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THE ERROR CONVERGENCE OF THE WEAK GALERKIN METHOD FOR TWO-DIMENSIONAL NON-LINEAR CONVECTION-DIFFUSION PROBLEM

ALA' N. ABDULLAH† AND HASHIM A. KASHKOOL

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Abstract . This paper presents the fully-discrete scheme for the solution of two-dimensional non-linear convection diffusion equations by using the Crank-Nicolson-Weak Galerkin finite element methods. We introduce and analyze stability. The error estimate and an optimal order of (L^2 and H^1)-norm are proved. We confirm the theoretical results with some numerical examples.

1. Introduction

The examination of convection-diffusion problems has piqued the interest of a growing number of specialists, engineers, and mathematicians. Due to challenges with proper resolution of the so-called boundary layers, special emphasis is dedicated to problems with convection dominating diffusion. Upwinding or artificial viscosity are commonly used in efficient finite difference or finite element methods. We consider the following the Dirichlet problem for non-linear convection-diffusion, which seeks an unknown function $u = u(x, t)$ satisfying [12]:

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(u)\nabla u) + b(\vec{u}) \cdot \nabla u + cu = f; \quad \Omega \times (0, T], \tag{1.1}$$

$$u(x, t) = g; \quad \Gamma \times (0, T], \tag{1.2}$$

$$u(x, 0) = u^0(x); \quad x \in \overline{\Omega}, \tag{1.3}$$

in a domain $\Omega \subset R^2$ with boundary Γ, where $x = (x_1, x_2)$ and $f = f(x, t) \in \Omega \times (0, T]$, and the diffusion coefficient and nonlinear function are denoted by $a(u)$, and the convection coefficient and nonlinear function are denoted by $b(u)$, and $c = c(x)$ is a

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† *Corresponding author*

scalar function on Ω. Suppose that the matrix $a(u)$ satisfies the following property: There exists a constant $\alpha > 0$ such that

$$\alpha \xi^T \xi \leq \xi^T a(u) \xi, \forall \xi \in R^2.$$

where ξ^T is the transpose of vector ξ. Numerical simulations and analysis for the problem (1.1 - 1.3) have been extensively studied in recent decades.

Liu and Liu [14] introduced the numerical identification of diffusion parameters in non-linear convection-diffusion equations. Chan et al [2] presented a dual Petrov-Galerkin finite element method for convection-diffusion problems of a higher order. Gao and Yuan [7] adopted an upwind finite volume element method for non-linear convection-diffusion problems. Fatehi et al [6] introduced a modified finite-volume Eulerian-Lagrangian localized adjoint method extended to non-linear convection-diffusion problems. Havle et al [9] applied the analysis of the discontinuous Galerkin finite element method to the space semidiscretization of a non-linear nonstationary convection-diffusion problem. Dolejší et al [5] applied the analysis of the discontinuous Galerkin finite element method to a non-linear convection-diffusion problem. Cockburn and Shu [4] studied the local discontinuous Galerkin method for non-linear time-dependent convection-diffusion systems. Xikui and Wenhua [19] presented the finite element procedure for the solution of a transient, multidimensional convection-diffusion problem. Salhi et al [16]presented the Galerkin-characteristic finite element method for the numerical solution of a time-dependent convection-diffusion problem.

In 2013, Wang and Ye [18] introduced the first weak Galerkin finite element method for the solution of general elliptic equations. Since then, there has been an active application of weak Galerkin methods for linear and non-linear convection-diffusion problems. Li and Wang [13] proposed a newly developed weak Galerkin method for the solution of a parabolic problem. Gao et al [8] presented numerical schemes based on the weak Galerkin finite element method for one class of Sobolev problems. Zhang and Lin[21] presented and analyzed the weak Galerkin finite element method for stationary Navier-Stokes equations. Cheichan [3] applied the weak Galerkin finite element method for the solution of the one-dimensional and two-dimensional nonlinear convection-diffusion equations and the two-dimensional linear convection-diffusion problem, respectively. Hussein [11] applied the weak Galerkin method for the solution of coupled Burger's equations (one-dimensional and two-dimensional). Song et al [17] introduced an over-penalized weak Galerkin finite element method for elliptic interface problems with non-homogeneous boundary conditions and discontinuous coefficients. In this paper, we present a detailed study of a weak Galerkin method for solving non-linear convection-diffusion problems, combined with the Crank-Nicolson scheme. Abdullah and Kashkool [1] described a nonlinear system of two couple PDE models for two-dimensional incompressible immiscible displacement fluids in porous media by the weak Galerkin mixed finite element method.

The rest of the paper is organized as follows. In Section 2, we will introduce the variational form for non-linear convection-diffusion equations using the weak Galerkin method. In Section 3, we will prove the stability of the method. In Section 4, we will give an error equation and show convergence and optimal rates in H^1- norm and L^2- norm in the view of non-linear assumptions for the convection-diffusion problems. In Section 5, we report numerical results to validate our theoretical results.

2. Preliminary

In this paper, we used the standard definitions and notations for the Sobolev spaces. In this section, we concentrate on the preliminaries, including some definitions of weak function, weak gradient, weak divergent, and approximation spaces.

The space of weak function associated with K [18]

$$W(K) = \left\{ v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K) \right\},$$

Let τ_h be a shaped regular and body-fitted partition of Ω (see [17]). Denote by $P_k(T)$ the set of polynomials on T with degree on more than k, and $P_k(e)$ the set of polynomials on e with degree no $e \in \mathcal{E}$ with degree no more than k, where $\mathcal{E}$ by a set of all edges in τ_h, in particular.

For any weak function $\omega = \{\omega^0, \omega^b\}$ on a polygon K with boundary ∂K satisfying $\omega^0 \in L^2(K), \omega^b \in H^{\frac{1}{2}}(\partial K)$, we define its weak gradient[18] denote by $\nabla_d \omega$, in the dual space of $H(div, K)$ and weak divergent[20] denote by $\nabla_d \cdot \omega$ in the dual space of $H^1(K)$, respectively.

$$\begin{aligned}
(\nabla_d \omega, q) &:= -(\omega^0 \nabla \cdot q)_K + \langle \omega^b q \cdot n \rangle_{\partial K}; \forall q \in H(div, K), \\
(\nabla_d \cdot \omega, \psi) &:= -(\omega^0, \nabla \psi)_K + \langle \omega^b \cdot n, \psi \rangle_{\partial K}; \forall \psi \in H^1(K),
\end{aligned}$$

where n is the outward normal direction to ∂K. Based on a weak function, the weak Galerkin finite element space is defined by

$$V_h := \{\omega = \{\omega^0, \omega^b\} : \omega^0 \in P_k(T), \omega^b \in P_k(e), \forall \; edge \; e \, \forall \; \partial T, T \in \tau_h\},$$

furthermore

$$V_h^0 = \{\{\omega^0, \omega^b\} \in V_h, \omega^b|_{\partial T \cap \partial \Omega} = 0, \forall \; T \in \tau_h\}.$$

Assume that $u \in H^{k+1}(\Omega); k \geq 1$ the exact solution of (1.1)-(1.3), and the following L^2 projections are introduced

$$\begin{aligned}
&Q_h : L^2(T) \to P_k(T); \forall T \in \tau_h, \\
&R_h : [L^2(T)]^2 \to [P_{k-1}(T)]^2; \forall T \in \tau_h.
\end{aligned}$$

3. The Fully-Discrete Weak Galerkin Method

The weak variational formulation for equations (1.1-1.3). Find $u = \{u_0, u_b\} \in W(K)$ and $v \in W(K)$ such that

$$\begin{aligned}
(u_{0,t}, v_0) + (a(u)\nabla_d u, \nabla_d v) - (u_0 b(u_0), \nabla_d v) - (u_0 \nabla_d b(u), v_0) + (cu_0, v_0) &= (f, v_0), \\
(u_0(x, 0), v) &= (u_0, v_0).
\end{aligned}$$

In a fully-discrete scheme, the time space $[0, T]$ is then uniformly discretized by $n+1$ points $t_n = N\tau, N = 0, 1, \cdots, n$, where the time step size is $\tau = T/n$. A fully-discrete weak Galerkin finite element method can be constructed as: For any $N \geq 1$. The Crank-Nicolson-weak Galerkin finite element solution. Find $u_h^{N+1} = (u_h^{N+1,0}, u_h^{N+1,b}) \in V_h$

satisfying

$$
\begin{aligned}
& (\frac{u_h^{N+1,0}-u_h^{N,0}}{\tau}, v^0) + (a(u_h^{N,0})\nabla_{d,r}\frac{u_h^{N+1}+u_h^N}{2}, \nabla_{d,r}v) + (c\frac{u_h^{N+1,0}+u_h^{N,0}}{2}, v^0) \\
- \;& (\frac{u_h^{N+1,0}+u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r}v) - (\frac{u_h^{N+1,0}+u_h^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0) \\
= \;& (\frac{f^{N+1}+f^N}{2}, v^0), \forall v \in V_h^0. \qquad (3.1)
\end{aligned}
$$

3.1. Stability. In this section, we show the stability of the Crank-Nicolson-weak Galerkin finite element by the following lemma:

Lemma 3.1. *For the solution u_h^{N+1} of equation (3.1) with a positive constant C_1, then*

$$
\begin{aligned}
\|u_h^{N+1}\|^2 \;\le\; & C_1 h\Big[(1+\tau+\tau^2)\|u_h^{N+1,0}\|^2 + (1+\tau^2)\|u_h^{N,0}\|^2 + \|(f^{N+1}+f^N)\|^2 \\
+ \;& \tau^2\|\nabla_{d,r}u_h^N\|^2\Big]. \qquad (3.2)
\end{aligned}
$$

