frac{6(6-1)}{2} = 15$ possible combinations), there exists a soft N-open set which soft contains exactly one of these points i.e., for example, that:

- $(F_1, \mathbb{E})$ soft contains $(h_{1_{e_1}}, \mathbb{E})$ but not $(h_{1_{e_2}}, \mathbb{E})$,
- $(F_1, \mathbb{E})$ soft contains $(h_{1_{e_1}}, \mathbb{E})$ but not $(h_{2_{e_1}}, \mathbb{E})$,
- $(F_3, \mathbb{E})$ soft contains $(h_{1_{e_1}}, \mathbb{E})$ but not $(h_{2_{e_2}}, \mathbb{E})$,
- ....
- $(F_4, \mathbb{E})$ soft contains $(h_{3_{e_1}}, \mathbb{E})$ but not $(h_{1_{e_2}}, \mathbb{E})$,
- $(F_2, \mathbb{E})$ soft contains $(h_{3_{e_2}}, \mathbb{E})$ but not $(h_{2_{e_1}}, \mathbb{E})$,
- etc.

and so that $(X, \mathcal{T}_i, 2, \mathbb{E})$ is a 2-wise soft T_0-space. However, neither the soft topological spaces $(X, \mathcal{T}_1, \mathbb{E})$ and $(X, \mathcal{T}_2, \mathbb{E})$ are soft T_0. In fact, $(X, \mathcal{T}_1, \mathbb{E})$ is a not a soft T_0-space since for $x = (h_{1_{e_1}}, \mathbb{E})$ and $y = (h_{2_{e_2}}, \mathbb{E})$, in $\mathcal{T}_1$, there is no soft open set that soft contains x but not y and no soft open set that soft contains y but not x. Similarly, $(X, \mathcal{T}_2, \mathbb{E})$ is a not a soft T_0-space since for $x = (h_{2_{e_1}}, \mathbb{E})$ and $y = (h_{2_{e_2}}, \mathbb{E})$, in $\mathcal{T}_2$, there is no soft open set that soft contains x but not y and no soft open set that soft contains y but not x.

Proposition 4.17. If the soft N-topological space $(X, \mathcal{T}_i, N, \mathbb{E})$ over X is an N-wise soft T_0-space, then the supremum soft topological space $\left(X, \bigvee_{i=1}^{N} \mathcal{T}_i, \mathbb{E}\right)$ is a soft T_0-space.

Proof. Suppose that $(X, \mathcal{T}_i, N, \mathbb{E})$ is an N-wise soft T_0-space. Then, for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$, by Definition 4.13 and Remark 4.4 there exists some soft N-open set $(A, \mathbb{E}) \in \bigcup_{i=1}^{N} \mathcal{T}_i$ such that $(x_\alpha, \mathbb{E}) \tilde{\in} (A, \mathbb{E})$

and $(y_\beta, \mathbb{E})\tilde{\notin}(A, \mathbb{E})$, or there exists some soft N-open set $(B, \mathbb{E}) \in \bigcup_{i=1}^{N} \mathcal{T}_i$ such that $(y_\beta, \mathbb{E})\tilde{\in}(B, \mathbb{E})$ and $(x_\alpha, \mathbb{E})\tilde{\notin}(B, \mathbb{E})$. Since, by Definition 3.9, the supremum soft topology $\bigvee_{i=1}^{N} \mathcal{T}_i$ of all soft topologies $\mathcal{T}_i$ (with $i = 1, \ldots N$) is the smallest soft topology on X containing the union $\bigcup_{i=1}^{N} \mathcal{T}_i$, we have that both $(A, \mathbb{E})$ and $(B, \mathbb{E})$ also belong to $\bigvee_{i=1}^{N} \mathcal{T}_i$ and hence that the supremum soft topological space $\left(X, \bigvee_{i=1}^{N} \mathcal{T}_i, \mathbb{E}\right)$ is a soft T_0-space. □

Remark 4.18. The converse of Proposition 4.14 is not true in general, that is can exist an N-wise soft T_0-space such that the supremum soft topological space of its soft topologies is not a soft T_0-space, as can be seen in the following counterexample.