Proof. In equation (3.1), select $v = u_h^{N+1}$

$$
\begin{aligned}
& ((u_h^{N+1,0}-u_h^{N,0}), u_h^{N+1,0}) + (\tau a(u_h^{N,0})\nabla_{d,r}\frac{u_h^{N+1}+u_h^N}{2}, \nabla_{d,r}u_h^{N+1}) \\
- \;& (\tau\frac{u_h^{N+1,0}+u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r}u_h^{N+1}) - (\tau\frac{u_h^{N+1,0}+u_h^{N,0}}{2}\nabla_{d,r}b(u_h^N), u_h^{N+1,0}) \\
+ \;& (\tau c\frac{u_h^{N+1,0}+u_h^{N,0}}{2}, u_h^{N+1,0}) = (\tau\frac{f^{N+1}+f^N}{2}, u_h^{N+1,0}).
\end{aligned}
$$

Suppose the above equation is

$$
\sum_{i=1}^{5} A^{(i)} = A^{(6)}. \qquad (3.3)
$$

Using the inequality $(\alpha-\beta,\alpha) \ge \frac{1}{2}(\|\alpha\|^2 - \|\beta\|^2)$ [10], we have

$$
|A^{(1)}| \ge \frac{1}{2}(\|u_h^{N+1,0}\|^2 - \|u_h^{N,0}\|^2).
$$

By using Cauchy-Schwarz inequality and Young's inequality for the above equation, we get

$$
\begin{aligned}
|A^{(2)}| \;=\; & |(\frac{a(u_h^{N,0})}{2}\nabla_{d,r}u_h^{N+1}, \nabla_{d,r}u_h^{N+1})| + |(\tau\frac{a(u_h^{N,0})}{2}\nabla_{d,r}u_h^N, \nabla_{d,r}u_h^{N+1})| \\
\le \;& \frac{\|a(u_h^{N,0})\|}{2}\|\nabla_{d,r}u_h^{N+1}\|^2 + \frac{\tau^2\|a(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}u_h^N\|^2 + \|\nabla_{d,r}u_h^{N+1}\|^2 \\
\le \;& Ch^{-1}\|u_h^{N+1}\|^2 + \frac{\tau^2\|a(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}u_h^N\|^2.
\end{aligned}
$$

$$
\begin{aligned}
|A^{(3)}| \;=\; & |(\tau\frac{b(u_h^{N,0})}{2}u_h^{N+1,0}, \nabla_{d,r}u_h^{N+1})| + |(\tau\frac{b(u_h^{N,0})}{2}u_h^{N,0}, \nabla_{d,r}u_h^{N+1})| \\
\le \;& \frac{2\|b(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}u_h^{N+1}\|^2 + \tau^2\|u_h^{N+1,0}\|^2 + \tau^2\|u_h^{N,0}\|^2 \\
\le \;& \alpha_1 h^{-1}\|u_h^{N+1}\|^2 + \tau^2\|u_h^{N+1,0}\|^2 + \tau^2\|u_h^{N,0}\|^2.
\end{aligned}
$$

$$
\begin{aligned}
|A^{(4)}| &= |(\tau\frac{\nabla_{d,r}b(u_h^N)}{2}u_h^{N+1,0}, u_h^{N+1,0})| + |(\tau\frac{\nabla_{d,r}b(u_h^N)}{2}u_h^{N,0}, u_h^{N+1,0})| \\
&\leq \frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}\|u_h^{N+1,0}\|^2 + \frac{\tau^2\|\nabla_{d,r}b(u_h^N)\|^2}{16}\|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2 \\
&\leq \alpha_2(\tau+\tau^2)\|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2.
\end{aligned}
$$

$$
\begin{aligned}
|A^{(5)}| &= |(\tau\frac{c}{2}u_h^{N+1,0}, u_h^{N+1,0})| + |(\tau\frac{c}{2}u_h^{N,0}, u_h^{N+1,0})| \\
&\leq \frac{\tau\|c\|}{2}\|u_h^{N+1,0}\|^2 + \frac{\tau^2\|c\|^2}{16}\|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2 \\
&\leq \alpha_3(\tau+\tau^2)\|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2.
\end{aligned}
$$

$$
|A^{(6)}| \leq \frac{\tau^2}{16}\|u_h^{N+1,0}\|^2 + \|(f^{N+1}+f^N)\|^2.
$$

Then the above equation becomes

$$
\begin{aligned}
& \frac{1}{2}(\|u_h^{N+1,0}\|^2 - \|u_h^{N,0}\|^2) + (C+\alpha_1)h^{-1}\|u_h^{N+1}\|^2 + \frac{\tau^2\|a(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}u_h^N\|^2 \\
+ & \tau^2\|u_h^{N+1,0}\|^2 + (\tau^2+1)\|u_h^{N,0}\|^2 + \alpha_2(\tau+\tau^2)\|u_h^{N+1,0}\|^2 + \alpha_3(\tau+\tau^2)\|u_h^{N+1,0}\|^2 \\
+ & \|u_h^{N,0}\|^2 \leq \frac{\tau^2}{16}\|u_h^{N+1,0}\|^2 + \|(f^{N+1}+f^N)\|^2,
\end{aligned}
$$

so,

$$
\begin{aligned}
(C+\alpha_1)h^{-1}\|u_h^{N+1}\|^2 &\leq (\frac{\tau^2}{16} - \frac{1}{2} - \tau^2 - \alpha_2(\tau+\tau^2) - \alpha_3(\tau+\tau^2))\|u_h^{N+1,0}\|^2 \\
&+ \|(f^{N+1}+f^N)\|^2 - \frac{\tau^2\|a(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}u_h^N\|^2 \\
&- (\frac{3}{2}+\tau^2)\|u_h^{N,0}\|^2,
\end{aligned}
$$

then

$$
\begin{aligned}
\|u_h^{N+1}\|^2 &\leq C_1 h\Big[(1+\tau+\tau^2)\|u_h^{N+1,0}\|^2 + (1+\tau^2)\|u_h^{N,0}\|^2 + \|(f^{N+1}+f^N)\|^2 \\
&+ \tau^2\|\nabla_{d,r}u_h^N\|^2\Big],
\end{aligned}
$$

where $C_1 = max\{-\frac{1}{2}, -(\alpha_2+\alpha_3), -(\frac{15}{16}+\alpha_2+\alpha_3), \frac{3}{2}, 1, \frac{-\|a(u_h^{N,0})\|^2}{16}\}/(C+\alpha_1)$.

□

4. Error Analysis of the Fully-Discrete Weak Galerkin Finite Element Scheme

In this section, we derive an error estimate for the weak Galerkin finite element method (3.1). Let us begin with the derivation of an error equation for a Crank-Nicolson-weak Galerkin approximation u_h^{N+1} and the L^2 projection of the exact solution u^{N+1} in the weak finite element space V_h, which will prove the error estimate for L^2 norm and H^1 norm and so, the finality optimal order error estimate for each.

4.1. The Error Equation of the Fully-Discrete Scheme. An error equation is given by the following lemma 4.1, for analyzing the difference between an L^2 projection of the exact solution u^{N+1} of the original problem and a numerical solution u_h^{N+1} from the weak Galerkin finite element approximation. Let $e^{N+1} = Q_h u^{N+1} - u_h^{N+1} = \{Q_h u^{N+1,0} - u_h^{N+1,0}, Q_h u^{N+1,b} - u_h^{N+1,b}\} = \{e^{N+1,0}, e^{N+1,b}\}$.

Lemma 4.1. *Let u_h^{N+1} be the solution of (3.1). Then for each $v \in V_h^0$, the error equation is*

$$
\begin{aligned}
& (\frac{e^{N+1,0} - e^{N,0}}{\tau}, v^0) + (a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1} + e^N}{2}, \nabla_{d,r}v) + (c\frac{e^{N+1,0} + e^{N,0}}{2}, v^0) \\
- & (\frac{e^{N+1,0} + e^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r}v) - (\frac{e^{N+1,0} + e^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0) \\
= & (\frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u^{N+1,0}}{\partial t}, v^0).
\end{aligned} \tag{4.1}
$$

Proof. From the bilinear equation (3.1) by using the operator projection Q_h

$$
\begin{aligned}
& \sum_{K\in T_h} (a(u_h^{N,0})\nabla_{d,r}\frac{Q_h u^{N+1} + Q_h u^N}{2}, \nabla_{d,r}v)_K + (c\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, v^0)_K \\
- & (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r}v)_K - (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0)_K \\
= & (a(u_h^{N,0})\nabla_{d,r}\nabla_{d,r}\frac{u_h^{N+1} + u_h^N}{2}, v) - (\nabla_{d,r}\frac{u_h^{N+1,0} + u_h^{N,0}}{2} b(u_h^{N,0}), v) \\
- & (\frac{u_h^{N+1,0} + u_h^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0) + (c\frac{u_h^{N+1,0} + u_h^{N,0}}{2}, v^0) \\
= & -(\frac{\partial u^{N+1,0}}{\partial t}, v^0) + (\frac{f^{N+1} + f^N}{2}, v^0).
\end{aligned}
$$

Replacing $(\frac{f^{N+1}+f^N}{2}, v^0)$ in the equation (3.1), we have

$$
\begin{aligned}
& \sum_{K\in T_h} (a(u_h^{N,0})\nabla_{d,r}\frac{Q_h u^{N+1} + Q_h u^N}{2}, \nabla_{d,r}v)_K + (c\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, v^0)_K \\
- & (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r}v)_K - (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0)_K \\
= & -(\frac{\partial u^{N+1,0}}{\partial t}, v^0) + (\frac{u_h^{N+1,0} - u_h^{N,0}}{\tau}, v^0) + (a(u_h^{N,0})\nabla_{d,r}\frac{u_h^{N+1} + u_h^N}{2}, \nabla_{d,r}v) \\
- & (\frac{u_h^{N+1,0} + u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r}v) - (\frac{u_h^{N+1,0} + u_h^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0) \\
+ & (c\frac{u_h^{N+1,0} + u_h^{N,0}}{2}, v^0),
\end{aligned}
$$

which leads to the error equation for the Crank-Nicolson-weak Galerkin scheme (3.1):

$$\begin{aligned} & (\frac{e^{N+1,0}-e^{N,0}}{\tau}, v^0) + (a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1}+e^N}{2}, \nabla_{d,r}v) + (c\frac{e^{N+1,0}+e^{N,0}}{2}, v^0) \\ - & (\frac{e^{N+1,0}+e^{N,0}}{2}b(u_h^{N,0}), \nabla_{d,r}v) - (\frac{e^{N+1,0}+e^{N,0}}{2}\nabla_{d,r}b(u_h^N), v^0) \\ = & (\frac{Q_hu^{N+1,0}-Q_hu^{N,0}}{\tau} - \frac{\partial u^{N+1,0}}{\partial t}, v^0). \end{aligned}$$