Example 4.19. Let $X = \{h_1, h_2, h_3\}$ be the universe set, $\mathbb{E} = \{e_1, e_2\}$ be the set of parameters and consider the soft 2-topological space $(X, \mathcal{T}_i, 2, \mathbb{E})$ defined in the Example 4.16. The supremum soft topology $\mathcal{T}_1 \vee \mathcal{T}_2$, being the smallest soft topology over X containing the crisp union $\mathcal{T}_1 \cup \mathcal{T}_2$, results to be:

$$\begin{aligned} \mathcal{T}_1 \vee \mathcal{T}_2 = \Big\{ & (\tilde{\emptyset}, \mathbb{E}), (F_1, \mathbb{E}), (F_2, \mathbb{E}), (F_3, \mathbb{E}), (F_4, \mathbb{E}), \\ & (F_5, \mathbb{E}), (F_6, \mathbb{E}), (F_7, \mathbb{E}), (F_8, \mathbb{E}), (F_9, \mathbb{E}), (\tilde{X}, \mathbb{E}) \Big\} \end{aligned}$$

where the new soft open sets $(F_i, \mathbb{E})$ (with $i = 6, \ldots 9$) are:

$$\begin{array}{llll} (F_6, \mathbb{E}) = (F_1, \mathbb{E})\tilde{\cup}(F_2, \mathbb{E}) & \text{defined by} & F_6(e_1) = \{h_1\}, & F_6(e_2) = \{h_2, h_3\}, \\ (F_7, \mathbb{E}) = (F_1, \mathbb{E})\tilde{\cup}(F_3, \mathbb{E}) & \text{defined by} & F_7(e_1) = \{h_1, h_3\}, & F_7(e_2) = X, \\ (F_8, \mathbb{E}) = (F_1, \mathbb{E})\tilde{\cap}(F_2, \mathbb{E}) & \text{defined by} & F_8(e_1) = \{h_1\}, & F_8(e_2) = \emptyset, \\ (F_9, \mathbb{E}) = (F_1, \mathbb{E})\tilde{\cap}(F_3, \mathbb{E}) & \text{defined by} & F_9(e_1) = \emptyset, & F_9(e_2) = \{h_2\}. \end{array}$$

However, the supremum soft topological space $(X, \mathcal{T}_1 \vee \mathcal{T}_2, \mathbb{E})$ is not a soft T_0-space, since for $x = (h_{1e_2}, \mathbb{E})$ and $y = (h_{3e_2}, \mathbb{E})$, in $\mathcal{T}_1 \vee \mathcal{T}_2$, there is no soft open set that soft contains x but not y and no soft open set that soft contains y but not x.

Although the following result can be directly proven, it is interesting to note that it can be achieved from Proposition 4.14 and Proposition 4.17.

Corollary 4.20. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X. If at least one soft topological space $(X, \mathcal{T}_j, \mathbb{E})$ (for some $j = 1, \ldots N$) is a soft T_0-space, then the supremum soft topological space $\left(X, \bigvee_{i=1}^{N} \mathcal{T}_i, \mathbb{E}\right)$ is a soft T_0-space.

The N-wise soft T_0 property is hereditary. In fact, we have the following proposition.

Proposition 4.21. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be an N-wise soft T_0-space over X and Y be a nonempty subset of X. Then the soft N-topological subspace $(Y, \mathcal{T}_{iY}, N, \mathbb{E})$ of $(X, \mathcal{T}_i, N, \mathbb{E})$ on Y is an N-wise soft T_0-space.

Proof. Suppose that $(X, \mathcal{T}_i, N, \mathbb{E})$ is an N-wise soft T_0-space and let Y be a nonempty subset of X. Then, for every pair of distinct soft points $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E}) \in \mathcal{SP}(Y)_{\mathbb{E}}$, it follows, in particular, that $(x_\alpha, \mathbb{E}), (y_\beta, \mathbb{E})$ are soft points in X too. So, by hypothesis, there exists some soft N-open set $(A, \mathbb{E})$ such that $(x_\alpha, \mathbb{E})\tilde{\in}(A, \mathbb{E})$ and $(y_\beta, \mathbb{E})\tilde{\notin}(A, \mathbb{E})$, or there exists some soft N-open set $(B, \mathbb{E})$ such that $(y_\beta, \mathbb{E})\tilde{\in}(B, \mathbb{E})$ and $(x_\alpha, \mathbb{E})\tilde{\notin}(B, \mathbb{E})$. Thus, by Definition 4.3, there exists some $j, k \in \{1, \ldots N\}$ such that $(A, \mathbb{E}) \in \mathcal{T}_j$ and $(B, \mathbb{E}) \in \mathcal{T}_k$. Hence, by Remark 2.44 and Definition 3.22, it follows that the soft intersections $(A,\mathbb{E})\tilde{\cap}(\tilde{Y},\mathbb{E}) = ({}^{Y}A,\mathbb{E})$ and $(B,\mathbb{E})\tilde{\cap}(\tilde{Y},\mathbb{E}) = ({}^{Y}B,\mathbb{E})$ belong to the soft relative topologies $(Y,\mathcal{T}_{j_Y},\mathbb{E})$ and $(Y,\mathcal{T}_{kY},\mathbb{E})$ respectively and they are such that $(x_\alpha,\mathbb{E})\tilde{\in}({}^{Y}A,\mathbb{E})$ and $(y_\beta,\mathbb{E})\tilde{\notin}({}^{Y}A,\mathbb{E})$, or such that $(y_\beta,\mathbb{E})\tilde{\in}({}^{Y}B,\mathbb{E})$ and $(x_\alpha,\mathbb{E})\tilde{\notin}({}^{Y}B,\mathbb{E})$. Finally, since by Definition 4.11, we have that both $({}^{Y}A,\mathbb{E})$ and $({}^{Y}B,\mathbb{E})$ belong to the relative soft N-topological space $(Y,\mathcal{T}_{iY},N,\mathbb{E})$, we have that Definition 4.13 holds, and so that the soft N-topological subspace $(Y,\mathcal{T}_{iY},N,\mathbb{E})$ is an N-wise soft T_0-space. $\square$

Remark 4.22. The converse of Proposition 4.21 is false, i.e. can exist a non N-wise soft T_0-space $(X,\mathcal{T}_i,N,\mathbb{E})$ which, for some $Y \subset X$, has a N-wise soft T_0-subspace $(Y,\mathcal{T}_{iY},N,\mathbb{E})$ as it is shown in the following counterexample.

Example 4.23. Let $X = \{h_1,h_2,h_3\}$ be the universe set and $\mathbb{E} = \{e_1,e_2\}$ be the set of parameters. Consider the soft 2-topological space $(X,\mathcal{T}_i,2,\mathbb{E})$ built by the following $N = 2$ soft topologies over X:

$$\begin{aligned}\mathcal{T}_1 &= \left\{(\tilde{\emptyset},\mathbb{E}),(\tilde{X},\mathbb{E}),(F_1,\mathbb{E})\right\},\\ \mathcal{T}_2 &= \left\{(\tilde{\emptyset},\mathbb{E}),(\tilde{X},\mathbb{E}),(F_2,\mathbb{E}),(F_3,\mathbb{E})\right\}\end{aligned}$$

where the soft sets $(F_i,\mathbb{E})$ (with $i = 1,\dots 3$) over X are respectively defined by setting:

$$\begin{aligned} &F_1(e_1) = \{h_1\}, && F_1(e_2) = \{h_2\},\\ &F_2(e_1) = \{h_2\}, && F_2(e_2) = \{h_1\},\\ &F_3(e_1) = \{h_2,h_3\}, && F_3(e_2) = \{h_1,h_2\}.\end{aligned}$$

Then $(X,\mathcal{T}_i,2,\mathbb{E})$ is not an N-wise soft T_0-space, since for $x = (h_{2_{e_1}},\mathbb{E})$ and $y = (h_{1_{e_2}},\mathbb{E})$, there is no soft N-open set that soft contains x but not y and no soft N-open set that soft contains y but not x.