□

Lemma 4.2. *For any u satisfying $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T)$, we have*

$$\left|\frac{u^{N+1}-u^N}{\tau} - \frac{\partial u^N}{\partial t}\right|^2 \le \frac{\tau}{12}\int_{t_N}^{t_{N+1}} \left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds. \tag{4.2}$$

Proof. By using Taylor expansions

$$u^{N+1} = u^N + \tau\frac{\partial}{\partial t}u^N + \frac{1}{2}\int_{t_N}^{t_{N+1}} \frac{\partial^2 u}{\partial t^2}\cdot(t_{N+1}-s)ds,$$

we have

$$\begin{aligned} \left|\frac{u^{N+1}-u^N}{\tau} - \frac{\partial u^N}{\partial t}\right|^2 &= \left|\frac{1}{2\tau}\int_{t_N}^{t_{N+1}} \frac{\partial^2 u}{\partial t^2}\cdot(t_{N+1}-s)ds\right|^2 \\ &\le \frac{1}{4\tau^2}\left(\int_{t_N}^{t_{N+1}} \left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds\right)\cdot\left(\int_{t_N}^{t_{N+1}} (t_{N+1}-s)^2\, ds\right) ds \\ &\le \frac{1}{4\tau^2}\left(\int_{t_N}^{t_{N+1}} \left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds\right)\cdot\frac{\tau^3}{3} = \frac{\tau}{12}\int_{t_N}^{t_{N+1}} \left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds, \end{aligned}$$

which conclude the proof.

□

4.2. L^2 Error Estimates of the Fully-Discrete Weak Galerkin Finite Element Scheme.

Lemma 4.3. *Let e^{N+1} be the error estimate equation (4.1) and C_1 be a constant. Then we have*

$$\begin{aligned} \|e^{N+1}\|^2 \quad \le \quad & C_1 h\Big[\|e^{N+1,0}\|^2 + \tau^2(\|e^{N,0}\|^2 + \|\nabla_{d,r}e^N\|^2) + \tau(\|e^{N,0}\|^2 + \|\nabla_{d,r}e^N\|^2) \\ + \quad & \|e^{N,0}\|^2 + \frac{\tau}{12}\int_{t_N}^{t_{N+1}} \left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds\Big]. \end{aligned} \tag{4.3}$$

Proof. When we use $v = e^N$ in the equation (4.1), we get

$$\begin{aligned} & (\frac{e^{N+1,0}-e^{N,0}}{\tau}, e^{N,0}) + (a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1}+e^N}{2}, \nabla_{d,r}e^N) \\ - & (\frac{e^{N+1,0}+e^{N,0}}{2}b(u_h^{N,0}), \nabla_{d,r}e^N) - (\frac{e^{N+1,0}+e^{N,0}}{2}\nabla_{d,r}b(u_h^N), e^{N,0}) \\ + & (c\frac{e^{N+1,0}+e^{N,0}}{2}, e^{N,0}) = (\frac{Q_hu^{N+1,0}-Q_hu^{N,0}}{\tau} - \frac{\partial u^{N+1,0}}{\partial t}, e^{N,0}), \end{aligned}$$

if we rewrite the above equation, we get

$$\begin{aligned}
& ((e^{N+1,0}-e^{N,0}),e^{N,0})+(\tau a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1}+e^{N}}{2},\nabla_{d,r}e^{N})\\
- & (\tau\frac{e^{N+1,0}+e^{N,0}}{2}b(u_h^{N,0}),\nabla_{d,r}e^{N})-(\tau\frac{e^{N+1,0}+e^{N,0}}{2}\nabla_{d,r}b(u_h^{N}),e^{N,0})\\
+ & (\tau c\frac{e^{N+1,0}+e^{N,0}}{2},e^{N,0})=(\tau\left(\frac{Q_hu^{N+1,0}-Q_hu^{N,0}}{\tau}-\frac{\partial u^{N+1,0}}{\partial t}\right),e^{N,0}).
\end{aligned}$$

Suppose the above equation is

$$\sum_{i=1}^{5}A^{(i)}=A^{(6)}. \tag{4.4}$$

By Cauchy-Schwarz inequality and Young's inequality, we get

$$|A^{(1)}|=|(e^{N+1,0},e^{N,0})|+|(e^{N,0},e^{N,0})|\leq\frac{1}{4}\|e^{N,0}\|^2+\|e^{N+1,0}\|^2+\lambda\|e^{N,0}\|^2.$$

$$\begin{aligned}
|A^{(2)}| &= |(\tau a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1}}{2},\nabla_{d,r}e^{N})|+|(\tau a(u_h^{N,0})\nabla_{d,r}\frac{e^{N}}{2},\nabla_{d,r}e^{N})|\\
&\leq \tau^2\frac{\|a(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}e^{N}\|^2+\|\nabla_{d,r}e^{N+1}\|^2+\frac{\lambda\ \tau}{2}\|a(u_h^{N,0})\|\ \|\nabla_{d,r}e^{N}\|^2.
\end{aligned}$$

$$\begin{aligned}
|A^{(3)}| &= |(\tau\frac{e^{N+1,0}}{2}b(u_h^{N,0}),\nabla_{d,r}e^{N})|+|(\tau\frac{e^{N,0}}{2}b(u_h^{N,0}),\nabla_{d,r}e^{N})|\\
&\leq \tau^2\frac{\|b(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}e^{N}\|^2+\|e^{N+1,0}\|^2+\tau^2\frac{\|b(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}e^{N}\|^2+\|e^{N,0}\|^2.
\end{aligned}$$

$$\begin{aligned}
|A^{(4)}| &= |(\tau c\frac{e^{N+1,0}}{2},e^{N,0})|+|(\tau c\frac{e^{N,0}}{2},e^{N,0})|\\
&\leq \tau^2\frac{\|c\|^2}{16}\|e^{N,0}\|^2+\|e^{N+1,0}\|^2+\tau\frac{\|c\|}{2}\lambda\ \|e^{N,0}\|^2.
\end{aligned}$$

$$\begin{aligned}
|A^{(5)}| &= |(\tau\frac{e^{N+1,0}}{2}\nabla_{d,r}b(u_h^{N}),e^{N,0})|+|(\tau\frac{e^{N,0}}{2}\nabla_{d,r}b(u_h^{N}),e^{N,0})|\\
&\leq \tau^2\frac{\|\nabla_{d,r}b(u_h^{N})\|^2}{6}\|e^{N,0}\|^2+\|e^{N+1,0}\|^2+\tau\frac{\|\nabla_{d,r}b(u_h^{N})\|}{2}\lambda\ \|e^{N,0}\|^2.
\end{aligned}$$

By Young's inequality and lemma (4.2), we get

$$\begin{aligned}
|A^{(16)}| &\leq \|\frac{Q_hu^{N+1,0}-Q_hu^{N,0}}{\tau}-\frac{\partial u^{N+1,0}}{\partial t}\|^2+\frac{\tau^2}{4}\|e^{N,0}\|^2\\
&\leq \frac{\tau}{12}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2u}{\partial t^2}\right|^2ds+\frac{\tau^2}{4}\|e^{N,0}\|^2.
\end{aligned}$$

Then the above equation is

$$\begin{aligned}
&\frac{1}{4}\|e^{N,0}\|^2 + \|e^{N+1,0}\|^2 + \lambda\|e^{N,0}\|^2 + \tau^2\frac{\|a(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}e^N\|^2 + \|\nabla_{d,r}e^{N+1}\|^2 \\
+\;& \frac{\tau}{2}\|a(u_h^{N,0})\|\lambda\;\|\nabla_{d,r}e^N\|^2 + \tau^2\frac{\|b(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}e^N\|^2 + \|e^{N+1,0}\|^2 \\
+\;& \tau^2\frac{\|b(u_h^{N,0})\|^2}{16}\|\nabla_{d,r}e^N\|^2 + \|e^{N,0}\|^2 + \tau^2\frac{\|c\|^2}{16}\|e^{N,0}\|^2 + \|e^{N+1,0}\|^2 \\
+\;& \tau\frac{\|c\|}{2}\lambda\;\|e^{N,0}\|^2 + \tau^2\frac{\|\nabla_{d,r}b(u_h^N)\|^2}{16}\|e^{N,0}\|^2 + \|e^{N+1,0}\|^2 \\
+\;& \tau\frac{\|\nabla_{d,r}b(u_h^N)\|}{2}\lambda\;\|e^{N,0}\|^2 \le \frac{\tau}{12}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds + \frac{\tau^2}{4}\|e^{N,0}\|^2,
\end{aligned}$$