However, if we consider the subset $Y = \{h_1,h_3\}$ of X, the sub soft sets $({}^{Y}F_i,\mathbb{E})$ (with $i = 1,\dots 3$) of the soft open set $(F_i,\mathbb{E})$ over Y results to be defined by:

$$\begin{aligned} &{}^{Y}F_1(e_1) = \{h_1\}, && {}^{Y}F_1(e_2) = \emptyset,\\ &{}^{Y}F_2(e_1) = \emptyset, && {}^{Y}F_2(e_2) = \{h_1\},\\ &{}^{Y}F_3(e_1) = \{h_3\}, && {}^{Y}F_3(e_2) = \{h_1\}.\end{aligned}$$

Thus, the soft 2-topological subspace $(Y,\mathcal{T}_{iY},2,\mathbb{E})$ of $(X,\mathcal{T}_i,2,\mathbb{E})$ on Y is formed by the following soft relative topologies $\mathcal{T}_{iY}$ of $\mathcal{T}_i$ on Y (with $i = 1,\dots 2$):

$$\begin{aligned}\mathcal{T}_{1Y} &= \left\{(\tilde{\emptyset},\mathbb{E}),(\tilde{Y},\mathbb{E}),({}^{Y}F_1,\mathbb{E})\right\}\\ \mathcal{T}_{2Y} &= \left\{(\tilde{\emptyset},\mathbb{E}),(\tilde{Y},\mathbb{E}),({}^{Y}F_2,\mathbb{E}),({}^{Y}F_3,\mathbb{E})\right\}\end{aligned}$$

and it is trivially checked that it is an N-wise soft T_0-space.

Definition 4.24. A soft N-topological space $(X,\mathcal{T}_i,N,\mathbb{E})$ over X is called an ***N-wise soft T_1-space*** if for every pair of distinct soft points $(x_\alpha,\mathbb{E}),(y_\beta,\mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ there exists a soft N-open set $(A,\mathbb{E})$ which soft contains one soft point but not the other one, that is $(x_\alpha,\mathbb{E})\tilde{\in}(A,\mathbb{E})$ and $(y_\beta,\mathbb{E})\tilde{\notin}(A,\mathbb{E})$.

Definition 4.25. A soft N-topological space $(X,\mathcal{T}_i,N,\mathbb{E})$ over X is called an ***N-wise soft T_2-space*** or ***N-wise soft Hausdorff space*** if for every pair of distinct soft points $(x_\alpha,\mathbb{E}),(y_\beta,\mathbb{E}) \in \mathcal{SP}(X)_{\mathbb{E}}$ there exist two soft N-open sets $(A,\mathbb{E}),(B,\mathbb{E})$ which are soft disjoints and soft contain the two soft points respectively, that is $(x_\alpha, \mathbb{E})\tilde{\in}(A, \mathbb{E})$, $(y_\beta, \mathbb{E})\tilde{\in}(B, \mathbb{E})$ and $(A, \mathbb{E})\tilde{\cap}(B, \mathbb{E}) = (\tilde{\emptyset}, \mathbb{E})$.

Proposition 4.26. Every N-wise soft T_2-space is an N-wise soft T_1-space and every N-wise soft T_1-space is an N-wise soft T_0-space

Proof. It easily follows from Definitions 4.13, 4.24 and 4.25. □

Proposition 4.27. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be a soft N-topological space over X. If at least one soft topological space $(X, \mathcal{T}_j, \mathbb{E})$ (for some $j = 1, \dots N$) is a soft T_1-space (respectively a soft T_2-space), then $(X, \mathcal{T}_i, N, \mathbb{E})$ is an N-wise soft T_1-space (resp. an N-wise soft T_2-space).

Proof. Similar to the proof of Proposition 4.14. □

Proposition 4.28. If the soft N-topological space $(X, \mathcal{T}_i, N, \mathbb{E})$ over X is an N-wise soft T_1-space (respectively an N-wise soft T_2-space), then the supremum soft topological space $\left(X, \bigvee_{i=1}^{N} \mathcal{T}_i, \mathbb{E}\right)$ is a soft T_1-space (resp. a soft T_2-space).