so,

$$\begin{aligned}
\|\nabla_{d,r}e^{N+1}\|^2 \;\le\;& -4\|e^{N+1,0}\|^2 + \left[\frac{\tau^2}{4} - \frac{5}{4} - \lambda - \tau^2\frac{\|c\|^2}{16} - \tau\frac{\|c\|}{2}\lambda\right. \\
-\;& \left.\tau^2\frac{\|\nabla_{d,r}b(u_h^N)\|^2}{16} - \tau\frac{\|\nabla_{d,r}b(u_h^N)\|}{2}\lambda\right]\|e^{N,0}\|^2 + \frac{\tau}{12}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds \\
-\;& \left[\tau^2\frac{\|a(u_h^{N,0})\|^2}{16} + \frac{\tau}{2}\|a(u_h^{N,0})\|\lambda + \tau^2\frac{\|b(u_h^{N,0})\|^2}{16}\right. \\
+\;& \left.\tau^2\lambda\frac{\|b(u_h^{N,0})\|^2}{16}\right]\|\nabla_{d,r}e^N\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\|e^{N+1}\|^2 \;\le\;& C_1 h\Big[\|e^{N+1,0}\|^2 + (\tau^2+\tau+1)\|e^{N,0}\|^2 + (\tau^2+\tau)\|\nabla_{d,r}e^N\|^2 \\
+\;& \frac{\tau}{12}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds\Big],
\end{aligned}$$

where
$C_1 = max\{-4, \frac{1}{4}, -\frac{5}{4}, -\lambda, -\frac{\|c\|^2}{16}, -\frac{\|c\|}{2}\lambda, -\frac{\|\nabla_{d,r}b(u_h^N)\|^2}{16}, -\frac{\|\nabla_{d,r}b(u_h^N)\|}{2}\lambda, -\frac{\|a(u_h^{N,0})\|^2}{16},$
$-\frac{1}{2}\|a(u_h^{N,0})\|\lambda, -\frac{\|b(u_h^{N,0})\|^2}{16}, -\lambda\frac{\|b(u_h^{N,0})\|^2}{16}\}/C.$

□

4.3. H^1 Error Estimates of the Fully-Discrete Weak Galerkin Finite Element Scheme.

Theorem 4.4. *Let* e^{N+1} *be the error estimate of equation (4.1) and a constant* C_1 *exist. Then we have*

$$\begin{aligned}
\|e^{N+1}\|^2 + \|\nabla_{d,r}e^{N+1}\|^2 \;\le\;& \beta_2 h\Big[\|e^{N+1,0}\|^2 + \|e^{N,0}\|^2 + \|\nabla_{d,r}e^N\|^2 \\
+\;& \tau^2\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds\Big].
\end{aligned} \tag{4.5}$$

Proof. Using the $v = \frac{e^{N+1}-e^N}{\tau}$ in equation (4.1), we get

$$\begin{aligned}
& (\frac{e^{N+1,0}-e^{N,0}}{\tau}, \frac{e^{N+1,0}-e^{N,0}}{\tau}) + (a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1}+e^N}{2}, \nabla_{d,r}\frac{e^{N+1}-e^N}{\tau}) \\
- & (\frac{e^{N+1,0}+e^{N,0}}{2}b(u_h^{N,0}), \nabla_{d,r}\frac{e^{N+1}-e^N}{\tau}) + (c\frac{e^{N+1,0}+e^{N,0}}{2}, \frac{e^{N+1,0}-e^{N,0}}{\tau}) \\
- & (\frac{e^{N+1,0}+e^{N,0}}{2}\nabla_{d,r}b(u_h^N), \frac{e^{N+1,0}-e^{N,0}}{\tau}) \\
= & (\frac{Q_h u^{N+1,0}-Q_h u^{N,0}}{\tau} - \frac{\partial u^{N+1,0}}{\partial t}, \frac{e^{N+1,0}-e^{N,0}}{\tau}),
\end{aligned}$$

rewritten the above equation

$$\begin{aligned}
& ((e^{N+1,0}-e^{N,0}),(e^{N+1,0}-e^{N,0})) + (\tau a(u_h^{N,0})\nabla_{d,r}\frac{e^{N+1}+e^N}{2}, \nabla_{d,r}(e^{N+1}-e^N)) \\
- & (\tau\frac{e^{N+1,0}+e^{N,0}}{2}b(u_h^{N,0}), \nabla_{d,r}(e^{N+1}-e^N)) + (\tau c\frac{e^{N+1,0}+e^{N,0}}{2}, (e^{N+1,0}-e^{N,0})) \\
- & (\tau\frac{e^{N+1,0}+e^{N,0}}{2}\nabla_{d,r}b(u_h^N), (e^{N+1,0}-e^{N,0})) \\
= & (\tau(\frac{Q_h u^{N+1,0}-Q_h u^{N,0}}{\tau} - \frac{\partial u^{N+1,0}}{\partial t}), (e^{N+1,0}-e^{N,0})).
\end{aligned}$$

Suppose the above equation is

$$\sum_{i=1}^{5} B^{(i)} = B^{(6)}. \tag{4.6}$$

By Cauchy-Schwarz inequality and Young's inequality, we get

$$|B^{(1)}| = |(e^{N+1,0}, e^{N+1,0})| + |(e^{N,0}, e^{N,0})| \le C\|e^{N+1,0}\|^2 + C\|e^{N,0}\|^2.$$

$$\begin{aligned}
|B^{(2)}| &= |(\frac{\tau}{2}a(u_h^{N,0})\nabla_{d,r}e^{N+1}, \nabla_{d,r}e^{N+1})| + |(\frac{\tau}{2}a(u_h^{N,0})\nabla_{d,r}e^{N}, \nabla_{d,r}e^{N})| \\
&\le \frac{\tau\|a(u_h^{N,0})\|}{2}\|\nabla_{d,r}e^{N+1}\|^2 + \frac{\tau\|a(u_h^{N,0})\|}{2}\|\nabla_{d,r}e^{N}\|^2 \\
&\le \tau C h^{-1}\|e^{N+1}\|^2 + \frac{\tau\|a(u_h^{N,0})\|}{2}\|\nabla_{d,r}e^{N}\|^2.
\end{aligned}$$

$$\begin{aligned}
|B^{(3)}| &= |(\frac{\tau}{2}e^{N+1,0}b(u_h^{N,0}), \nabla_{d,r}e^{N+1})| + |(\frac{\tau}{2}e^{N,0}b(u_h^{N,0}), \nabla_{d,r}e^{N})| \\
&\le \frac{\tau\|b(u_h^{N,0})\|^2}{16}\|e^{N+1,0}\|^2 + \tau\|\nabla_{d,r}e^{N+1}\|^2 + \frac{\tau\|b(u_h^{N,0})\|^2}{16}\|e^{N,0}\|^2 + \tau\|\nabla_{d,r}e^{N}\|^2.
\end{aligned}$$

$$\begin{aligned}
|B^{(4)}| &= |(\frac{\tau}{2}e^{N+1,0}\nabla_{d,r}b(u_h^N), e^{N+1,0})| + |(\frac{\tau}{2}e^{N,0}\nabla_{d,r}b(u_h^N), e^{N,0})| \\
&\le \frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}\|e^{N+1,0}\|^2 + \frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}\|e^{N,0}\|^2.
\end{aligned}$$

$$|B^{(5)}| = |(\frac{\tau}{2}c\,e^{N+1,0}, e^{N+1,0})| + |(\frac{\tau}{2}c\,e^{N,0}, e^{N,0})| \le \frac{\tau\|c\|}{2}\|e^{N+1,0}\|^2 + \frac{\tau\|c\|}{2}\|e^{N,0}\|^2.$$

By Young's inequality and lemma (4.2), we get

$$\begin{aligned} |B^{(6)}| &= |(\frac{Q_h u^{N+1,0}-Q_h u^{N,0}}{\tau}-\frac{\partial u^{N+1,0}}{\partial t},\tau e^{N+1,0})| \\ &+ |(\frac{Q_h u^{N+1,0}-Q_h u^{N,0}}{\tau}-\frac{\partial u^{N+1,0}}{\partial t},\tau e^{N,0})| \\ &\le \tau^2\|e^{N+1,0}\|^2+\tau^2\|e^{N,0}\|^2+2\|\frac{Q_h u^{N+1,0}-Q_h u^{N,0}}{\tau}-\frac{\partial u^{N+1,0}}{\partial t}\|^2 \\ &\le C\|e^{N+1,0}\|^2+C\|e^{N,0}\|^2+\frac{\tau^3}{6}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds. \end{aligned}$$

Then, from the above equation, we get

$$\begin{aligned} & C\|e^{N+1,0}\|^2+\tau\|\nabla_{d,r}e^{N+1}\|^2+(\tau+\frac{\tau\|a(u_h^{N,0})\|}{2})\|\nabla_{d,r}e^N\|^2+\tau Ch^{-1}\|e^{N+1}\|^2 \\ + & \frac{\tau\|b(u_h^{N,0})\|^2}{16}\|e^{N+1,0}\|^2+(C+\frac{\tau\|b(u_h^{N,0})\|^2}{16})\|e^{N,0}\|^2+\frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}\|e^{N+1,0}\|^2 \\ + & \frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}\|e^{N,0}\|^2+\frac{\tau\|c\|}{2}\|e^{N+1,0}\|^2+\frac{\tau\|c\|}{2}\|e^{N,0}\|^2\le\frac{\tau^3}{6}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds \\ + & C\|e^{N+1,0}\|^2+C\|e^{N,0}\|^2, \end{aligned}$$

so,

$$\begin{aligned} & \tau Ch^{-1}\|e^{N+1}\|^2+\tau\|\nabla_{d,r}e^{N+1}\|^2\le\frac{\tau^3}{6}\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds \\ - & (\frac{\tau\|b(u_h^{N,0})\|^2}{16}+\frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}+\frac{\tau\|c\|}{2})\|e^{N,0}\|^2-(\frac{\tau\|a(u_h^{N,0})\|}{2}+\tau)\|\nabla_{d,r}e^N\|^2 \\ - & (\frac{\tau\|b(u_h^{N,0})\|^2}{16}+\frac{\tau\|\nabla_{d,r}b(u_h^N)\|}{2}+\frac{\tau\|c\|}{2})\|e^{N+1,0}\|^2, \end{aligned}$$

then

$$\|e^{N+1}\|^2+\|\nabla_{d,r}e^{N+1}\|^2\le\beta_2 h\left[\|e^{N+1,0}\|^2+\|e^{N,0}\|^2+\|\nabla_{d,r}e^N\|^2+\tau^2\int_{t_N}^{t_{N+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 ds\right],$$

where
$\beta_2=max\{-(\frac{\|b(u_h^{N,0})\|^2}{16}+\frac{\|\nabla_{d,r}b(u_h^N)\|}{2}+\frac{\|c\|}{2}),-(\frac{\|b(u_h^{N,0})\|^2}{16}+\frac{\|\nabla_{d,r}b(u_h^N)\|}{2}+\frac{\|c\|}{2}),-(\frac{\|a(u_h^{N,0})\|}{2}+1),\frac{1}{6}\}/min\{C,1\}$.

□

4.4. Estimates of the L^2- Normal Optimal Order Error. We derived the optimal order error estimate in the L^2-norm for discrete time weak Galerkin finite element methods in this subsection. Let $u^{N+1}\in H_0^1(\Omega)\bigcap H^2(\Omega)$ and $\Pi_h u^{N+1}$ denote the elliptic projection of u_h^{N+1} onto finite element space V_h^0.

Theorem 4.5. *Suppose that the exact solution of equation (3.1) is so regular that $u^{N+1}\in H^{k+1}(\Omega)$ and there exists a constant β. Then*

$$
\begin{aligned}
\|\theta^{N+1}\|^2 \quad &\leq \quad \beta h \Bigg[(1+\tau^2)\|\theta^{N+1,0}\|^2 + (1+\tau)\|\theta^{N,0}\|^2 + (1+\tau+\tau^2)\|\nabla_{d,r}\theta^N\|^2 \\
&+ \quad \tau^2 (\|\Pi_h u^{N+1,0}\|^2 + \|\Pi_h u^{N,0}\|^2 + \|\Pi_h \nabla_{d,r} u^{N+1}\|^2 + \|\Pi_h \nabla_{d,r} u^N\|^2) \Bigg].
\end{aligned} \tag{4.7}
$$

Proof. In equation (3.1), choose $u_h^N = \Pi_h u^N$ and $u_h^N = Q_h u^N$ respectively.

$$
\begin{aligned}
& (\frac{\Pi_h u^{N+1,0} - \Pi_h u^{N,0}}{\tau}, v^0) + (a(u_h^N)\Pi_h \nabla_{d,r} \frac{u^{N+1} + u^N}{2}, \nabla_{d,r} v) \\
- \quad & (\frac{\Pi_h u^{N+1,0} + \Pi_h u^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v) - (\frac{\Pi_h u^{N+1,0} + \Pi_h u^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0) \\
+ \quad & (c\frac{\Pi_h u^{N+1,0} + \Pi_h u^{N,0}}{2}, v^0) = (\frac{f^{N+1} + f^N}{2}, v^0),
\end{aligned} \tag{4.8}
$$

and,

$$
\begin{aligned}
& (\frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau}, v^0) + (a(u_h^{N,0})\nabla_{d,r} \frac{Q_h u^{N+1} + Q_h u^N}{2}, \nabla_{d,r} v) \\
- \quad & (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v) - (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0) \\
+ \quad & (c\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, v^0) = (\frac{f^{N+1} + f^N}{2}, v^0),
\end{aligned} \tag{4.9}
$$

we get by subtracting equation (4.9) from (4.8)

$$
\begin{aligned}
& (\frac{\Pi_h u^{N+1,0} - \Pi_h u^{N,0}}{\tau}, v^0) - (\frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau}, v^0) \\
+ \quad & (a(\Pi_h u^{N,0})\Pi_h \nabla_{d,r} \frac{u^{N+1} + u^N}{2}, \nabla_{d,r} v) - (a(Q_h u^{N,0})\nabla_{d,r} \frac{Q_h u^{N+1} + Q_h u^N}{2}, \nabla_{d,r} v) \\
- \quad & (\frac{\Pi_h u^{N+1,0} + \Pi_h u^{N,0}}{2} b(\Pi_h u^{N,0}), \nabla_{d,r} v) + (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} b(Q_h u^{N,0}), \nabla_{d,r} v) \\
- \quad & (\frac{\Pi_h u^{N+1,0} + \Pi_h u^{N,0}}{2} \Pi_h \nabla_{d,r} b(u^N), v^0) + (c\frac{\Pi_h u^{N+1,0} + \Pi_h u^{N,0}}{2}, v^0) \\
+ \quad & (\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} \nabla_{d,r} b(Q_h u^N), v^0) - (c\frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, v^0) = 0,
\end{aligned} \tag{4.10}
$$

rewrite the equation (4.10), and use $v = \theta^N = \Pi_h u^N - Q_h u^N$, we have

$$
\begin{aligned}
&(\theta^{N+1,0}, \theta^{N,0}) - (\theta^{N,0}, \theta^{N,0}) + (\tau a(\Pi_h u^{N,0})\nabla_{d,r} \frac{\theta^{N+1}}{2}, \nabla_{d,r}\theta^N) + (\tau c\frac{\theta^{N,0}}{2}, \theta^{N,0}) \\
+ \quad & (\tau a(\Pi_h u^{N,0})\nabla_{d,r} \frac{\theta^{N}}{2}, \nabla_{d,r}\theta^N) + (\tau\frac{a(\Pi_h u^{N,0}) - a(Q_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^{N+1}, \nabla_{d,r}\theta^N) \\
+ \quad & (\tau\frac{(a(\Pi_h u^{N,0}) - a(Q_h u^{N,0}))}{2}\Pi_h\nabla_{d,r}u^{N}, \nabla_{d,r}\theta^N) - (\tau\frac{\theta^{N+1,0}}{2}b(\Pi_h u^{N,0}), \nabla_{d,r}\theta^N) \\
- \quad & (\tau\frac{\theta^{N,0}}{2}b(\Pi_h u^{N,0}), \nabla_{d,r}\theta^N) - (\tau\frac{b(\Pi_h u^{N,0}) - b(Q_h u^{N,0})}{2}\Pi_h u^{N+1,0}, \nabla_{d,r}\theta^N) \\
- \quad & (\tau\frac{b(\Pi_h u^{N,0}) - b(Q_h u^{N,0})}{2}\Pi_h u^{N,0}, \nabla_{d,r}\theta^N) - (\tau\frac{\theta^{N+1,0}}{2}\Pi_h\nabla_{d,r}b(u^N), \theta^{N,0}) \\
- \quad & (\tau\frac{\theta^{N,0}}{2}\Pi_h\nabla_{d,r}b(u^N), \theta^{N,0}) - (\tau\frac{\nabla_{d,r}(b(\Pi_h u^N) - b(Q_h u^N))}{2}\Pi_h u^{N+1,0}, \theta^{N,0}) \\
- \quad & (\tau\frac{\nabla_{d,r}(b(\Pi_h u^N) - b(Q_h u^N))}{2}\Pi_h u^{N,0}, \theta^{N,0}) + (\tau c\frac{\theta^{N+1,0}}{2}, \theta^{N,0}) = 0.
\end{aligned}
$$

Suppose the above equation is

$$
\sum_{i=1}^{16} C^{(i)} = 0. \tag{4.11}
$$

By Cauchy-Schwarz inequality and Young's inequality, we get

$$
|C^{(1)}| \le \|\theta^{N+1,0}\|^2 + \frac{1}{4}\ \|\theta^{N,0}\|^2.
$$

$$
|C^{(2)}| \le C\|\theta^{N,0}\|^2.
$$

$$
|C^{(3)}| \le \tau^2\frac{\|a(\Pi_h u^{N,0})\|^2}{16}\|\nabla_{d,r}\theta^N\|^2 + \|\nabla_{d,r}\theta^{N+1}\|^2 \le \alpha\tau^2\|\nabla_{d,r}\theta^N\|^2 + Ch^{-1}\|\theta^{N+1}\|^2.
$$

$$
|C^{(5)}| \le \tau\frac{\|a(\Pi_h u^{N,0})\|}{2}\|\nabla_{d,r}\theta^N\|^2.
$$

$$
\begin{aligned}
|C^{(6)}| \quad &= \quad |(\tau\frac{a(\Pi_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^{N+1}, \nabla_{d,r}\theta^N)| + |(\tau\frac{a(Q_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^{N+1}, \nabla_{d,r}\theta^N)| \\
&\le \quad \frac{\tau^2(\|a(\Pi_h u^{N,0})\|^2 + \|a(Q_h u^{N,0})\|^2)}{16}\|\Pi_h\nabla_{d,r}u^{N+1}\|^2 + 2\,\|\nabla_{d,r}\theta^N\|^2 \\
&\le \quad \alpha_1\tau^2\|\Pi_h\nabla_{d,r}u^{N+1}\|^2 + 2\|\nabla_{d,r}\theta^N\|^2.
\end{aligned}
$$

$$
\begin{aligned}
|C^{(7)}| \quad &\le \quad |(\tau\frac{(a(\Pi_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^{N}, \nabla_{d,r}\theta^N)| + |(\tau\frac{a(Q_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^{N}, \nabla_{d,r}\theta^N)| \\
&\le \quad \frac{\tau^2(\|a(\Pi_h u^{N,0})\|^2 + \|a(Q_h u^{N,0})\|^2)}{16}\|\Pi_h\nabla_{d,r}u^{N}\|^2 + 2\,\|\nabla_{d,r}\theta^N\|^2 \\
&\le \quad \alpha_2\tau^2\|\Pi_h\nabla_{d,r}u^{N}\|^2 + 2\|\nabla_{d,r}\theta^N\|^2.
\end{aligned}
$$

$$
|C^{(8)}| \le \frac{\tau^2\|b(\Pi_h u^{N,0})\|^2}{2}\|\nabla_{d,r}\theta^N\|^2 + \|\theta^{N+1,0}\|^2.
$$