Proof. Similar to the proof of Proposition 4.17. □

Proposition 4.29. Let $(X, \mathcal{T}_i, N, \mathbb{E})$ be an N-wise soft T_1-space (respectively an N-wise soft T_2-space) over X and Y be a nonempty subset of X. Then the soft N-topological subspace $(Y, \mathcal{T}_{iY}, N, \mathbb{E})$ on Y is an N-wise soft T_1-space (resp. an N-wise soft T_2-space).

Proof. Similar to the proof of Proposition 4.21. □

5. Conclusion

We have defined the new notion of soft N-topological space as generalization both of the concepts of soft topological space given by Shabir and Naz [33] and that of N-topological space as introduced by Tawfiq and Majeed [35] and, independently, by Khan [12] and we have investigated some basic properties of such class of spaces with particular regard to its subspaces and to the parameterized families of crisp topologies generated by the soft N-space. We have also introduced and studied some new separation axioms called N-wise soft T_0, N-wise soft T_1, and N-wise soft T_2.

This paper is just a beginning of the investigation of a new kind of structure. So, it will be necessary to continue the study and carry out more theoretical research in order to build a general framework for practical applications.

References

[1] N. Çağman, Contributions to the theory of soft sets, Journal of New Results in Science, 4(2014) **4**, 33-41.

[2] M. Chiney and S.K. Samanta, Soft topology redefined, arXiv:1701.00466 [math.GM], Dec 2016.

[3] S. Das and S.K. Samanta, Soft metric, Annals of Fuzzy Mathematics and Informatics, 6(2013), no. 1, 77-94.

[4] D. N. Georgiou, A. C. Megaritis and V. I. Petropoulos, On soft topological spaces, Applied Mathematics & Information Sciences, 7(2013), no. 5, 1889-1901.

[5] A. F. Hassan, Soft tritopological spaces, International Journal of Computer Applications, 176(2017), no. 9, 26-30.

[6] H. Hazra, P. Majumdar and S.K. Samanta, Soft topology, Fuzzy information and Engineering, 4(2012), no. 1, 105-115.
[7] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Computers & Mathematics with Applications, 62(2011), no. 11, 4058-4067.
[8] S. Hussain and B. Ahmad, Soft separation axioms in soft topological spaces, Hacettepe Journal of Mathematics and Statistics, 44(2015), no. 3, 559-568.
[9] B. M. Ittanagi, Soft Bitopological spaces, International Journal of Computer Applications, 107(2014), no. 7, 1-4.
[10] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc., 13(1963), 71-81.
[11] M. A. K. Al-Khafaj and M.H. Mahmood, Some Properties of Soft Connected Spaces and Soft Locally Connected Spaces, IOSR Journal of Mathematics, 10(2014), no. 5, 102-107.
[12] K. A. Khan, On the possibility of N-topological space, International Journal of Mathematical Archive, 3(2012), no. 7, 2520-2523.
[13] A. M. Khattak, A. Zaighum, M. Ishfaq, S. Jbeen and I. Hussain, Some contribution of soft regular open sets to soft separation axioms in soft quad topological spaces, Journal of New Theory, 19(2017), 82-107.
[14] A.M. Khattak, S. Abdullah, A. Zaighum, F. Jamal, M.A. Rana, Z. Anjum, M. Zamir and M. Ishfaq, Characterization of soft semi separation axioms in soft quad topological spaces, Iraqi Journal of Science, 59(2018), 552-563.
[15] A. Luay, Tritopological spaces, Journal of Babylon University, 9(2003), 33-45.
[16] Z. Ma, B. Yang and B. Hu, Soft set theory based on its extension, Fuzzy Information and Engineering, 2(2010), no. 4, 423-432.
[17] P. K. Maji and A. R. Roy, An application of soft sets in a decision making problem, Computers & Mathematics with Applications, 44(2002), 1077-1083.
[18] P. K. Maji, R. Biswas R and A. R. Roy, Soft set theory, Computers & Mathematics with Applications, 45(2003), 555-562.
[19] M. Matejdes, Soft topological questions and answers, International Journal of Pure and Applied Mathematics, 104(2015), no. 2, 237-247.
[20] M. Matejdes, Some remarks on soft separation axioms, International Journal of Pure and Applied Mathematics, 116(2017), no. 1, 191-195.
[21] A. Mehmood, N. Hanif, F. Nadeem, M. Zamir, C. Park, G. Nordo and S. Jabeen, Soft b-separation axioms in neutrosophic soft topological structures, Annals of Fuzzy Mathematics and Informatics, 18(2019), no. 1, 93-105.
[22] A. Mehmood, G. Nordo, F. Nadeem, H. Kalsoom, S. Jabeen and M. Gul., Neutrosophic Soft s-Open Sets, Sohag Journal of Mathematics, 7(2020), no. 3, 53-61.
[23] A. Mehmood, F. Nadeem, C. Park, G. Nordo, H. Kalsoom, M. R. Khan and N. Abbas, Neutrosophic Soft alpha-Open Set in Neutrosophic Soft Topological Spaces, Journal of Algorithms and Computation, 52(2020), no. 1, 37-66.
[24] A. Mehmood, F. Nadeem, G. Nordo, M. Zamir, C. Park, H. Kalsoom, S. Jabeen and M. I. Khan, Generalized Neutrosophic Separation Axioms in Neutrosophic Soft Topological Spaces, Neutrosophic Sets and Systems, 32(2020), 38-51.
[25] A. Mehmood, S. A. Ghour, F. Afzal, G. Nordo and N. Saleem, Comprehensive Note on Characterization of Neutrosophic Soft P-open Sets in Neutrosophic Soft Quad-topological Space, Journal of Intelligent & Fuzzy Systems, 43(2022), no. 1, 1519-1540.
[26] W. K. Min, A note on soft topological spaces, Computers & Mathematics with Applications, 62(2011), 3524-3528.
[27] D. Molodtsov, Soft set theory – first result, Computers & Mathematics with Applications, 37(1999), 19-31.
[28] D. V. Mukundan, Introduction to quad topological spaces (4-tuple topology), International Journal of Scientific & Engineering Research, 4(2013), no. 7, 2483-2485.