$$|C^{(9)}| \leq \frac{\tau^2 \|b(\Pi_h u^{N,0})\|^2}{2} \|\nabla_{d,r}\theta^N\|^2 + \|\theta^{N,0}\|^2.$$

$$\begin{aligned}
|C^{(10)}| &= |(\tau \frac{b(\Pi_h u^{N,0})}{2} \Pi_h u^{N+1,0}, \nabla_{d,r}\theta^N)| + |(\tau \frac{b(Q_h u^{N,0})}{2} \Pi_h u^{N+1,0}, \nabla_{d,r}\theta^N)| \\
&\leq \frac{\tau^2 (\|b(\Pi_h u^{N,0})\|^2 + \|b(Q_h u^{N,0})\|^2)}{16} \|\Pi_h u^{N+1,0}\|^2 + 2\,\|\nabla_{d,r}\theta^N\|^2 \\
&\leq \alpha_3 \tau^2 \|\Pi_h u^{N+1,0}\|^2 + 2\|\nabla_{d,r}\theta^N\|^2.
\end{aligned}$$

$$\begin{aligned}
|C^{(11)}| &= |(\tau \frac{b(\Pi_h u^{N,0})}{2} \Pi_h u^{N,0}, \nabla_{d,r}\theta^N)| + |(\tau \frac{b(Q_h u^{N,0})}{2} \Pi_h u^{N,0}, \nabla_{d,r}\theta^N)| \\
&\leq \frac{\tau^2 \|b(\Pi_h u^{N,0})\|^2}{16} \|\Pi_h u^{N,0}\|^2 + 2\,\|\nabla_{d,r}\theta^N\|^2 + \frac{\tau^2 \|b(Q_h u^{N,0})\|^2}{16} \|\Pi_h u^{N,0}\|^2 \\
&\leq \alpha_3 \tau^2 \|\Pi_h u^{N,0}\|^2 + 2\|\nabla_{d,r}\theta^N\|^2.
\end{aligned}$$

$$|C^{(12)}| \leq \frac{\tau^2 \|\Pi_h \nabla_{d,r} b(u^N)\|^2}{16} \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 \leq C\tau^2 \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2.$$

$$|C^{(13)}| \leq \frac{\tau \|\Pi_h \nabla_{d,r} b(u^N)\|}{2} \|\theta^{N,0}\|^2 \leq C\tau \|\theta^{N,0}\|^2.$$

$$\begin{aligned}
|C^{(14)}| &= |(\tau \frac{\nabla_{d,r} b(\Pi_h u^N)}{2} \Pi_h u^{N+1,0}, \theta^{N,0})| + |(\tau \frac{\nabla_{d,r} b(Q_h u^N)}{2} \Pi_h u^{N+1,0}, \theta^{N,0})| \\
&\leq \frac{\tau^2 (\|\nabla_{d,r} b(\Pi_h u^N)\|^2 + \|\nabla_{d,r} b(Q_h u^N)\|^2)}{16} \|\Pi_h u^{N+1,0}\|^2 + 2\,\|\theta^{N,0}\|^2 \\
&\leq \alpha_5 \tau^2 \|\Pi_h u^{N+1,0}\|^2 + 2\|\theta^{N,0}\|^2.
\end{aligned}$$

$$\begin{aligned}
|C^{(15)}| &= |(\tau \frac{\nabla_{d,r} b(\Pi_h u^N)}{2} \Pi_h u^{N,0}, \theta^{N,0})| + |(\tau \frac{\nabla_{d,r} b(Q_h u^N)}{2} \Pi_h u^{N,0}, \theta^{N,0})| \\
&\leq \frac{\tau^2 \|\nabla_{d,r} b(\Pi_h u^N)\|^2}{16} \|\Pi_h u^{N,0}\|^2 + 2\,\|\theta^{N,0}\|^2 + \frac{\tau^2 \|\nabla_{d,r} b(Q_h u^N)\|^2}{16} \|\Pi_h u^{N,0}\|^2 \\
&\leq \alpha_5 \tau^2 \|\Pi_h u^{N,0}\|^2 + 2\|\theta^{N,0}\|^2.
\end{aligned}$$

$$|C^{(16)}| \leq \tau^2 \frac{\|c\|^2}{16} \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2.$$

$$|C^{(4)}| \leq \tau \frac{\|c\|}{2} \|\theta^{N,0}\|^2.$$

Then, the above equation becomes, we get

$$
\begin{aligned}
& 2\|\theta^{N+1,0}\|^2 + \frac{1}{4}\ \|\theta^{N,0}\|^2 + C\|\theta^{N,0}\|^2 + \alpha\tau^2\|\nabla_{d,r}\theta^N\|^2 + Ch^{-1}\|\theta^{N+1}\|^2 \\
+ & \ \tau\frac{\|a(\Pi_h u^{N,0})\|}{2}\|\nabla_{d,r}\theta^N\|^2 + \alpha_1\tau^2\|\Pi_h\nabla_{d,r}u^{N+1}\|^2 + \alpha_2\tau^2\|\Pi_h\nabla_{d,r}u^N\|^2 \\
+ & \ 4\|\nabla_{d,r}\theta^N\|^2 + \frac{\tau^2\|b(\Pi_h u^{N,0})\|^2}{2}\|\nabla_{d,r}\theta^N\|^2 + \frac{\tau^2\|b(\Pi_h u^{N,0})\|^2}{2}\|\nabla_{d,r}\theta^N\|^2 \\
+ & \ \|\theta^{N,0}\|^2 + \alpha_3\tau^2\|\Pi_h u^{N+1,0}\|^2 + 2\|\nabla_{d,r}\theta^N\|^2 + \alpha_3\tau^2\|\Pi_h u^{N,0}\|^2 + 2\|\nabla_{d,r}\theta^N\|^2 \\
+ & \ C\tau^2\|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + C\tau\|\theta^{N,0}\|^2 + \alpha_5\tau^2\|\Pi_h u^{N+1,0}\|^2 + 4\|\theta^{N,0}\|^2 \\
+ & \ \alpha_5\tau^2\|\Pi_h u^{N,0}\|^2 + \tau^2\frac{\|c\|^2}{16}\|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \tau\frac{\|c\|}{2}\|\theta^{N,0}\|^2 \le 0,
\end{aligned}
$$

so,

$$
\begin{aligned}
Ch^{-1}\|\theta^{N+1}\|^2 \ \le \ & -(2 + C\tau^2 + \tau^2\frac{\|c\|^2}{16})\|\theta^{N+1,0}\|^2 - (7\frac{1}{4} + C + C\tau + \tau\frac{\|c\|}{2})\|\theta^{N,0}\|^2 \\
- & \ (\alpha\tau^2 + \tau\frac{\|a(\Pi_h u^{N,0})\|}{2} + 8 + \tau^2\|b(\Pi_h u^{N,0})\|^2)\|\nabla_{d,r}\theta^N\|^2 \\
- & \ (\alpha_3 + \alpha_5)\tau^2(\|\Pi_h u^{N+1,0}\|^2 + \|\Pi_h u^{N,0}\|^2) - \alpha_1\tau^2\|\Pi_h\nabla_{d,r}u^{N+1}\|^2 \\
- & \ \alpha_2\tau^2\|\Pi_h\nabla_{d,r}u^N\|^2,
\end{aligned}
$$

then

$$
\begin{aligned}
\|\theta^{N+1}\|^2 \ \le \ & \beta h\Big[(1+\tau^2)\|\theta^{N+1,0}\|^2 + (1+\tau)\|\theta^{N,0}\|^2 + (1+\tau+\tau^2)\|\nabla_{d,r}\theta^N\|^2 \\
+ & \ \tau^2(\|\Pi_h u^{N+1,0}\|^2 + \|\Pi_h u^{N,0}\|^2) + \tau^2\|\Pi_h\nabla_{d,r}u^{N+1}\|^2 + \tau^2\|\Pi_h\nabla_{d,r}u^N\|^2\Big],
\end{aligned}
$$

where
$\beta = max\{-2, -(1+C), -\frac{29}{4} - C, -C - \frac{\|c\|}{2}, -\alpha - \|b(\Pi_h u^{N,0})\|^2, -8, \frac{-\|a(\Pi_h u^{N,0})\|}{2}, -\alpha_1,$ $-\alpha_2, -(\alpha_3+\alpha_5)\}/C.$

□

4.5. Estimates of the H^1- Norm's Optimal Order Error. We derived the optimal order error estimate in the H^1- norm for discrete time weak Galerkin finite element methods in this subsection. Let $u^{N+1} \in H_0^1(\Omega)\bigcap H^2(\Omega)$ and $\Pi_h u^{N+1}$ denote the elliptic projection of u_h^{N+1} onto finite element space V_h^0.

Theorem 4.6. *Suppose that the exact solution of equation (3.1) is so regular that $u^{N+1} \in H^{k+1}(\Omega)$ and there exists a constant β_1. Then*

$$
\begin{aligned}
\|\theta^{N+1}\|^2 + \|\nabla_{d,r}\theta^{N+1}\|^2 \ \le \ & \beta_1 h\Big[\tau^2(\|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2 + \|\nabla_{d,r}u_h^{N+1}\|^2 + \|\nabla_{d,r}u_h^N\|^2 \\
+ & \ \|\nabla_{d,r}\theta^N\|^2) + (1+\tau^2)\|\theta^{N,0}\|^2 + (1+\tau+\tau^2)\|\theta^{N+1,0}\|^2\Big].
\end{aligned}
\tag{4.12}
$$

Proof. Rewrite equation (4.10) and using $v = \theta^{N+1} = \Pi_h u^{N+1} - Q_h u^{N+1}$, we have

$$\begin{aligned}
& (\theta^{N+1,0}, \theta^{N+1,0}) + (\tau \frac{(a(\Pi_h u^{N,0}) - a(Q_h u^{N,0}))}{2} \Pi_h \nabla_{d,r} u^N, \nabla_{d,r} \theta^{N+1}) \\
- \; & (\theta^{N,0}, \theta^{N+1,0}) + (\tau \frac{a(\Pi_h u^{N,0}) - a(Q_h u^{N,0})}{2} \Pi_h \nabla_{d,r} u^{N+1}, \nabla_{d,r} \theta^{N+1}) \\
+ \; & (\tau a(\Pi_h u^{N,0}) \nabla_{d,r} \frac{\theta^{N+1}}{2}, \nabla_{d,r} \theta^{N+1}) + (\tau c \frac{\theta^{N,0}}{2}, \theta^{N+1,0}) + (\tau c \frac{\theta^{N+1,0}}{2}, \theta^{N+1,0}) \\
- \; & (\tau \frac{\theta^{N,0}}{2} b(\Pi_h u^{N,0}), \nabla_{d,r} \theta^{N+1}) - (\tau \frac{b(\Pi_h u^{N,0}) - b(Q_h u^{N,0})}{2} \Pi_h u^{N+1,0}, \nabla_{d,r} \theta^{N+1}) \\
- \; & (\tau \frac{b(\Pi_h u^{N,0}) - b(Q_h u^{N,0})}{2} \Pi_h u^{N,0}, \nabla_{d,r} \theta^{N+1}) - (\tau \frac{\theta^{N+1,0}}{2} b(\Pi_h u^{N,0}), \nabla_{d,r} \theta^{N+1}) \\
- \; & (\tau \frac{\Pi_h u^{N,0} - Q_h u^{N,0}}{2} \Pi_h \nabla_{d,r} b(u^N), \theta^{N+1,0}) + (\tau a(\Pi_h u^{N,0}) \nabla_{d,r} \frac{\theta^N}{2}, \nabla_{d,r} \theta^{N+1}) \\
- \; & (\tau \frac{\nabla_{d,r}(b(\Pi_h u^N) - b(Q_h u^N))}{2} \Pi_h u^{N+1,0}, \theta^{N+1,0}) \\
- \; & (\tau \frac{\nabla_{d,r}(b(\Pi_h u^N) - b(Q_h u^N))}{2} \Pi_h u^{N,0}, \theta^{N+1,0}) \\
- \; & (\tau \frac{\Pi_h u^{N+1,0} - Q_h u^{N+1,0}}{2} \Pi_h \nabla_{d,r} b(u^N), \theta^{N+1,0}) = 0.
\end{aligned}$$

Suppose the above equation is

$$\sum_{i=1}^{16} D^{(i)} = 0. \tag{4.13}$$

By Cauchy-Schwarz inequality and Young's inequality, we get

$$|D^{(1)}| \le C \|\theta^{N+1,0}\|^2.$$

$$|D^{(2)}| \le \frac{1}{4} \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2.$$

$$|D^{(6)}| \le \tau \frac{\|a(\Pi_h u^{N,0})\|}{2} \|\nabla_{d,r} \theta^{N+1}\|^2.$$

$$|D^{(4)}| \le \tau^2 \frac{\|a(\Pi_h u^{N,0})\|^2}{16} \|\nabla_{d,r} \theta^N\|^2 + \|\nabla_{d,r} \theta^{N+1}\|^2.$$

$$\begin{aligned}
|D^{(3)}| \; &= \; |(\tau \frac{a(\Pi_h u^{N,0})}{2} \Pi_h \nabla_{d,r} u^{N+1}, \nabla_{d,r} \theta^{N+1})| \\
& + \; |(\tau \frac{a(Q_h u^{N,0})}{2} \Pi_h \nabla_{d,r} u^{N+1}, \nabla_{d,r} \theta^{N+1})| \\
& \le \; \tau^2 \frac{(\|a(\Pi_h u^{N,0})\|^2 + \|a(Q_h u^{N,0})\|^2)}{16} \|\Pi_h \nabla_{d,r} u^{N+1}\|^2 + 2 \, \|\nabla_{d,r} \theta^{N+1}\|^2 \\
& \le \; \tau^2 \alpha_1 \|\nabla_{d,r} u_h^{N+1}\|^2 + 2C h^{-1} \|\theta^{N+1}\|^2.
\end{aligned}$$

$$\begin{aligned}
|D^{(5)}| &= |(\tau\frac{(a(\Pi_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^N, \nabla_{d,r}\theta^{N+1})| \\
&+ |(\tau\frac{a(Q_h u^{N,0})}{2}\Pi_h\nabla_{d,r}u^N, \nabla_{d,r}\theta^{N+1})| \\
&\leq \tau^2\frac{(\|a(\Pi_h u^{N,0})\|^2 + \|a(Q_h u^{N,0})\|^2)}{16}\|\Pi_h\nabla_{d,r}u^N\|^2 + 2\,\|\nabla_{d,r}\theta^{N+1}\|^2 \\
&\leq \alpha_1\tau^2\|\nabla_{d,r}u_h^N\|^2 + 2Ch^{-1}\|\theta^{N+1}\|^2.
\end{aligned}$$

$$|D^{(11)}| \leq \tau^2\frac{\|b(\Pi_h u^{N,0})\|^2}{2}\|\theta^{N+1,0}\|^2 + \|\nabla_{d,r}\theta^{N+1}\|^2.$$

$$|D^{(8)}| \leq \tau^2\frac{\|b(\Pi_h u^{N,0})\|^2}{2}\|\theta^{N,0}\|^2 + \|\nabla_{d,r}\theta^{N+1}\|^2.$$

$$\begin{aligned}
|D^{(9)}| &= |(\tau\frac{b(\Pi_h u^{N,0})}{2}\Pi_h u^{N+1,0}, \nabla_{d,r}\theta^{N+1})| \\
&+ |(\tau\frac{b(Q_h u^{N,0})}{2}\Pi_h u^{N+1,0}, \nabla_{d,r}\theta^{N+1})| \\
&\leq \tau^2\frac{(\|b(\Pi_h u^{N,0})\|^2 + \|b(Q_h u^{N,0})\|^2)}{16}\|\Pi_h u^{N+1,0}\|^2 + 2\,\|\nabla_{d,r}\theta^{N+1}\|^2 \\
&\leq \tau^2\alpha_2\|u_h^{N+1,0}\|^2 + 2Ch^{-1}\|\theta^{N+1}\|^2.
\end{aligned}$$

$$\begin{aligned}
|D^{(10)}| &= |(\tau\frac{b(\Pi_h u^{N,0})}{2}\Pi_h u^{N,0}, \nabla_{d,r}\theta^{N+1})| \\
&+ |(\tau\frac{b(Q_h u^{N,0})}{2}\Pi_h u^{N,0}, \nabla_{d,r}\theta^{N+1})| \\
&\leq \tau^2\frac{(\|b(\Pi_h u^{N,0})\|^2 + \|b(Q_h u^{N,0})\|^2)}{16}\|\Pi_h u^{N,0}\|^2 + 2\,\|\nabla_{d,r}\theta^{N+1}\|^2 \\
&\leq \alpha_2\tau^2\|u_h^{N,0}\|^2 + 2Ch^{-1}\|\theta^{N+1}\|^2.
\end{aligned}$$

$$\begin{aligned}
|D^{(7)}| &\leq |(\tau\frac{\Pi_h u^{N+1,0}}{2}\Pi_h\nabla_{d,r}b(u^N), \theta^{N+1,0})| \\
&+ |(\tau\frac{Q_h u^{N+1,0}}{2}\Pi_h\nabla_{d,r}b(u^N), \theta^{N+1,0})| \\
&\leq \tau^2\frac{(\|\Pi_h u^{N+1,0}\|^2 + \|Q_h u^{N+1,0}\|^2)}{16}\|\Pi_h\nabla_{d,r}b(u^N)\|^2 + 2\,\|\theta^{N+1,0}\|^2 \\
&\leq \alpha_4\tau^2\|\nabla_{d,r}u_h^N\|^2 + \|\theta^{N+1,0}\|^2.
\end{aligned}$$

$$\begin{aligned}
|D^{(12)}| &= |(\tau\frac{\Pi_h u^{N,0}}{2}\Pi_h\nabla_{d,r}b(u^N), \theta^{N+1,0})| \\
&+ |(\tau\frac{Q_h u^{N,0}}{2}\Pi_h\nabla_{d,r}b(u^N), \theta^{N+1,0})| \\
&\leq \tau^2\frac{(\|\Pi_h u^{N,0}\|^2 + \|Q_h u^{N,0}\|^2)}{16}\|\Pi_h\nabla_{d,r}b(u^N)\|^2 + 2\,\|\theta^{N+1,0}\|^2 \\
&\leq \alpha_5\tau^2\|\nabla_{d,r}u_h^N\|^2 + \|\theta^{N+1,0}\|^2.
\end{aligned}$$

$$\begin{aligned}|D^{(13)}| &\leq \tau^2\frac{\|\Pi_h u^{N+1,0}\|^2}{16}\|\nabla_{d,r}(b(\Pi_h u^N)-b(Q_h u^N))\|^2+\|\theta^{N+1,0}\|^2\\ &\leq \alpha_6\tau^2\|\nabla_{d,r}\theta^N\|^2+\|\theta^{N+1,0}\|^2.\end{aligned}$$

$$\begin{aligned}|D^{(15)}| &\leq \tau^2\frac{\|\Pi_h u^{N,0}\|^2}{16}\|\nabla_{d,r}(b(\Pi_h u^N)-b(Q_h u^N))\|^2+\|\theta^{N+1,0}\|^2\\ &\leq \alpha_7\tau^2\|\nabla_{d,r}\theta^N\|^2+\|\theta^{N+1,0}\|^2.\end{aligned}$$

$$|D^{(16)}|\leq \tau\frac{\|c\|}{2}\|\theta^{N+1,0}\|^2.$$

$$|D^{(14)}|\leq \tau^2\frac{\|c\|^2}{16}\|\theta^{N,0}\|^2+\|\theta^{N+1,0}\|^2.$$