[29] M. Naz, M. Shabir and M. I. Ali, Separation axioms in bi-soft topological spaces, eprint arXiv:1509.00866 [math.GN], Aug 2015.

[30] S. Nazmul and S. K. Samanta, Neighborhood properties of soft topological spaces, Annals of Fuzzy Mathematics and Informatics, 6(2013), no. 1, 1-15.

[31] T. J. Neog and D. K. Sut, A new approach to the theory of soft sets, International Journal of Computer Applications, 32(2011), no. 2, 1-6.

[32] Z. Pawlak, Hard and soft sets, In: Proceedings of the RSKD'93 International Workshop on Rough Sets and Knowledge Discovery: Rough Sets, Fuzzy Sets and Knowledge Discovery, 1994, London: Springer-Verlag, pp. 130-135.

[33] M. Shabir and M. Naz., On soft topological spaces, Computers & Mathematics with Applications, 61(2011), 1786-1799.

[34] F. G. Shi and B. Pang, A note on soft topological spaces, Iranian Journal of Fuzzy Systems, 12(2015), no. 5, 149-155.

[35] L. N. M. Tawfiq and R.N. Majeed, N-topological space and its applications in artificial neural networks, Ibn al-Haytham Journal for Pure & Applied Science, 24(2011), no. 1, 1-11.

[36] M. L. Thivagar, V. Ramesh and A. Dasan, On new structure of N-topology, Cogent Math. 3 (2016), Art. ID 1204104, 10 pp.

[37] B. P. Varol and H. Aygün, On soft Hausdorff spaces, Annals of Fuzzy Mathematics and Informatics, 5(2013), no. 1, 15-24.

[38] N. Xie, Soft points and the structure of soft topological spaces, Annals of Fuzzy Mathematics and Informatics, 10(2015), no. 2, pp. 309-322.

[39] L. A. Zadeh, Fuzzy sets, Information and Control, 8(1965), no. 3, 338-353.

[40] İ. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Annals of Fuzzy Mathematics and Informatics, 3(2012), no. 2, 171-185.

(Giorgio Nordo) Dipartimento di Scienze Matematiche e Informatiche, scienze Fisiche e scienze della Terra (MIFT), Messina University, Messina, Italy

E-mail address: `giorgio.nordo@unime.it`