Then, the above equation becomes, we get

$$\begin{aligned}&(C+1)\|\theta^{N+1,0}\|^2+\frac{1}{4}\|\theta^{N,0}\|^2+\tau\frac{\|a(\Pi_h u^{N,0})\|}{2}\|\nabla_{d,r}\theta^{N+1}\|^2+2\|\nabla_{d,r}\theta^{N+1}\|^2\\ +\;&\tau^2\frac{\|a(\Pi_h u^{N,0})\|^2}{16}\|\nabla_{d,r}\theta^N\|^2+\tau^2\alpha_1\|\nabla_{d,r}u_h^{N+1}\|^2+\alpha_1\tau^2\|\nabla_{d,r}u_h^N\|^2\\ +\;&4Ch^{-1}\|\theta^{N+1}\|^2+\tau^2\frac{\|b(\Pi_h u^{N,0})\|^2}{2}\|\theta^{N+1,0}\|^2+\tau^2\frac{\|b(\Pi_h u^{N,0})\|^2}{2}\|\theta^{N,0}\|^2\\ +\;&\|\nabla_{d,r}\theta^{N+1}\|^2+\tau^2\alpha_2\|u_h^{N+1,0}\|^2+4Ch^{-1}\|\theta^{N+1}\|^2+\alpha_2\tau^2\|u_h^{N,0}\|^2\\ +\;&\|\theta^{N+1,0}\|^2+\alpha_4\tau^2\|\nabla_{d,r}u_h^N\|^2+\|\theta^{N+1,0}\|^2+\alpha_5\tau^2\|\nabla_{d,r}u_h^N\|^2+\alpha_6\tau^2\|\nabla_{d,r}\theta^N\|^2\\ +\;&\|\theta^{N+1,0}\|^2+\|\theta^{N+1,0}\|^2+\alpha_7\tau^2\|\nabla_{d,r}\theta^N\|^2+\tau\frac{\|c\|}{2}\|\theta^{N+1,0}\|^2+\tau^2\frac{\|c\|^2}{16}\|\theta^{N,0}\|^2\\ +\;&\|\theta^{N+1,0}\|^2\leq 0,\end{aligned}$$

so,

$$\begin{aligned}&8Ch^{-1}\|\theta^{N+1}\|^2+(\frac{\tau\|a(\Pi_h u^{N,0})\|}{2}+3)\|\nabla_{d,r}\theta^{N+1}\|^2\\ \leq\;&-(\alpha_7+\alpha_6+\frac{\|a(\Pi_h u^{N,0})\|^2}{16})\tau^2\|\nabla_{d,r}\theta^N\|^2-\tau^2\alpha_2\|u_h^{N+1,0}\|^2-\alpha_2\tau^2\|u_h^{N,0}\|^2\\ -\;&(C+6+\tau\frac{\|c\|}{2}+\tau^2\frac{\|b(\Pi_h u^{N,0})\|^2}{2})\|\theta^{N+1,0}\|^2-(\alpha_1+\alpha_4+\alpha_5)\tau^2\|\nabla_{d,r}u_h^N\|^2\\ -\;&\alpha_1\tau^2\|\nabla_{d,r}u_h^{N+1}\|^2-(\frac{1}{4}+\tau^2\frac{\|b(\Pi_h u^{N,0})\|^2}{2}+\tau^2\frac{\|c\|^2}{16})\|\theta^{N,0}\|^2,\end{aligned}$$

then

$$\begin{aligned}\|\theta^{N+1}\|^2+\|\nabla_{d,r}\theta^{N+1}\|^2 \leq\;& \beta_1 h\Bigg[\tau^2(\|u_h^{N+1,0}\|^2+\|u_h^{N,0}\|^2+\|\nabla_{d,r}u_h^{N+1}\|^2+\|\nabla_{d,r}u_h^N\|^2\\ +\;&\|\nabla_{d,r}\theta^N\|^2)+(1+\tau^2)\|\theta^{N,0}\|^2+(1+\tau+\tau^2)\|\theta^{N+1,0}\|^2\Bigg],\end{aligned}$$

where
$\beta_1=max\{-\alpha_1,-\alpha_2,-(\alpha_1+\alpha_4+\alpha_5),-(\alpha_7+\alpha_6+\frac{\|a(\Pi_h u^{N,0})\|^2}{16}),\frac{-1}{4},-(\frac{\|b(\Pi_h u^{N,0})\|^2}{2}+\frac{\|c\|^2}{16}),-(C+6),-\frac{\|c\|}{2},-\frac{\|b(\Pi_h u^{N,0})\|^2}{2}\}/min\{8C,(\frac{\|a(\Pi_h u^{N,0})\|}{2}+3)\}$. $\square$

5. Numerical Experiment

In order to demonstrate the effectiveness and reliability of the weak Galerkin finite element method for solving non-linear convection-diffusion problems, we present a numerical example. MATLAB is used as a software development environment.

Here, we consider the equations (1.1-1.3),[15]

$$\frac{\partial u}{\partial t} - \nabla \cdot (\varepsilon \nabla u) + u \cdot \nabla u + cu = f; \Omega \times (0, T] \tag{5.1}$$

$$u(x,t) = 0; \Gamma \times (0, T] \tag{5.2}$$

$$u(x,0) = u^0(x); x \in \overline{\Omega}. \tag{5.3}$$

In the example, the uniform triangle partition is employed, where $(h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64})$ denotes the spatial mesh size. Let $\Omega = [0,1]^2$, and f and g are calculated by determining the exact solution.

$$u(x,y,t) = e^{-t} x(1-x)y(1-y).$$

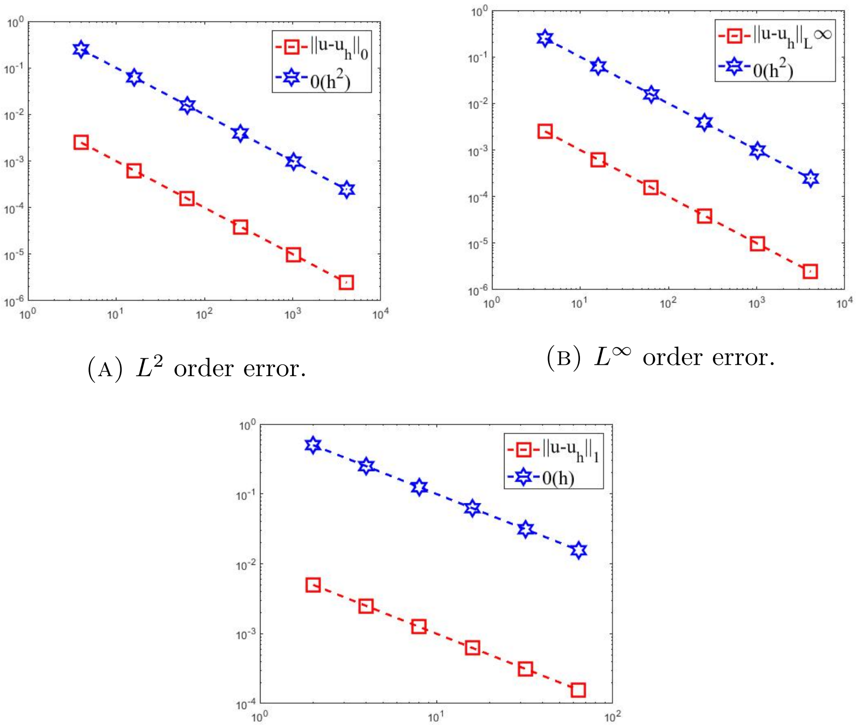
We took $t = 0.2$, $\Delta t = 0.001$ and $\varepsilon = 0.0001$. We calculated the error value in a table 1 and fig. (1) demonstrate the order error scheme.

h	$\|u-u_h\|_{H^1}$	Order	$\|u-u_h\|_{L^2}$	Order	$\|u-u_h\|_{L^\infty}$	Order
$\frac{1}{2}$	5.0027e-03	0	2.5000e-03	0	2.5177e-03	0
$\frac{1}{4}$	2.5018e-03	9.9974e-01	6.2501e-04	2.00e+0	6.2634e-04	2.0071e+0
$\frac{1}{8}$	1.2510e-03	9.9985e-01	1.5625e-04	2.00e+0	1.5634e-04	2.0022e+0
$\frac{1}{16}$	6.2554e-04	9.9993e-01	3.9063e-05	2.00e+0	3.9068e-05	2.0006e+0
$\frac{1}{32}$	3.1278e-04	9.9996e-01	9.7657e-06	2.00e+0	9.7660e-06	2.0002e+0
$\frac{1}{64}$	1.5639e-04	9.9998e-01	2.4414e-06	2.00e+0	2.4413e-06	2.0001e+0

TABLE 1. Errors of the weak Galerkin method with fixed $t = 0.2$, $\Delta t = 0.001$ and $\varepsilon = 0.0001$.

6. Conclusions

In this paper, we introduce the weak Galerkin method, including the stabilizer of weak functions and the Crank-Nicolson scheme for the solution of a non-linear convection-diffusion problem. We have analyzed the stability and assured the error equation and the convergence rates are optimal in the H^1 and L^2 norms. The convergence and flexibility of the weak Galerkin method result in good approximations to the problem (see fig. 1), while the theory has been verified with numerical examples.



(A) L^2 order error.

(B) L^∞ order error.

(C) H^1 order error.

FIGURE 1. Order error for the velocity with H^1-, L^2- and $L^\infty-$ norms.

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(Ala' N. Abdullah) Department of mathematics, college of education for pure sciences, university of basrah AND Department of mathematics, college of education, university of misan, basrah, iraq.
E-mail address: pgs2207@uobasrah.edu.iq; mr.ala_najim@uomisan.edu.iq.

(Hashim A. Kashkool) Department of mathematics, college of education for pure sciences, University of basrah, basrah, iraq.
E-mail address: hkashkool@yahoo.com.